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# PRIME ONE-SIDED IDEALS OF A FINITE NORMALIZING EXTENSION

Dedicated to Professor Hisao Tominaga on his 60th birthday

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## Introduction

Throughout the present paper, $R$ will represent a ring with identity 1. Let $I$ be a right ideal of $R$, and $b_R(I)=\{r\in R\mid Rr\subset I\}$. Then, $b_R(I)$ is the largest ideal of $R$ contained in $I$. We shall call that $I$ is a *prime right ideal* provided that if $X$ and $Y$ are right ideals of $R$ with $XY\subset I$, then either $X\subset I$ or $Y\subset I$. It is clear that a maximal right ideal is a prime right ideal. If $I$ is a prime right ideal, then $b_R(I)$ is a prime ideal. Next, let $S$ be a ring extension of a ring $R$ with the same identity 1. $S$ is said to be a *left torsionfree* $R$-bimodule if $r_S(X)=0$ for every essential ideal $X$ of $R$, where $r_S(X)$ is the right annihilator of $X$ in $S$ (cf. [1]). Right torsionfree is defined similarly, and $S$ is said to be *torsionfree* if it is both left and right torsionfree. Moreover, $S$ is said to be *fully torsionfree* if, for every prime ideal $P$ of $S$, $S/P$ is a right torsionfree $R/(P\cap R)$-bimodule (cf. [3]). Furthermore, we say that $S$ is a *finite normalizing extension* (resp. a *liberal extension*) of $R$ if there exists a finite subset $\{a_1, a_2, \cdots, a_n\}$ of $S$ such that $S=\sum_{i=1}^n Ra_i$ and $Ra_i=a_iR$ for all $i=1, 2, \cdots, n$ (resp. $ra_i=a_ir$ for all $r\in R$ and for all $i=1, 2, \cdots, n$). A ring extension $T$ of $R$ is said to be an *intermediate normalizing extension* (resp. an *intermediate extension*) if there exists a finite normalizing extension (resp. a liberal extension) $S$ of $R$ containing $T$.

Recently, Heinicke and Robson [1, 2], Lorenz [5], Jabbour [3] and others, gave some descriptions of the relationship between the prime ideals of $R$ and any intermediate normalizing extension $T$. In this paper, we shall verify that there is a similar relationship between the prime right ideals of $R$ and $T$. In Section 1, we shall prove a "lying over" theorem for a liberal extension, and a "lying inside" theorem and a "lying outside" theorem for an intermediate extension. In Sections 2 and 3, we shall prove a "cutting down" theorem for a fully torsionfree finite normalizing extension and an intermediate normalizing extension of a fully torsionfree finite normalizing extension.

## 1. Prime right ideals of a liberal extension

In this section, we discuss the relationship of prime right ideals of a liberal extension and an intermediate extension.

**Theorem 1.1** (Lying over). *Let $S$ be a liberal extension of a ring $R$. If $K$ is a prime right ideal of $R$, then there exists a prime right ideal $I$ of $S$ such that $I \cap R=K$. When this is the case, there hold $b_R(I) \cap R=b_R(K)$ and $I \cap R=KS \cap R=K$.*

Proof. Since $b_R(K)$ is a prime ideal, there exists a prime ideal $P$ of $S$ such that $P \cap R=b_R(K)$ and $P$ is a maximal with respect to $P \cap R=b_R(K)$ by [9, Theorem 4.1]. By [9, Lemma 3.2], $S/P$ is a liberal extension of $R/b_R(K)$. Hence we may assume that $S$ is a prime liberal extension of a prime ring $R$ such that $B \cap R \neq 0$ for each non-zero ideal $B$ of $S$, and $K$ is a prime right ideal of $R$ with $b_R(K)=0$. Since, by [9, Lemma 3.5], there is a non-zero ideal $A$ of $S$ such that $R+A$ is contained in a full matrix ring $M_m(R)$, we have $KA \subset M_m(K)$, and so $KA \cap R \subset K$. Consequently, by Zorn's Lemma, there exists a right ideal $I$ of $S$ which is maximal with respect to $I \cap R \subset K$ and $I \supset KA$. Let $X$ and $Y$ be right ideals of $S$ with $XY \subset I$ and $Y \not\subset I$. Then we have $((X+I) \cap R)(SY \cap R) \subset K$. Since $SY \neq 0$, $SY \cap R$ is a non-zero ideal of $R$, and so $SY \cap R \not\subset K$. Therefore $(X+I) \cap R \subset K$, and so $X \subset I$. This implies that $I$ is a prime right ideal of $S$. According to [9, Theorem 4.6], it is clear that $b_S(I)=0$ and $A \not\subset I$. Since $KSA=KA \cap I$, we have $KS \subset I$ and $KS \cap R=I \cap R=K$.

