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NON-SOLVABLE GROUPS, WHOSE CHARACTER DEGREES ARE PRODUCTS OF AT MOST TWO PRIME NUMBERS*)

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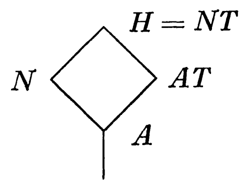
1. Introduction

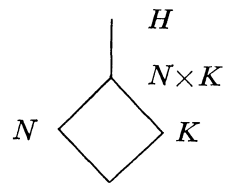
If $n \in N$ has the prime-number-decomposition $n=\prod_{i=1}^{k} p_i^{a_i}$, we define $\omega(n)=\sum_{i=1}^{k} a_i$. If Irr$(G)$ is furthermore the set of irreducible complex characters of the finite group G, we define $\omega(G)=\max_{\chi \in \mathrm{Irr}(G)} \omega(\chi(1))$.
Suppose first that $\omega(G)=1$, which means that all non-linear characters have prime-number-degrees. By a theorem of M. Isaacs and D. Passman (cf. Isaacs [6], 14.4), G must be solvable. But this conclusion does not hold, if $\omega(G)=2$; for example $cd(A_5)=\{1, 3, 4, 5\}$ and $cd(A_7)=\{1, 6, 10, 14, 15, 21, 35\}$ (cf. McKay [8]; cd=character degrees).
There seem to be many solvable groups G with $\omega(G)=2$. In a later paper we shall consider these; in particular we shall show that they have derived length at most 4.**)
The class of non-solvable groups G with $\omega(G)=2$ is quite small. It is completely described by the following theorem.

Theorem. *Suppose that G is non-solvable. Then $\omega(G)=2$ if and only if G is a direct product of an abelian group with a group H of the following type:*
(1) $H \cong A_7$.
(2) $H \cong A_5$.
(3) $H=NT$, where N is a normal abelian 2-subgroup of H, $T \cong A_5$, $N=N_0 \times A$, where A is a the natural module for $SL(2, 4) \cong A_5$ and $[N, T] \leq A$.



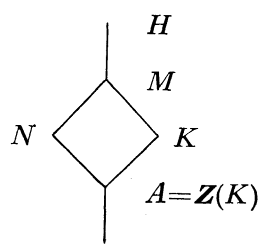
*) This paper is a contribution to the research project "Darstellungstheorie" of the DFG.
**) Arch. Math. 46 (1986), 387–392.

(*4*) *H has a normal subgroup $M=N\times K$ of index 2, where $K\cong A_5$ and N is an abelian 2-group. Further $H/N\cong S_5$ and H/K is abelian.*



(*5*) *H is a central product of $SL(2, 5)$ with an abelian 2-group.*

(*6*) *H has a normal subgroup M of index 2. M is a central product of $K\cong SL(2, 5)$ with an abelian 2-group N, where $N\leq \mathbf{Z}(G)$ and H/K is abelian.*



2. Proof of the Theorem

We start considering a simple group G, here we use the classification of all finite simple groups.

Lemma 1. *If G is simple, non-abelian with $\omega(G)=2$, then*

$$G\cong A_5 \quad or \quad G\cong A_7 .$$

Proof. (*1*) By [10], G isn't sporadic.

(*2*) Let $G\cong A_n$ $(n\geq 5)$. We consider the Young-tableau, corresponding to the partition (n-3, 1, 1, 1).

n	$n-4$	....	2	1
3				
2				
1				

As the hook-product $H_{(n-3,1,1,1)}$ is $n\cdot(n-4)!\cdot 3\cdot 2$, the

character $\chi_{(n-3,1,1,1)}\in \mathrm{Irr}(S_n)$ has degree

$(n!)/H_{(n-3,1,1,1)} = (n-1)(n-2)(n-3)/6$ (cf. Müller [9] 6.36).

