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DENSITY PROPERTIES OF COMPLEX LIE GROUPS

TAKASHI IWAMOTO

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0. Introduction

Let G be a locally compact group. A subgroup H of G is called of *finite covolume* if H is closed and G/H has a finite G-invariant Radon measure.

A. Borel studied properties of subgroups of finite covolume in semi-simple Lie groups without compact factors [1] and proved:

Theorem (Borel's density theorem). *Let G be a semi-simple Lie group without compact factors and H a subgroup of finite covolume in G. Let f be a finite dimensional linear representation of G. Then every $f(H)$-invariant vector subspace is $f(G)$-invariant.*

H. Furstenberg showed that Borel's theorem holds for minimally almost periodic groups and those subgroups of finite covolume [2].

In this paper, applying Furstenberg's idea to some more general situations, we shall prove complex Lie group version of Borel's theorem, that is:

Theorem. *Let G be a complex analytic group and H a subgroup of finite covolume in G. Let f be a holomorphic representation of G on a finite dimensional complex vector space. Then every $f(H)$-invariant vector subspace is $f(G)$-invariant.*

Using this theorem, we obtain properties of subgroups of finite covolume in a complex analytic group [see Section 3].

1. Preliminary results

Let G be a locally compact group and V a finite dimensional vector space over the field K, where K is the real number field R or the complex number field C. Let f be a continuous representation of G on V.

DEFINITION 1. (G, f) is said to have *property* (A) if the following conditions are satisfied:

(1) G has no closed subgroup of finite index.

(2) For any $f(G)$-invariant subspace W of V, $f(G)|_W \subset K \cdot 1_W$ or

$$\{|\det f(g)|_W|^{-1/\dim W} \cdot f(g)|_W ; g \in G\}$$

is unbounded in $\mathrm{GL}(W)$, where $f(g)|_W$ and $\det f(g)|_W$ are the restriction of $f(g)$ to W and its determinant, respectively.

Let $P(V)$ denote the projective space corresponding to V. For a subset $A\subset V$, $\bar{A}$ denotes the canonical image of A in $P(V)$. For a vector subspace $W\subset V$, $\bar{W}$ is called a linear subvariety. Following Furstenberg's terminology, we call a finite union of linear subvarieties a *quasi-linear subvariety*. For simplicity we denote a quasi-linear subvariety by "*q.l.v.*". By the descending chain condition for all the algebraic sets, we have that for any subset $B\subset P(V)$ there exists a minimal *q.l.v.* containing B. In this case this *q.l.v.* is determined uniquely. We denote it by $q(B)$. For a linear map t of a subspace $W\subset V$ to V, $\bar{t}$ denotes the map of $\bar{W}\backslash\overline{\ker t}$ to $P(V)$ corresponding to t. The following lemma is essentialy due to Furstenberg.

Lemma 1. *Let $\{t_k\}_{k=1}^{\infty}$ be in $GL(V)$ such that*

$$|\det t_k|/||t_k||^n \to 0 \quad as \quad k\to\infty$$

where $n=\dim W$ and $||\ \ ||$ is a suitable norm on $\mathrm{End}(V)$.

Then there exist a transformation T on $P(V)$ whose range is a proper $q.l.v.\subsetneqq P(V)$ and a suitable subsequence $\{t'_k\}_{k=1}^{\infty}$ of $\{t_k\}_{k=1}^{\infty}$ such that

$$\bar{t}'_k(x)\to T(x) \quad as \quad k\to\infty$$

for any $x\in P(V)$.

Proof. Let W be a subspace of V. Passing to a subsequence and taking suitable constants a_k's such that $||a_k\cdot t_k||=1$ (where $||\ \ ||$ is a suitable norm on $\mathrm{Hom}(W, V)$), we may assume that $a_k\cdot t_k$ converges to a nonzero linear map h of W into V with respect to the natural topology in $\mathrm{Hom}(W, V)$. We note that for $v\in W\backslash\ker h$

$$\bar{t}_k(\bar{v})\to\bar{h}(\bar{v}) \quad \text{as} \quad k\to\infty\,.$$

Set $W_0=V$. There exist a subsequence $\{t_k(0)\}$ of $\{t_k\}$ and a linear map h_0 of $W_0=V$ to V such that $\{\overline{t_k(0)}\}$ converges pointwise to $\bar{h}_0$ on $P(V)\backslash\overline{\ker h_0}$.