By making use of the same methods as in the proof of the above theorem, we readily obtain the following

**Corollary 1.2** (Going up). *Let $S$ be a liberal extension of $R$. If $K_0 \supset K$ are prime right ideals of $R$ and $I$ is a prime right ideal of $S$ with $I \cap R=K$, then there exists a prime right ideal $I_0$ of $S$ such that $I_0 \supset I$ and $I_0 \cap R=K_0$.*

If $P$ and $Q$ are prime ideals of $S$ such that $P \supset Q$ and $P \cap R=Q \cap R$, then $P=Q$ ([1, Theorem 5.10]). We shall now present some examples of liberal extensions in which there does not hold an "incomparability" theorem for prime right ideals.

Example 1.3. Let $D$ be a division ring, and $S=\begin{pmatrix} D & D \\ D & D \end{pmatrix}$. Then $S$ is a liberal extension of $D=\left\{\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \middle| d \in D\right\}$, and $I=\begin{pmatrix} 0 & 0 \\ D & D \end{pmatrix}$ is a maximal right ideal of $S$ with $b_S(I)=0$. But $I \cap D=0$ which is a prime ideal of $D$.

EXAMPLE 1.4. Let $A$ be a simple ring with a non-zero maximal right ideal $M$, and $S=\begin{pmatrix} A & A \\ A & A \end{pmatrix}$. Then $I_1=\begin{pmatrix} M & M \\ A & A \end{pmatrix}$ and $I_2=\begin{pmatrix} M & M \\ M & M \end{pmatrix}$ are prime right ideals of $S$ such that $I_1 \supsetneqq I_2$, but $I_1 \cap A=M=I_2 \cap A$, which is a prime right ideal but not an ideal.

In the rest of this section, we investigate the relationship of the prime right ideals between $R$ and any intermediate extension.

**Theorem 1.5** (Lying outside). *Let $T$ be a intermediate extension of $R$ with a liberal extension $S$ of $R$ containing $T$, and $J$ a prime right ideal of $T$. Then $J \cap R$ is a prime right ideal of $R$, and there exists a prime right ideal $I$ of $S$ such that $I \cap T \subset J$ and $b_S(I) \cap R=b_T(J) \cap R=b_R(J \cap R)=b_R(I \cap R)$.*

Proof. By Zorn's Lemma, there exists an ideal $P$ of $S$ which is maximal with respect to the property $P \cap T \subset b_T(J)$. Since $b_T(J)$ is a prime ideal of $T$, $P$ is prime and $P \cap R=b_T(J) \cap R$ (cf. [7, Theorem 12.3] and [8, Theorem 3.2]). Then, since $S/P \supset T/(P \cap T) \supset R/(P \cap R)$, we may assume that $S$ is a prime liberal extension of a prime ring $R$, and $T$ is a subring of $S$ containing $R$, and $J$ is a prime right ideal of $T$ such that $b_T(J) \cap R=0$ and $Q \cap T \not\subset J$ for each non-zero ideal $Q$ of $S$. By Zorn's Lemma, there is a right ideal $I$ which is maximal with respect to the property $I \cap T \subset J$. It is clear that $b_S(I)=0$. Suppose that $X$ and $Y$ are right ideals of $S$ with $XY \subset I$ and $Y \not\subset I$. Then $((X+I) \cap T) \times (SY \cap T) \subset I \cap T \subset J$ and $SY$ is a non-zero ideal of $S$. Therefore we obtain $(X+I) \cap T \subset J$, and so $X \subset I$.Thus $I$ is a prime right ideal of $S$. Next we claim that $J \cap R$ is a prime right ideal of $R$. To prove this, assume that $X_1$ and $X_2$ are right ideals of $R$ with $X_1 X_2 \subset J \cap R$ and $X_2 \not\subset J \cap R$. Now, by [8, Proposition 2.5], there exist a liberal extension $S'=\sum_{j=1}^{p} b_j CR$ of $CR$ and a non-zero ideal $X$ of $CR$ such that $XS' \subset CT \subset S' \subset CS$, where $C$ is the center of the complete ring of quotients of $R$, and $b_1, b_2, \cdots, b_p \in V_{CS}(CR)$. Moreover, by [8, Lemma 4.1], there exist non-zero ideals $Y_1$, $Y_2$ of $R$ such that $\sum_{j=1}^{p} b_j Y_2$ is a ring (without 1) and $TY_1T \subset \sum_{j=1}^{p} b_j Y_2 \subset T$. Then we have $X_1 TY_1 TX_2 Y_2 T \subset X_1 \sum_j b_j Y_2 X_2 Y_2 T \subset X_1 Y_2 X_2 \sum_j b_j Y_2 T \subset X_1 X_2 T \subset J$. Since $Y_1 \neq 0$, $Y_2 \neq 0$ and $X_2 \neq 0$, $Y_1 X_2 Y_2$ is a non-zero ideal of $R$ contained in the ideal $TY_1 TX_2 Y_2 T$ of $T$. Since $b_T(J) \cap R=0$, we have $X_1 T \subset J$. Hence $J \cap R$ is a prime right ideal. Once again, using [8, Lemma 4.1], we obtain that $TY_1 T b_R(J \cap R) Y_2 T \subset J$, and so $b_R(J \cap R) Y_2 T \subset J$. This implies that $b_R(J \cap R)=0$. The rest is clear.