But for $n \neq 7$ the partition $(n-3, 1, 1, 1)$ isn't self-associated and we obtain $(n-3)(n-2)(n-1)/6 \in \mathrm{cd}(A_n)$. The hypothesis $\omega(G)=2$ yields that one of $n-1$, $n-2$ and $n-3$ divides 6 and consequently $n \leq 9$. As $8 \in \mathrm{cd}(A_6)$, $20 \in \mathrm{cd}(A_8)$ and $8 \in \mathrm{cd}(A_9)$ (cf. McKay [8]), we conclude $G \cong A_5$ or $G \cong A_7$.
(*3*) Suppose now that G is a Chevalley-group or a twisted type in characteristic p, say. Then G has the Steinberg-character σ of degree $\sigma(1)=p^a$, where p^a is the exact p-part of $|G|$. An inspection of those Chevalley-type groups, whose orders have p-parts at most p^2, yields $G \cong \mathrm{PSL}(2, p)$ or $G \cong \mathrm{PSL}(2, p^2)$. We set $q=p$ or $q=p^2$. Now $\mathrm{PSL}(2, q)$ has character degrees $q-1$ and $q+1$ and as $\omega(G)=2$ and $q>3$, there exists a prime r such that $q-1=2r$. Therefore $q+1=2(r+1)$ and $r+1$ must also be a prime number, hence $r=2$. This yields $q=5$ and $G \cong \mathrm{PSL}(2,5) \cong A_5$.

The composition structure of a group G, satisfying $\omega(G)=2$, is not too complicated, namely

Lemma 2. *If G is a non-solvable group with $\omega(G)=2$, then G has a solvable normal subgroup N such that*

$$G/N \cong A_7 \text{ or } A_5 \text{ or } S_5 .$$

Proof. Let M be the solvable residue of G and M/L a chief-factor of G. By lemma 1, $M/L \cong A_5$ or A_7. We define $N/L = \mathbf{C}_{G/L}(M/L)$. As $\mathrm{Aut}(A_5) \cong S_5$ and $\mathrm{Aut}(A_7) \cong S_7$, $G/N \cong A_5$, S_5, A_7 or S_7. But S_7 can be ruled out, because $20 \in \mathrm{cd}(S_7)$ (cf. Kerber-James [7] page 350). Suppose that N is non-solvable, then we obtain a chief-factor $S/R \cong A_5$ or A_7 with $S \leq N$. If $C/R = \mathbf{C}_{G/R}(S/R)$, we have $C/R \times S/R \trianglelefteq G/R$, where C/R involves a composition factor A_5 or A_7. This, however, yields $\omega(C/R \times S/R) > 2$, a contradiction.

As lemma 2 indicates, there are two cases to consider, namely the case, where A_7 is involved and where A_5 is involved. We start with the first situation, which turns out to be the simplest one. We remind the reader that by Huppert-Manz [5] the group A_7 has the following subgroups $U < A_7$ with $\omega(|A_7 : U|) \leq 2$:

type	index
PSL (2, 7)	$3 \cdot 5$
A_6	7
S_5	$3 \cdot 7$
$(A_4 \times Z_3) \cdot Z_2$	$5 \cdot 7$

Notice that A_7 has no subgroup of index 3^2 (cf. Huppert-Blackburn [4], XII

10.12).

Lemma 3. *Let $\omega(G)=2$, $N \trianglelefteq G$, N solvable and $G/N \cong A_7$. Then $G=N\times E$, where $N'=1$ and $E\cong A_7$.*