We shall define for $i=1, 2, 3, \cdots$, subspaces W_i, subsequences $\{t_k(i)\}$ of $\{t_k\}$ and linear maps h_i of W_i to V, inductively. Set $W_i=\ker h_{i-1}$. Take a subsequence $\{t_k(i)\}$ of $\{t_k(i-1)\}$ and a linear map h_i such that for $v\in W_i\backslash\ker h_i$

$$\overline{t_k(v)}\,(\bar{v})\to\bar{h}_i(\bar{v}) \quad \text{as} \quad k\to\infty\,.$$

Since $\dim V<+\infty$, there exists an integer m such that $\dim W_{m+1}=\dim\ker\ h_m=0$.

Set $T(\bar{v})=\bar{h}_i(\bar{v})$ for $v\in W_i\backslash W_{i+1}$, where $i=0, 1, 2, \cdots$. Let $\{t'_k\}=\{t_k(m)\}$.

Then the range of T is $\cup \bar{h}_i(\bar{W}_i)$ and $\{\bar{t}'_k\}$ converges pointwise to T on $P(V)$. In order to show that the range of T is proper, it is sufficient to prove det $h_0=0$. By the assumption we have that

$$\begin{aligned}|\det h_0| &= |\det \lim a_k \cdot t_k(0)| \\ &= \lim |\det a_k \cdot t_k(0)| / \|a_k \cdot t_k(0)\|^n \\ &= \lim |\det t_k| / \|t_k\|^n = 0\end{aligned}$$

where $\| \ \|$ is a norm on End $(V)=\mathrm{Hom}(W_0, V)$ and a_k is a scalar constant. q.e.d.

Lemma 2. *Let W_i for $i=1, 2, 3, \cdots, k$, be a subspace of V. If a subspace $W\subset V$ is contained in $\cup_{i=1}^k W_i$, there exists an integer $1\leqq i'\leqq k$ such that $W\subset W_{i'}$.*

Proof. Suppose that W is not contained in any W_i. Then for every $i=1, 2, 3, \cdots, k$, there exists a nonzero vector $v_i\in W$ which is not contained in W_i.

We shall prove that for $j=1, 2, 3, \cdots, k$, there exist j real numbers t_i's such that $\sum_{i=1}^j t_i\cdot v_i$ is not contained in $\cup_{i=1}^j W_i$, by induction on j. By the assumption of induction we can find $(j-1)$ real numbers t_i's such that $u=\sum_{i=1}^{j-1} t_i\cdot v_i$ is not contained in $\cup_{i=1}^{j-1} W_i$. If $u\notin W_j$, set $t_j=0$. Assume that $u\in W_j$. Since $\cup_{i=1}^{j-1} W_i$ is closed, we can find a sufficiently small number t_j such that $u+t_j\cdot v_j \notin \cup_{i=1}^{j-1} W_i$. Since $u\in W_j$ and $t_j\cdot v_j\notin W_j$, we have that $\sum_{i=1}^j t_i\cdot v_i=u+t_j\cdot v_j\notin W_j$. Consequently we can find k real numbers t_i's such that $\sum_{i=1}^k t_i\cdot v_i\notin \cup_{i=1}^k W_i$.

However $\sum_{i=1}^k t_i\cdot v_i\in W\subset \cup_{i=1}^k W_i$ leads to a contradiction. q.e.d.

Lemma 3. *Assume that (G,f) has a property (A). Let $\bar{f}$ be a representation of G on $P(V)$ induced by f and μ a finite $\bar{f}(G)$-invariant Radon measure on $P(V)$. Then the support of μ consists of $\bar{f}(G)$-fixed points.*

Proof. If $f(G)\subset K\cdot 1_V$, there is nothing to prove. Hence we may assume that there exists a sequence $\{g_k\}\subset G$ such that

$$\{|\det f(g_k)|^{-1/\dim V}\cdot f(g_k);\ k=1, 2, 3, \cdots\}$$

is unbounded in End(V). If necessary taking a subsequence we may assume, by Lemma 1, that there exists a transformation T on $P(V)$ whose range is a proper *q.l.v.* Q and that $\bar{f}(g_k)$ converges pointwise to T.

Let $D(x)$ be the distance from x to Q for some metric on $P(V)$. By the bounded convergence theorem, we have that for any $x\in P(V)$

$$\begin{aligned}0 &= \int_{P(V)} D(T(x))\, d\mu \\ &= \int_{P(V)} \lim D(\bar{f}(g_k)(x))\, d\mu\end{aligned}$$

$$= \lim \int_{P(V)} D(\bar{f}(g_k)(x))\, d\mu$$
$$= \int_{P(V)} D(x)\, d\mu$$

This implies that supp $\mu \subset Q$.