**Corollary 1.6.** *Let $R$, $T$, $S$ and $J$ be as in the above theorem.If $(J \cap R)S \cap T \subset J$, then there exists a prime right ideal $I$ of $S$ such that $I \cap T \subset J$ and $I \cap R=J \cap R$. In this case, there holds that $b_S(I) \cap R=b_T(J) \cap R=b_R(J \cap R)=b_R(I \cap R)$.*

Let $T$ be an intermediate extension of $R$, and $S$ a fixed liberal extension of $R$ containing $T$. Let $K$ be a prime right ideal of $R$ and $I$ a prime right ideal of $S$ with $I\cap R=K$. Then, by Zorn's Lemma, there exists a right ideal $J$ of $T$ which is maximal with respect to the property $J\cap R=K$ and $J\supset I\cap T$. In this situation, we shall prove the following

**Lemma 1.7.** *$b_R(K)=b_T(J)\cap R=b_S(I)\cap R$ and $b_S(I)\cap T\subset b_T(J)$.*

Proof. Obviously we obtain $b_S(I)\cap T\subset b_T(J)$. Since $b_R(K)S$ is an ideal of $S$ contained in $I$, this implies $b_R(K)\subset b_R(K)S\cap R\subset b_S(I)\cap R\subset b_T(J)\cap R\subset b_R(K)$.

**Proposition 1.8** (Lying inside). *$J$ is a prime right ideal of $T$ if and only if $b_T(J)$ is a prime ideal of $T$.*

Proof. If $b_T(J)$ is prime, then $b_T(J)$ is an ideal $Q$ of $T$ which is maximal with respect to $Q\cap R=b_R(K)$ and $Q\supset b_S(I)\cap T$ (cf. [7, Theorem 12.7 and 8, Theorem 3.3]). Suppose that $X$ and $Y$ are right ideals of $T$ with $XY\subset I$ and $Y\not\subset J$. Then $TY+b_T(J)\neq b_T(J)$ and $((X+J)\cap R)((TY+b_T(J))\cap R)\subset J\cap R=K$. Hence it follows from the maximality of $b_T(J)$ that $(X+J)\cap R\subset K$, and so $J$ is a prime right ideal.

The following examples show that whether $J$ is a prime right ideal or not.

EXAMPLE 1.9. Let $A$, $M$ and $S$ be as in Example 1.4, and let $T=\begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$. Then $I=\begin{pmatrix} M & M \\ A & A \end{pmatrix}$ is a prime right ideal of $S$ with $I\cap A=M$. Since $\begin{pmatrix} M & A \\ 0 & A \end{pmatrix}\begin{pmatrix} A & A \\ 0 & 0 \end{pmatrix}\subset I\cap T$, $I\cap T$ is not prime right. $\begin{pmatrix} M & A \\ 0 & A \end{pmatrix}$ is the required $J$, and which is a prime right ideal of $T$.

EXAMPLE 1.10. Let $A$ be a simple ring having at least two maximal right ideals, and let $M$ and $N$ be distinct maximal right ideals. Let us put $S=\begin{pmatrix} A & A & A \\ A & A & A \\ A & A & A \end{pmatrix}$ and $T=\begin{pmatrix} A & A & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}$. Since $A$ is two-sided simple, $I=\begin{pmatrix} M & M & M \\ A & A & A \\ N & N & N \end{pmatrix}$ is a prime right ideal of $S$ and $I\cap A=M\cap N$ is a prime right ideal of $A$. Hence, $\begin{pmatrix} M & A & 0 \\ 0 & A & 0 \\ 0 & 0 & N \end{pmatrix}$ is the required $J$. However, since $\begin{pmatrix} M & A & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}\begin{pmatrix} A & A & 0 \\ 0 & A & 0 \\ 0 & 0 & 0 \end{pmatrix}\subset J$, $J$ is not a prime right ideal.

## 2. Prime right ideals of a finite normalizing extension

In the rest of our study, suppose that $S$ is a finite normalizing extension of $R$.

**Proposition 2.1** (Lying over). *Suppose that $S$ is a finite normalizing extension of $R$. If $K$ is a prime right ideal of $R$, then there exists a prime right ideal $I$ of $S$ such that $I \cap R \subset K$ and $b_R(K)$ is a minimal prime ideal over $b_S(I) \cap R$.*

Proof. Since $b_R(K)$ is a prime right ideal of $R$, there exists a prime ideal $Q$ of $S$ such that $b_R(K)$ is a minimal prime ideal over $Q \cap R$. Hence we may assume that $S$ is a prime finite normalizing extension of $R$ and $K$ is a prime right ideal of $R$ such that $A \cap R \not\subset b_R(K)$ for each non-zero ideal $A$ of $S$ and $b_R(K)$ is minimal prime. We next claim that there is a prime right ideal $I$ of $S$ which satisfies $I \cap R \subset K$ and $b_S(I) \cap R = 0$. By Zorn's Lemma, there exists a right ideal $I$ of $S$ which is maximal with respect to $I \cap R \subset K$. Let $X$ and $Y$ be right ideals of $S$ such that $XY \subset I$ and $Y \not\subset I$. Since $((X+I) \cap R) \times (SY \cap R) \subset I \cap R \subset K$ and $SY$ is a non-zero ideal of $S$, we have $X \subset I$. Thus $I$ is a prime right ideal of $R$. Clearly we have $b_S(I) = 0$.