Proof. By a trivial induction argument, we may suppose that N is an irreducible G-module.
(*1*) We may assume that N is faithful, because the Schur-extensions of A_7 by Z_2 and Z_3 have character degrees 20 and 24, respectively (cf. Humphreys [2]).
(*2*) Let $1\neq\lambda\in\mathrm{Irr}(N)$. Then $\boldsymbol{T}(\lambda)/N$ must be one of those subgroups, listed above (Huppert [3] 17.11).
(*3*) Suppose that $\boldsymbol{T}(\lambda)/N\cong\mathrm{PSL}(2, 7)$, S_5 or $(A_4\times Z_3)\cdot Z_2$. We consider $\lambda^{\boldsymbol{T}(\lambda)}=\sum_i e_i\chi_i$, $\chi_i\in\mathrm{Irr}(\boldsymbol{T}(\lambda))$. Then $\chi_i^G\in\mathrm{Irr}(G)$, $\chi_i(1)=e_i$ and $\chi_i^G(1)=e_i\cdot 3\cdot 5$ or $e_i\cdot 3\cdot 7$ or $e_i\cdot 5\cdot 7$. As $\omega(G)=2$, we conclude that $e_i=1$ and χ_1 is an extension of λ. By a theorem of Gallagher (cf. Isaacs [6] 6.17), we have $\{\chi_i\}=\{\chi_1\varphi\mid\varphi\in\mathrm{Irr}(\boldsymbol{T}(\lambda)/N\}$. But this contradicts the fact that in each case $\boldsymbol{T}(\lambda)/N$ has non-linear characters.
(*4*) It remains to investigate the situation, where all $\boldsymbol{T}(\lambda)/N\cong A_6$. As the subgroups of type A_6 are conjugate under the action of A_7, we can define $p^s=|\boldsymbol{C}_{\mathrm{Irr}(N)}(U)|$, where $U\cong A_6$ and $|N|=p^n$. A double counting yields

$$7\cdot(p^s-1)=|\{(\lambda, U)\mid 1\neq\lambda\in\mathrm{Irr}(N),\ U\cong A_6,\ U=\boldsymbol{T}(\lambda)/N\}|=p^n-1$$

and consequently $7=1+p^s+\cdots+p^{(n/s-1)s}$. This, however, yields a faithful A_7-module of type (2,2,2), a contradiction.

It remains to deal with the case that A_5 is involved. For this purpose we list the subgroups $U<A_5$ with $\omega(|A_5\colon U|)\leq 2$:

type	index
A_4	5
D_{10}	$2\cdot 3$
D_6	$2\cdot 5$
$Z_2\times Z_2$	$3\cdot 5$

Lemma 4. *Let $\omega(G)=2$, $M\trianglelefteq G$ and M an irreducible non-trivial module for G/M of type $(\underbrace{p,\cdots,p}_{n})$. Furthermore let $G/M\cong A_5$ or $G/M\cong SL(2, 5)$, which means there is a central subgroup L/M of G/M of order at most 2. Then we have $p=2$ and $n=4$.*

Proof. (*1*) Of course $\omega(|G\colon\boldsymbol{T}(\lambda)|)\leq 2$ for all $1\neq\lambda\in\mathrm{Irr}(M)$. In particular $2\mid|\boldsymbol{T}(\lambda)/M|$, if $G/M\cong SL(2, 5)$. As SL(2, 5) has only one involution, we have $L\leq\boldsymbol{T}(\lambda)$.
(*2*) $\boldsymbol{T}(\lambda)/L\not\cong D_{10}$ and $\not\cong D_6$:

If not, λ would be extendible to $\hat{\lambda}\in \mathrm{Irr}(\boldsymbol{T}(\lambda))$, because the Sylow-subgroups of $\boldsymbol{T}(\lambda)/M$ are cyclic (Isaacs [6] 11.31). This, however, means $\lambda^{\boldsymbol{T}(\lambda)}=\hat{\lambda}\varphi+\cdots$ with $\varphi\in \mathrm{Irr}(\boldsymbol{T}(\lambda)/M)$, $\varphi(1)=2$ and $(\hat{\lambda}\varphi)^G(1)=|G\colon \boldsymbol{T}(\lambda)|\cdot 2$, in both cases a contradiction.