Let X be the unique minimal *q.l.v.* containing supp μ. X can be denoted by $X = \cup_{i=1}^{m} \bar{W}_i$ where W_i is a subspace of V. We may assume that there is no inclusion relation among W_i's. Remark that X is also proper in $P(V)$. Since supp μ is $\bar{f}(G)$-invariant and X is the smallest *q.l.v.* containing supp μ, X is $\bar{f}(G)$-invariant. For every $g \in G$, $f(g)\, W_i \subset \cup_{i=1}^{m} W_i$. By Lemma 2, there exists for every $i=1, 2, 3, \cdots, m$, there exists an integer $s(i)$ such that $f(g)\, W_i \subset W_{s(i)}$. Since there is no inclusion relation among W_i's, $\cup_{i=1}^{m} W_i = f(g) \cup_{i=1}^{m} W_i \subset \cup_{i=1}^{m} W_{s(i)} \subset \cup_{i=1}^{m} W_i$ implies that $f(g)\, W_i = W_{s(i)}$ for $i=1, 2, 3, \cdots, m$. Thus G permutes W_i's. Since G has no closed subgroup of finite index, G leaves each W_i invariant.

We shall show that $f(G)|_{W_i} \subset K \cdot 1_{W_i}$ for $i=1, 2, 3, \cdots, m$.

Suppose that there exists $W_{i'}$ such that $f(G)|_{W_{i'}} \not\subset K \cdot 1_{W_{i'}}$. The same argument as above with respect to $\bar{W}_{i'}$, $f|_{W_{i'}}$, and $\mu | \bar{W}_{i'}$ shows that there exists a *q.l.v.* X' contained properly in $\bar{W}_{i'}$ such that supp $\mu|_{\bar{W}_{i'}} \subset X'$. Thus X contains $(\cup_{i \neq i'} W_i) \cup X'$ properly. This contradicts the definition of X.

Therefore $f(G)|_{W_i} \subset K \cdot 1_{W_i}$ for $i=1, 2, 3, \cdots, m$. q.e.d.

2. Main theorem

Let G be a locally compact group and f a continuous representation of G on a finite dimensional vector space V over K.

Lemma 4. *Assume that (G, f) has property (A). Let H be a subgroup of finite covolume in G. Then for 1-dimensional subspace W of V, W is $f(G)$-invariant if and only if W is $f(H)$-invariant.*

Proof. In order to prove the lemma it is sufficient to show "if" part. Set $p = \bar{W} \in P(V)$. Define the map π of G/H to $P(V)$ by

$$\pi\colon G/H \ni g\,H \mapsto \overline{f(g)\,p} \in P(V)\,.$$

Then π carries a finite G-invariant measure on G/H to a finite $\bar{f}(G)$-invariant measure on $P(V)$. Since p is contained in the support of this measure, by Lemma 3, p is a $\bar{f}(G)$-fixed point. q.e.d.

For a representation f of G on a vector space V, the k-th exterior product representation $\Lambda_k f$ of f on $\Lambda_k V$ is defined by

$$\Lambda_k f(g)\,(v_1 \Lambda v_2 \Lambda v_3 \Lambda \cdots \Lambda v_k)$$
$$= f(g) v_1 \Lambda f(g) v_2 \Lambda f(g) v_3 \Lambda \cdots \Lambda f(g) v_k$$

where $g\in G$ and $v_i\in V$.

DEFINITION 2. (G,f) is said to have *property* (B) if for $k=1, 2, 3, \cdots,$ dim V, $(G, \Lambda_k f)$ has property (A).

Theorem 1. *Assume that (G,f) has property (B). Let H be a subgroup of finite covolume in G. Then for any subspace W of V, W is $f(G)$-invariant if and only if W is $f(H)$-invariant.*

Proof. In order to prove the theorem it is sufficient to show "if" part. Let k be dim W. Taking k-th exterior product of f, we can reduce the proof to Lemma 4.

Proposition 1. *Let G be a complex analytic group and f a holomorphic representation of G on a finite dimensional complex vector space V. Then (G,f) has property (B).*

Proof. If f is holomorphic, so is $\Lambda_k f$. Thus, in order to prove the proposition, it is sufficient to show that (G,f) has property (A). Since G is connected G has no closed subgroup of finite index. In order to show the condition (2) of Definition 1 we may assume that $W=V$, for the restriction $f|_W$ of f to a invariant subspace W is also holomorphic.

Let G' denote the complex linear group $f(G)$ and $\hat{G}'$ its Lie algebra. For $A=(a_{ij})\in \mathrm{End}(V)$, we define the norm of A by $\|A\|=(\sum_{i,j}|a_{ij}|^2)^{1/2}$.