**Lemma 2.2.** *Let $S$ be a torsionfree finite normalizing extension of $R$. If $Y$ is an essential ideal of $R$, then $b_S(YS) \neq 0$.*

Proof. If $X$ is an $R$-$S$-subbimodule of $S$ with $YS \cap X = 0$, then $YX = 0$. Since $Y \neq 0$, there holds $X = 0$. Hence it follows that $YS$ is an essential $R$-$S$-subbimodule of $S$. By [6, Lemma 4], we have $b_S(YS) \neq 0$.

**Proposition 2.3.** *Let $S$ be a prime torsionfree finite normalizing extension of a prime ring $R$. If $I$ is a prime right ideal of $S$ with $b_S(I) = 0$, then $I \cap R$ is a prime right ideal of $R$ with $b_R(I \cap R) = 0$.*

Proof. Assume that $X$ and $Y$ are right ideals of $R$ with $XY \subset I \cap R$ and $Y \not\subset I \cap R$. Then we obviously obtain $XSb_S(RYS) \subset I$, and hence we have either $XS \subset I$ or $b_S(RYS) \subset I$. On the other hand, since $R$ is prime, $RY$ is an essential ideal of $R$, and so, $b_S(RYS)$ is a non-zero ideal of $S$ by Lemma 2.2. Hence there holds $XS \subset I$. Therefore, it follows that $I \cap R$ is a prime right ideal of $R$. The rest of the proof is clear from Lemma 2.2.

If $R$ is not prime, then it may happen that $I \cap R$ is not a prime right ideal of $R$.

EXAMPLE 2.4. Let $A$, $M$ and $S$ be as in Example 1.4. Putting $R = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, $S$ is a prime torsionfree finite normalizing extension of $R$ and $R$ is not prime.

Considering the prime right ideal $I=\begin{pmatrix} M & M \\ M & M \end{pmatrix}$ of $S$, $I\cap R=\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$ is not a prime right ideal of $R$.

Let $S=\sum_{i=1}^{n} Ra_i$ $(Ra_i=a_iR)$ be a prime torsionfree finite normalizing extension of $R$. Let $Q(S)$ be the right Martindale quotient of $S$. Then, there exist orthogonal idempotents $f_1, f_2, \cdots, f_m$ in $V_{Q(S)}(R)=\{q\in Q(S) \mid rq=qr$ for all $r\in R\}$ such that $f_1+f_2+\cdots+f_m=1$ and $m\leqq n$. We set here $P_i=r_R(f_i)$, $i=1, 2, \cdots, m$. Then, the $P_i$ are $m$ distinct minimal prime ideals of $R$ such that $\cap_{i=1}^{m} P_i=0$, and $R/P_i\cong R/P_1$ for all $i$. Let us set $D_i=\cap_{j=1, j\neq i}^{m} P_j$, for all $i$. Then each $D_i$ is a non-zero ideal of $R$ with $f_id=d$ (for all $d\in D_i$), and so $D_i$ is an essential ideal of $f_iR$. Since $f_i\in Q(S)$, there exists an essential ideal $B$ of $S$ such that $f_iB\subset S$ for all $i$ (cf. [1] and [5]). By [1, Theorem 5.7], each $f_iSf_i$ $(1\leqq i\leqq m)$ is a prime torsionfree finite normalizing extension of the prime ring $f_iR$. Now, let $I$ be a prime right ideal of $S$ with $b_S(I)=0$. Let us set $g_i(I)=\{f_isf_i\in f_iSf_i \mid f_isf_iB\subset I\}$. Then we have $f_iSf_iB\subset f_iB\subset S$ by [1, Proposition 5.5], and so $g_i(I)$ is a right ideal of $f_iSf_i$. Then $g_i(I)=f_iSf_i$ if and only if $f_iB\supset I$. Under this situation, we shall prove the following

**Lemma 2.5.** *There exists an $f_i$ such that $g_i(I)\neq f_iSf_i$. Such an $f_i$ is independent of a choice of an essential ideal $B$.*

Proof. If $g_i(I)=f_iSf_i$ for $i=1, 2, \cdots, m$, then we have $B\subset f_1B+f_2B+\cdots+f_mB\subset I$. This is a contradiction. To prove the rest, for essential ideals $B$, $B'$ of $S$, we assume that $f_iB\not\subset I$ and $f_iB'\subset I$. Since $f_iBB'\subset f_iB'\subset I$ and $f_iB\not\subset I$, we have $B'\subset I$, which contradicts $b_S(I)=0$. Hence $f_iB\not\subset I$ if and only if $f_iB\not\subset I$.

By Lemma 2.5, we may assume that $f_iB\not\subset I$ if $i=1, 2, \cdots, t$, and $f_iB\subset I$ if $i=t+1, \cdots, m$.