(*3*) By (*2*), $\boldsymbol{T}(\lambda)/L$ contains just one subgroup of type $Z_2\times Z_2$, hence $\boldsymbol{T}(\lambda)/M$ one Sylow-subgroup of G/M. Therefore

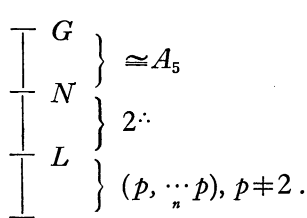
$$
\begin{aligned}
&|\mathrm{Syl}_2(G/M)|\cdot(|\boldsymbol{C}_{\mathrm{Irr}(M)}(Q)|-1) = \\
&|\{(\lambda, Q)\,|\,1\neq\lambda\in \mathrm{Irr}(M),\ Q\in \mathrm{Syl}_2(G/M),\ Q\leq \boldsymbol{T}(\lambda)/M\}| = \\
&|\mathrm{Irr}(M)|-1 = p^n-1\,.
\end{aligned}
$$

If we put $p^s=|\boldsymbol{C}_{\mathrm{Irr}(M)}(Q)|$, we obtain $5\cdot(p^s-1)=p^n-1$. This yields $s|n$, $5=1+p^s+\cdots+p^{(n/s-1)s}$ and consequently $p=2$, $s=2$ and $n=4$.

The lemma above handles the case that A_5 acts on an irreducible module. We suppose now that A_5 acts on an arbitrary solvable group.

Lemma 5. *Let $\omega(G)=2$, $G=G'$ and $N\trianglelefteq G$ with $G/N\cong A_5$. Then there is an abelian 2-group A, such that $A\leq N$, $A\trianglelefteq G$ and either $N=A$ or $G/A\cong SL(2, 5)$.*

Proof. *a*) We first show that N is a 2-group. Put $L=\boldsymbol{O}^2(N)$ and suppose $L\neq 1$. We choose a chief factor L/M; then L/M is of type $(p, \overset{n}{\cdots}, p)$ for an odd prime p. We can assume further on that $M=1$.



(*1*) As $\omega(N)\leq 2$, we have $\mathrm{cd}(N)\subseteq\{1, 2, 4\}$.

(2) $\mathrm{cd}(N)\subseteq\{1, 2\}$: Suppose there is $\tau\in\mathrm{Irr}(N)$ with $\tau(1)=4$. Then τ is fixed under the action of G and consequently $\tau^G=\sum_i e_i\,\chi_i$, where $\chi_i\in\mathrm{Irr}(G)$ and $(\chi_i)_N=e_i\,\tau$. Now $\omega(G)=2$ forces $e_i=1$ and τ is extendible to G. By Gallagher's theorem, $\tau(1)\cdot d\in\mathrm{cd}(G)$ for all $d\in\mathrm{cd}(A_5)$, a contradiction.

(*3*) $\mathrm{cd}(N)=\{1, 2\}$: Suppose $N'=1$, which means $N=S\times L$ with $S\in\mathrm{Syl}_2(N)$. If we consider G/S, lemma 4 implies the trivial action of $G/N\cong A_5$ on L, a contradiction to $G=G'$.

(*4*) By Isaacs [6] 12.11, we have one of the following assertions:

(*i*) N has a characteristic abelian subgroup U of index 2. As $G=G'$, we have $G/U\cong\mathrm{SL}(2, 5)$ and we obtain the same contradiction as in (*3*), using lemma 4.

(*ii*) $|N/\boldsymbol{Z}(N)|=2^2$ or 2^3. But as A_5 has no irreducible GF(2)-module of dimension 2 or 3, $N/\boldsymbol{Z}(N)$ is central in $G/\boldsymbol{Z}(N)$, a contradiction to $G=G'$, because the Schur-multiplier of A_5 has order 2.

Altogether we have shown that N is a 2-group.

b) It remains to show that N has a characteristic abelian subgroup A of index at most 2.

(*1*) $\mathrm{cd}(N)\subseteq\{1, 2\}$: Use the arguments of *a*) (*1*) and (*2*).

(*2*) By Isaacs [6] 12.11, we have to rule out $|N/\boldsymbol{Z}(N)|=2^2$ or 2^3. But this is done as in *a*) (*4*) (*ii*).

For our later arguments we need the knowledge of those extensions G of A_5 by an irreducible GF(2)-module which have $\omega(G)=2$.