For nonzero $X\in\hat{G}'$, set

$$f_X(z)=\|\exp n\, z\, X\|/|\det \exp z\, X|$$

where $n=\dim V$ and $z\in C$. We note that $f_X(z)$ can be written the form;

$$f_X(z)=(|f_1(z)|^2+|f_2(z)|^2+\cdots+|f_m(z)|^2)$$

where $m=n^2$ and $f_i(z)$ is a holomorphic function of z. Thus there exist only two possible cases:

Case 1. There exists a nonzero $X\in\hat{G}'$ such that $f_X(z)$ is unbounded. Since $f_X(z)=\|\exp n\, z\, X\|/|\det \exp z\, X|\leqq\|\exp z\, X\|^n/|\det \exp z\, X|$, we have that

$$\{|\det \exp z\, X|^{-1/n}\cdot\exp z\, X;\ z\in C\}$$

is unbounded.

Case 2. For every $X\in\hat{G}'$, $f_X(z)$ is constant. In this case each element of the matrix $(\exp n\, z\, X)/(\det \exp z\, X)$ is constant. Substituting 0 for z, we have that

$$(\exp n\, z\, X)/(\det \exp z\, X)=1_V$$

for all $z\in C$ and all $X\in\hat{G}$. Consequently we have that

$$\exp z X \in C \cdot 1_V \quad \text{for all} \quad X \in \hat{G} .$$

Since $f(G) = G'$ is connected, $f(G) \subset C \cdot 1_V$. q.e.d.

From Theorem 1 and Proposition 1, it follows that:

Theorem 2. *Let G be a complex analytic group and H a subgroup of finite covolume in G. Let f be a holomorphic representation of G on a finite dimensional complex vector space. Then every $f(H)$-invariant subspace is $f(G)$-invariant.*

REMARK. There are several other cases in which property (B) holds. If G is minimally almost periodic and f is an arbitrary representation, or if G is an analytic group and f is a unipotent representation (G, f) has property (B). In both the cases Theorem 1 holds [2, 4].

3. Density properties

In this section G always denotes a complex analytic group, H a subgroup of finite covolume in G and f a holomorphic representation of G on a finite dimensional complex vector space V.

Corollary 1. *Every element of $f(G)$ is a linear combination of elements of $f(H)$.*

Proof. Let W be the subspace spanned by the elements of $f(H)$ in $\mathrm{End}(V)$. The action of G on $\mathrm{End}(V)$

$$G \times \mathrm{End}(V) \ni (g, A) \mapsto f(g) \circ A \in \mathrm{End}(V)$$

defines a holomorpnic representation of G on $\mathrm{End}(V)$. Since W is H-invariant under this action Theorem 2 concludes that W is G-invariant. Thus we have that for evrey $g \in G$

$$f(g) = f(g) \circ 1_V \in W . \qquad \text{q.e.d.}$$

From Corollary 1, it follows immediately that:

Corollary 2. *The centralizer of $f(H)$ in* $\mathrm{GL}(V)$ *coincides with the centralizer of $f(G)$.*

Corollary 3. *$f(G)$ and $f(H)$ has the same Zariski closure in* $\mathrm{GL}(V)$.

Proof. Let G' and H' be the Zariski closures of $f(G)$ and $f(H)$, respectively. Clearly $H' \subset G'$. By Chevalley's theorem we can find a rational representation r of $\mathrm{GL}(V)$ on a complex vector space E and a nonzero vector $v \in E$ such that

$$H' = \{x \in \mathrm{GL}(V);\ r(x)v \in C \cdot v\} .$$

Since $r\circ f$ is a holomorphic representation of G and $C\cdot v$ is $r\circ f(H)$-invariant, by Theorem 2, $C\cdot v$ is $r\circ f(G)$-invariant. Thus we have that $f(G)\subset H'$. q.e.d.

Appendix. Professor Goto pointed a criterion for property (A). This criterion seems to make the meaning of property (A) clear.

Let V be a finite dimensional complex vector space. An endmorphism A on V is called *conformal* if A is semi-simple and the real part of every eigen value of A is equal to each other. Let the totality of the conformal endomorphisms on V be denoted by $c(V)$. For a representation f of a Lie group by df we denote the associated representation of its Lie algebra.

Proposition (M. Goto). *Let G be an analytic group and $\hat{G}$ its Lie algebra. Let f be a representation of G on a finite dimensional complex vector space V. Assume that for every $f(G)$-invariant subspace W of V $df(\hat{G})|_W\subset c(W)$ implies that $df(\hat{G})|_W\subset C\cdot 1_W$. Then (G,f) has property (A).*

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Department of Mathematics
Kyushu University 33
Hakozaki, Fukuoka 812
Japan