**Lemma 2.6.** *For each $i=1, 2, \cdots, t$, $g_i(I)$ is a prime right ideals of $f_iSf_i$ with $b_{f_iSf_i}(g_i(I))=0$.*

Proof. For $s=\sum_{j=1}^{n} r_ja_j\in S(r_j\in R)$, we put $s^{\sharp(i)}=\sum_{j\in\sharp(i)} r_ja_j$, where $\sharp(i)=\{j \mid f_ia_jf_i\neq 0\}$, and then $f_isf_i=s^{\sharp(i)}f_i$ ([1, Proposition 5.4]). Let $s^{\sharp(i)}f_i$ and $s'^{\sharp(i)}f_i$ be any elements of $f_iSf_i$ such that $s^{\sharp(i)}f_if_iSf_is'^{\sharp(i)}f_i\subset g_i(I)$ and $s'^{\sharp(i)}f_i\notin g_i(I)$. Then we obtain $s^{\sharp(i)}f_iB\cdot Ss'^{\sharp(i)}f_iB\subset s^{\sharp(i)}f_iSs'^{\sharp(i)}f_iB\subset I$. Since $f_is'f_i\notin g_i(I)$, it follows that $Ss'^{\sharp(i)}f_iB\subset I$ and so $f_isf_iB\subset I$. Thus, $g_i(I)$ is a prime right ideal of $f_iSf_i$. Next, if $f_isf_i\in g_i(I)$ and $f_iSf_if_isf_i\subset g_i(I)$, then $f_iB\cdot Sf_isf_iB\subset f_iSf_isf_iB\subset I$. Since $f_iB\not\subset I$, we have $Sf_isf_iB\subset I$. This implies $Sf_isf_iB=0$, and so $f_isf_iB=0$. Since $f_isf_i\in Q(S)$ and $B$ is an essential ideal of $S$,it follows that $b_{f_iSf_i}(g_i(I))=0$.

Combining Proposition 2.3 with Lemma 2.6, we obtain the following

**Corollary 2.7.** *$g_i(I)\cap f_iR$ is a prime right ideal of $f_iR$ such that $b_{f_iR}(g_i(I)\cap f_iR)=0$ for $i=1, 2, \cdots, t$.*

**Theorem 2.8** (Cutting down). *Let $S$ be a prime torsionfree finite normalizing extension of $R$. If $I$ is a prime right ideal of $S$ such that $b_S(I)=0$, then there exist prime right ideals $K_1, K_2, \cdots, K_t$ of $R$ such that $\cap_{i=1}^t K_i=I\cap R$, $b_R(K_i)=P_i$ for $i=1, 2, \cdots, t$. In this case, there holds $b_R(I\cap R)=\cap_{i=1}^t P_i$.*

Proof. By Lemma 2.5, we may assume that $f_iB\not\subset I$ for $i=1, 2, \cdots, t$ and $f_iB\subset I$ for $i=t+1, \cdots, m$. By Corollary 2.7, $g_i(I)\cap f_iR$ $(1\leqq i\leqq t)$ is a prime right ideal of $f_iR$ such that $b_{f_iR}(g_i(I)\cap f_iR)=0$. Here we set $K_i=\{r\in R\mid f_ir\in g_i(I)\cap f_iR\}$ $(1\leqq i\leqq t)$. Then, it is easily seen that each $K_i$ is a prime right ideal of $R$ such that $b_R(K_i)=P_i$. Now we claim that $\cap_{i=1}^t K_i=I\cap R$. Actually, if $r\in\cap_{i=1}^t K_i$, then $f_ir\in g_i(I)\cap f_iR$ for $i=1, 2, \cdots, t$, and so $f_irB\subset I$. On the other hand, for $i=t+1, \cdots, m$, $f_irB\subset f_iB\subset I$. Hence $rB\subset f_1rB+f_2rB+\cdots+f_trB+f_{t+1}rB+\cdots+f_mrB\subset I$. Since $I$ is the prime right ideal of $S$ with $b_S(I)=0$, we have therefore $r\in I\cap R$. Thus $\cap_{i=1}^t K_i\subset I\cap R$. Conversely, for $r\in I\cap R$, we have $rf_iB\subset I$, which implies that $f_irf_i\in g_i(I)\cap f_iR$ for $i=1, 2, \cdots, t$, and so $r\in\cap_{i=1}^t K_i$. Therefore, $\cap_{i=1}^t K_i=I\cap R$. The rest of the proof is clear.