EXAMPLE 6. The group A_5 has three irreducible modules over GF(2), namely the trivial module M_0, the augmentated permutation module M_1 and the module M_2, belonging to the representation $A_5\cong\mathrm{SL}(2, 4)$. Let $M\trianglelefteq G$, $G/M\cong A_5$ and M an irreducible GF(2)A_5-module.

a) $M\cong M_0$: If the extension is non-splitting, we have $G\cong\mathrm{SL}(2, 5)$. As $\mathrm{cd}(\mathrm{SL}(2, 5))=\{1, 2, 3, 4, 5, 6\}$ (cf. Dornhoff [1], page 228), $\omega(G)=2$ holds.

b) $M\cong M_1$: In this case we have $M\cong\{\sum_{i=1}^{5} k_i v_i \mid k_i\in\mathrm{GF}(2), \sum_{i=1}^{5} k_i=0\}$, where $\bigoplus_{i=1}^{5}\mathrm{GF}(2)v_i$ is the permutation module for A_5. Obviously, the stabilizer of v_1+v_2 in A_5 is isomorphic to S_3. As $M\cong M_1$ is self-dual, we have $M\cong\mathrm{Irr}(M)$ and therefore there is $\lambda\in\mathrm{Irr}(M)$, such that $\boldsymbol{T}(\lambda)/M\cong S_3$. As in lemma 4 (*2*), λ is extendible to $\boldsymbol{T}(\lambda)$ and we obtain $2\cdot|G\colon \boldsymbol{T}(\lambda)|=2^2\cdot 5\in\mathrm{cd}(G)$. This shows $\omega(G)>2$.

c) $M\cong M_2$: Now SL(2, 4) acts transitively on $M\backslash 1$. Therefore $|G\colon \boldsymbol{T}(\lambda)|=15$ for all $1\neq\lambda\in\mathrm{Irr}(M)$, because $M\cong M_2$ is self-dual. Remark that $\boldsymbol{T}(\lambda)/M$ is a Kleinian four-group. By Prince [11], G must split over M; hence the linear characters λ are extendible to $\hat{\lambda}\in\mathrm{Irr}(\boldsymbol{T}(\lambda))$, So we have $\lambda^{\boldsymbol{T}(\lambda)}=\sum_{\alpha\in\mathrm{Irr}(\boldsymbol{T}(\lambda)/M)}\alpha(1)\alpha\hat{\lambda}$, where $(\alpha\hat{\lambda})^G\in\mathrm{Irr}(G)$ has degree $(\alpha\hat{\lambda})(1)\cdot|G\colon \boldsymbol{T}(\lambda)|=15$. From this we conclude $\mathrm{cd}(G)=\{1, 3, 4, 5, 15\}$, hence $\omega(G)=2$.

The notation M_0, M_1, M_2 for the irreducible modules of A_5 over GF(2) will be used from now on.

Lemma 7. *Suppose* $\omega(G)=2$, $G=G'$ *and* $G/A\cong SL(2,5)$. *Then* $A=1$.

Proof. By lemma 5, A is an abelian 2-group. Suppose $A\neq 1$. Let A/B be a chief-factor of G. We can assume $B=1$. As SL(2,5) has trivial Schur-multiplier, $A\neq M_0$. Define $N/A=\boldsymbol{Z}(G/A)$, then N centralizes A, hence N is abelian. Moreover A, considered as an G/N-module, is one of the modules M_1 and M_2.

(*1*) $A\cong M_1$. A Sylow-5-subgroup S of A_5 has fixed points on N/A, but not on $A\cong M_1$. Hence N is elementary abelian. As M_1 is of defect 0, $N\cong A\oplus C$, where $C\cong M_0$ as A_5-modules. But by example 6 *b*), we have the contradiction $\omega(G/C)>2$.