**Corollary 2.9.** *Let $S$ be an arbitrary fully torsionfree finite normalizing extension of $R$. If $I$ is a prime right ideal of $S$, then there exist prime right ideals $K_1, K_2, \cdots, K_t$ of $R$ such that $I\cap R=\cap_{i=1}^t K_i$, and each $b_R(K_i)$ $(1\leqq i\leqq n)$ is a minimal prime ideal of $R$ over $b_S(I)\cap R$. In this case, there holds $b_R(I\cap R)=\cap_{i=1}^t b_R(K_i)\supset b_S(I)\cap R$.*

EXAMPLE 2.10. Let $A$ and $M$ be as in Example 1.4. Let us set $S=\begin{pmatrix} A & A & A\\ A & A & A\\ A & A & A\end{pmatrix}$ and $R=\begin{pmatrix} A & 0 & 0\\ 0 & A & 0\\ 0 & 0 & A\end{pmatrix}$. Since $R$ is an only essential ideal of $R$, $S$ is a prime torsionfree finite normalizing extension of $R$. For the prime right ideal $I=\begin{pmatrix} M & M & M\\ M & M & M\\ A & A & A\end{pmatrix}$ of $S$ with $b_S(I)=0$, we immediately obtain that $I\cap R=\begin{pmatrix} M & 0 & 0\\ 0 & M & 0\\ 0 & 0 & A\end{pmatrix}$ is a right ideal of $R$ which is not prime and not an ideal. On the other hand, $P_1=\begin{pmatrix} 0 & 0 & 0\\ 0 & A & 0\\ 0 & 0 & A\end{pmatrix}$, $P_2=\begin{pmatrix} A & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & A\end{pmatrix}$ and $P_3=\begin{pmatrix} A & 0 & 0\\ 0 & A & 0\\ 0 & 0 & 0\end{pmatrix}$ are the all minimal prime ideals of $R$. Moreover, $K_1=\begin{pmatrix} M & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{pmatrix}$ and $K_2=\begin{pmatrix} A & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & A \end{pmatrix}$ are prime right ideals of $R$ such that $I \cap R=K_1 \cap K_2$ and $b_R(I \cap R)=b_R(K_1) \cap b_R(K_2)=P_1 \cap P_2 \neq 0$ and $t=2 \neq 3=m$.

## 3. Prime right ideals of an intermediate normalizing extension

In this section, we shall prove a "cutting down" theorem for prime right ideals of an intermediate normalizing extension which corresponds to that of Section 2. Throughout this section, suppose that $T$ is an intermediate normalizing extension of $R$, and $S$ is a fixed finite normalizing extension of $R$ containing $T$.

**Lemma 3.1.** *Let $S$ be a torsionfree $R$-bimodule. If $Y$ is an essential ideal of $R$, then $YT$ is an essential $R$-submodule of $T$ and there exists a non-zero ideal $A$ of $S$ with $0 \neq A \cap T \subset b_T(YT)$.*

Proof. Since $Y$ be an essential ideal of $R$, by making use of the same methods as in the proof of Lemma 2.2, we readily obtain that $YT$ is an essential $R$-subbimodule of $T$. Let $T^*$ be a relative complement of $T$ in the $R$-bimodule $S$. Then, by [6, Lemma 4], $YT+T^*$ contains a non-zero ideal $A$ of $S$ which is an essential $R$-subbimodule of $S$, and so $0 \neq A \cap T \subset b_T(YT)$.

Now, let $Q$ be a prime ideal of $T$. Then, by [3, Proposition 5.6], there exists a prime ideal $P$ of $S$ such that $P \cap T \subset Q$ and $A \cap T \not\subset Q$ for all ideals $A \supsetneqq P$ of $S$. Obviously, $S/P$ is a finite normalizing extension of $R/(P \cap R)$ and $Q/(P \cap T)$ is a prime ideal of an intermediate normalizing extension $T/(P \cap T)$ of $R/(P \cap R)$ such that $B/P \cap T/(P \cap T) \not\subset Q/(P \cap T)$ for each non-zero ideal $B/P$ of $S/P$. As in [2], $Q$ will be called a *standard setting* if $S$ is a prime ring and $Q$ satisfies $B \cap T \not\subset Q$ for each non-zero ideal $B$ of $S$.

**Proposition 3.2.** *Let $S$ be a prime torsionfree finite normalizing extension of a prime ring $R$. If $J$ is a prime right ideal of $T$ such that $b_T(J)$ is a standard setting, then $J \cap R$ is a prime right ideal of $R$ with $b_R(J \cap R)=0$.*

Proof. Let $X$ and $Y$ be right ideals of $R$ with $XY \subset J \cap R$ and $Y \not\subset J \cap R$. Since $R$ is prime, $RY$ is an essential ideal of $R$, and so, by Lemma 3.1, there exist a non-zero ideal $A$ of $S$ with $0 \neq A \cap T \subset b_T(RYT)$. Hence, we have $b_T(RYT) \not\subset J$ since $b_T(J)$ is a standard setting. Noting $XTb_T(RYT) \subset J$, we obtain $X \subset XT \cap R \subset J \cap R$. The assertion $b_R(J \cap R)=0$ is clear by Lemma 3.1.

Throughout the rest of our study, we assume that $S$ is a prime torsionfree finite normalizing extension of $R$. The notations in Section 2 will be used again here. As was seen, each $f_iSf_i$ ($1 \leqq i \leqq m$) is a prime torsionfree finite normalizing extension of the prime ring $f_iR$. Now, by $T_{[i]}$, we denote the subring of the prime ring $f_iSf_i$ which is generated by $f_iTf_i$. Then, by [3, Proposition 5.1 (2)], there exists an ideal $V_{(i)}$ of $T_{[i]}$ such that $V_{(i)}\subset T$ and $V_{(i)}$ is an essential $f_iR$-subbimodule of $T_{[i]}$. Then $V_{(i)}$ can be regarded as an essential $R$-subbimodule of $T_{[i]}$. Hence, $\sum_{i=1}^{m} V_{(i)}=\sum_{i=1}^{m}\oplus V_{(i)}$ is an essential $R$-subbimodule of $\sum_{i=1}^{m}\oplus T_{[i]}$. It is obvious that $(\sum_{i=1}^{T} T_{[i]})\cap R=R$. Moreover, for a prime right ideal $J$ of $T$ such that $b_T(J)$ is a standard setting, we set $h_i(J)=\{q\in T_{[i]}\,|\,qV_{(i)}\subset J\}$. Then, $h_i(J)=T_{[i]}$ if and only if $V_{(i)}T\subset J$. Using a similar argument to Lemma 2.5 making use of the above remark and Lemma 3.1, we obtain the following

**Lemma 3.3.** *$(\sum_{i=1}^{m} V_{(i)})\cap R$ is an essential $R$-subbimodule of $R$, and $V_{(i)}T\not\subset J$ for some $f_i$.*

By Lemma 3.3, we may assume that $V_{(i)}T\not\subset J$ for $i=1, 2, \cdots, s$, and $V_{(i)}T\subset J$ for $i=s+1, \cdots, m$. In this situation, we shall prove the following

**Lemma 3.4.** $b_T(J)\cap R\subset P_1\cap P_2\cap\cdots\cap P_s$.

Proof. Let $1\leqq i\leqq s$. Since $V_{(i)}T\not\subset J$, we obtain $TV_{(i)}T\not\subset J$ and so $TV_{(i)}T\not\subset b_T(J)$. If $TV_{(i)}V_{(i)}T\subset b_T(J)$, then we have $TV_{(i)}TV_{(i)}T\subset TV_{(i)}f_iTf_iV_{(i)}T\subset TV_{(i)}V_{(i)}T\subset b_T(J)$ and so $TV_{(i)}T\subset b_T(J)$, which contradicts $TV_{(i)}T\not\subset b_T(J)$. Hence we have $TV_{(i)}V_{(i)}T\not\subset b_T(J)$. We set here $P'_{(i)}=\{t_i\in T_{[i]}\,|\,TV_{(i)}t_iV_{(i)}T\subset b_T(J)\}$. Then, by the correspondence of prime ideals in a Morita contest

$$C_i=\begin{pmatrix} T & TV_{(i)} \\ V_{(i)}T & T_{[i]} \end{pmatrix},$$

$P'_{(i)}$ is a prime ideal of $T_{[i]}$ such that $P'_{(i)}\not\supset V_{(i)}TV_{(i)}$. We now claim that $A'\cap T_{[i]}\not\subset P'_{(i)}$ for all non-zero ideals $A'$ of $f_iSf_i$. Let $A'$ be a non-zero ideal of $f_iSf_i$ such that $A'\cap T_{[i]}\subset P'_{(i)}$, and let $A=\{s\in S\,|\,f_iSsSf_i\subset A'\}$. Then $A$ is an ideal of $S$. Since $f_iSf_iA'f_iB\subset f_iSf_iB\subset f_iB\subset S$ and $f_iS(f_iSf_iA'f_iB)Sf_i\subset f_iSf_iA'f_iBf_i\subset A'$, we have $f_iSf_iA'f_iB\subset A$. By the Morita context $C_i$, $b_T(J)$ is the prime ideal of $T$ corresponding to the prime ideal $P'_{(i)}$ of $T_{[i]}$. Clearly, $V_{(i)}T(A\cap T)TV_{(i)}\subset f_iSASf_i\cap T_{[i]}\subset A'\cap T_{[i]}\subset P'_{(i)}$. This implies $A\cap T\subset b_T(J)$. Since $b_T(J)$ is a standard setting, we have $A=0$, and so $f_iSf_iA'f_iBf_i=0$. Recalling that $f_iSf_i$ is a prime ring, we have $A'=0$, which is contradictory to $A'\neq 0$. Hence we obtain that $A'\cap T_{[i]}\not\subset P'_{(i)}$ for all non-zero ideals $A'$ of $f_iSf_i$. If $P'_{(i)}\cap f_iR\neq 0$, then, by Lemma 3.1, there exists a non-zero ideal $A'$ of $f_iSf_i$ such that $0\neq A'\cap T_{[i]}\subset(P'_{(i)}\cap f_iR)T_{[i]}\subset P'_{(i)}$, which is a contradiction. Therefore we have $P'_{(i)}\cap f_iR=0$. Since $TV_{(i)}f_i(b_T(J)\cap R)f_iV_{(i)}T\subset b_T(J)$, it follows that $f_i(b_T(J)\cap R)f_i\subset P'_{(i)}\cap f_iR=0$, and hence $b_T(J)\cap R\subset r_R(f_i)=P_i$. This implies $b_T(J)\cap R\subset P_1\cap P_2\cap\cdots\cap P_s$, completing the proof.