(*2*) Now we assume $A\cong M_2$. We proceed as in example 6 *c*). Again, A_5 operates transitively on $A\backslash 1$ and also on $\mathrm{Irr}(A)\backslash 1$. If $1\neq\lambda\in\mathrm{Irr}(A)$, then $|G\colon \boldsymbol{T}(\lambda)|=15$. As before $\lambda^{\boldsymbol{T}(\lambda)}=\sum_i e_i\,\chi_i$, $\chi_i(1)=e_i$, $\chi_i^G\in\mathrm{Irr}(G)$ and $\chi_i^G(1)=15\cdot e_i$. As $\omega(G)=2$, all $e_i=1$. Hence χ_1 is an extension of λ to $\boldsymbol{T}(\lambda)$ and $\lambda^{\boldsymbol{T}(\lambda)}=\chi_1\sum_{\varphi\in\mathrm{Irr}(\boldsymbol{T}(\lambda)/A)}\varphi(1)\,\varphi$. But as $\boldsymbol{T}(\lambda)/A$ now is a quaternion group of order 8, it has an irreducible character φ of degree 2. Then $(\chi_1\varphi)^G$ is irreducible and has degree 30.

Lemma 8. *Let* $\omega(G)=2$, $G=G'$ *and* $G/N\cong A_5$. *If* $N\neq 1$, *then either* $G\cong SL(2,5)$ *or G is the splitting extension of* A_5 *with* M_2.

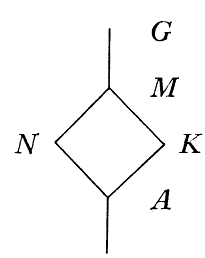
Proof. If G has a factor group isomorphic SL(2,5), then by lemma 7 $G\cong\mathrm{SL}(2,5)$. Hence by lemma 5, we have $G/A\cong A_5$, A an abelian 2-group, and every chief factor A/B is not isomorphic to M_0. By example 6 *b*), also $A/B\ncong M_1$, so $A/B\cong M_2$. To prove $B=1$ we can assume that B is an irreducible A_5-module.

We claim $\boldsymbol{T}(\lambda)<G$ for all $1\neq\lambda\in\mathrm{Irr}(A)$: This follows from the fact that $\mathrm{Irr}(A)$ has no submodule isomorphic to M_0, because A has no factor module isomorphic to M_0.

Let $1\neq\lambda\in\mathrm{Irr}(A)$. As $\omega(|G\colon\boldsymbol{T}(\lambda)|)\le 2$, the arguments of lemma 4 show that $\boldsymbol{T}(\lambda)/A$ contains exactly one Sylow-2-subgroup of G/A. Again, a double counting yields

$$5\cdot(|\boldsymbol{C}_{\mathrm{Irr}(A)}(Q)|-1)=$$
$$|\{(\lambda, Q)\mid 1\neq\lambda\in\mathrm{Irr}(A),\ Q\in\mathrm{Syl}_2(G/A),\ Q\le\boldsymbol{T}(\lambda)/A\}|=$$
$|\mathrm{Irr}(A)|-1$. We put $2^s=|\boldsymbol{C}_{\mathrm{Irr}(A)}(Q)|$. Then
$$5\cdot(2^s-1)=\begin{cases}2^8-1 & \text{if } B\ncong M_0\\ 2^5-1 & \text{if } B\cong M_0\end{cases},\ \text{a contradiction.}$$

Lemma 9. *Let* $\omega(G)=2$ *and suppose that* A_5 *is involved. Then G has the following normal series:*



where

(*i*) $K/A \cong A_5$, $|G/M| \leq 2$ *and* $G/N \cong S_5$ *in case of* $|G/M|=2$.

(*ii*) $(N/A)'=1$.

(*iii*) $A=1$; *or* $A \cong M_0$ *and* $K \cong SL(2,5)$; *or* $A \cong M_2$ *and* K *splits over* A.

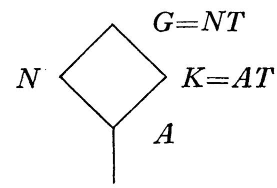
Proof. Let K be the solvable residue of G. If K/A is a chief-factor, then $K/A \cong A_5$ (by lemma 2). We put $N/A = \boldsymbol{C}_{G/A}(K/A)$. Then $M/A := N/A \times K/A$ is a normal subgroup of G/A of index at most 2, where in case of $|G/M|=2$ we have $G/N \cong S_5$. As $\omega(M/A)=2$, we conclude that N/A is abelian. The application of lemma 8 to K finally yields (*iii*).