**Lemma 3.5.** *Let $J$ be a prime right ideal of $T$ such that $b_T(J)$ is a standard setting. Then, for each $i=1, 2, \cdots, s$, $h_i(J)$ is a prime right ideal of $T_{[i]}$ such that $b_{T_{[i]}}(h_i(J))$ is a standard setting in the extension $f_iSf_i$ of $f_iR$.*

Proof. It is clear that $h_i(J)$ is a right ideal of $T_{[i]}$. Let $X$ and $Y$ be right ideals of $T_{[i]}$ with $XY\subset h_i(J)$ and $Y\not\subset h_i(J)$. Then, we have $YV_{(i)}T\subset J$ and $XV_{(i)}TYV_{(i)}T\subset J$. Hence $XV_{(i)}\subset J$, and so $X\subset h_i(J)$. Therefore $h_i(J)$ is a prime right ideal of $T_{[i]}$. Next we shall show that $b_{T_{[i]}}(h_i(J))\cap f_iR=0$. Now, let $f_ir$ be an arbitrary element in $b_{T_{[i]}}(h_i(J))\cap f_iR$ $(r\in R)$. Then $T_{[i]}f_irV_{(i)}T\subset J$, and $V_{(i)}Tf_i=V_{(i)}f_iTf_i\subset T_{[i]}$. Hence we have $V_{(i)}TrV_{(i)}T=V_{(i)}Tf_irV_{(i)}T\subset J$, and so $rV_{(i)}\subset TrV_{(i)}T\subset b_T(J)\subset J$. Since the ideal $D_i$ of $R$ is an essential ideal of $f_iR$, we obtain $r(D_i\cap V_{(i)})\subset b_T(J)\cap R\subset P_i$ by Lemma 3.4. Noting that $D_i\cap V_{(i)}\neq 0$ and $f_ir(D_i\cap V_{(i)})\subset f_iP_i=0$, we have $f_ir=0$. Thus $b_{T_{[i]}}h_i(J)\cap f_iR=0$. If $b_{T_{[i]}}(h_i(J))$ is not a standard setting, then there exists a non-zero ideal $A$ of $f_iSf_i$ with $A\cap T_{[i]}\subset b_{T_{[i]}}(h_i(J))$. By [1, Theorem 5.10], $A\cap f_iR\neq 0$, this is a contradiction to $b_{T_{[i]}}(h_i(J))\cap f_iR=0$. This completes the proof.

Combining Lemma 3.5 with Proposition 3.2, we obtain the following

**Corollary 3.6.** *If $J$ is a prime right ideal of $T$ such that $b_T(J)$ is a standard setting, then $h_i(J)\cap f_iR$ is a prime right ideal of $f_iR$ with $b_{f_iR}h_i(J(\cap f_iR)=0$ for all $i=1, 2, \cdots, s$.*

Now we arrived at the position to prove the following theorem which corresponds to Theorem 2.8.

**Theorem 3.7** (Cutting down). *Let $S$ be a prime torsionfree finite normalizing extension of a ring $R$, and $T$ a ring with $R\subset T\subset S$. If $J$ is a prime right ideal of $T$ such that $b_T(J)$ is a standard setting, then there exist prime right ideals $K_1, K_2, \cdots, K_s$ of $R$ such that $J\cap R=\cap_{i=1}^s K_i$, $b_R(K_i)=P_i$ for $i=1, 2, \cdots, s$, and $b_R(J\cap R)=\cap_{i=1}^s P_i\supset b_T(J)\cap R$.*

Proof. By Lemma 3.3, we may assume that $V_{(i)}T\not\subset J$ for $i=1, 2, \cdots, s$, and $V_{(i)}T\subset J$ for $i=s+1, \cdots, m$. Then, by Lemma 3.4, we have $b_R(J\cap R)\subset \cap_{i=1}^s P_i$. Let us set $K_i=\{r\in R \mid f_ir\in h_i(J)\cap f_iR\}$ for $i=1, 2, \cdots, s$. Then, by Corollary 3.6, $h_i(J)\cap f_iR$ is a prime right ideal of $f_iR$ with $b_{f_iR}(h_i(J)\cap f_iR)=0$. Hence it follows that $K_i$ is a prime right ideal of $R$ with $b_R(K_i)=P_i$. By making use of the same methods as in the proof of Theorem 2.8, we obtain $J\cap R=\cap_{i=1}^s K_i$.

**Corollary 3.8.** *Let $S$ be an arbitrary fully torsionfree finite normalizing extension of $R$, and $T$ a ring with $R\subset T\subset S$. If $J$ is a prime right ideal of $T$,* *then there exist prime right ideals* $K_1, K_2, \cdots, K_r$ *of* $R$ *such that* $J\cap R=\cap_{i=1}^{s} K_i$; $b_R(J\cap R)=\cap_{i=1}^{s} b_R(K_i)\supset b_T(J)\cap R$, $b_R(K_i)=P_i$ *for all* $i=1, 2, \cdots, s$, *and the* $P_i$ *are minimal prime over* $b_T(J)\cap R$.

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