We shall use the notation of lemma 9.

Lemma 10. *The case* $|G/M|=2$ *and* $A \cong M_2$ *does not occur.*

Proof. As $K/A \cong A_5$ acts transitively on M_2, we have $|G\colon \boldsymbol{T}(\lambda)|=15$ for all $1 \neq \lambda \in \mathrm{Irr}(A)$; in particular $\boldsymbol{T}(\lambda)/A$ contains a Sylow-2-subgroup of G/A. As the Sylow-2-subgroups of S_5 are isomorphic to D_8, $\boldsymbol{T}(\lambda)/A$ is non-abelian. Let $\lambda^{\boldsymbol{T}(\lambda)} = \sum_i e_i \chi_i$ with $\chi_i \in \mathrm{Irr}(\boldsymbol{T}(\lambda))$. Then $\chi_i^G \in \mathrm{Irr}(G)$, $\chi_i^G(1) = 15 \cdot \chi_i(1)$; hence $\omega(G)=2$ does imply $\chi_i(1)=1$. Therefore χ_1 extends λ and $\{\chi_i\} = \{\chi_1 \varphi \mid \varphi \in \mathrm{Irr}\, \boldsymbol{T}(\lambda)/A\}$. As $\boldsymbol{T}(\lambda)/A$ is non-abelian, we obtain a contradiction to $\chi_i(1) = 1$.

Lemma 11. *Now we assume that* $\omega(G)=2$ *and that the solvable residue* K *of* G *is the splitting extension of* $A \cong M_2$ *by* A_5. *We also assume that* $G=M$.



Then N *is abelian and* $N = N_1 \times A$. *Neglecting abelian direct factors of* G, *we can assume that* N *is a* 2-*group.*

Proof. *a*) Let T be a complement of A in K. Then obviously T also is a complement of N in G. Certainly, $\mathrm{Hom}_T(A, A) \cong \mathrm{GF}(4)$. Hence $N/\boldsymbol{C}_N(A)$ has order 1 or 3. We assume at first that $|N/\boldsymbol{C}_N(A)|=3$. Suppose $S/A \in \mathrm{Syl}_3(N/A)$ and $|S/A| = 3^m$. We consider the normal subgroup $R=SK=ST$ of G. Obviously $\omega(R)=2$. As T operates transitively on the characters $(\neq 1)$ of A, we have $|R\colon \boldsymbol{T}_R(\lambda)|=15$ for every $1 \neq \lambda \in \mathrm{Irr}(A)$. $\boldsymbol{T}_R(\lambda)$ splits over A, for the Sylow-2-subgroup of $\boldsymbol{T}_R(\lambda)$ does so (Gaschütz's theorem). Hence there exists an extension $\hat{\lambda}$ of λ to $\boldsymbol{T}_R(\lambda)$, and we obtain $\lambda^R = (\sum_i \hat{\lambda}\, \psi_j(1)\, \psi_j)^R$ $(\psi_j \in \mathrm{Irr}(\boldsymbol{T}_R(\lambda)/A))$. Also $(\hat{\lambda}\psi_j)^R \in \mathrm{Irr}(R)$. As $\omega(R)=2$ and $(\hat{\lambda}\psi_j)^R(1) = |R\colon \boldsymbol{T}_R(\lambda)|\psi_j(1)$

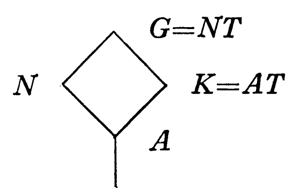
$=15\,\psi_j(1)$, this forces $\psi_j(1)=1$. Hence $\boldsymbol{T}_R(\lambda)/A$ is abelian. We write $\bar{H}=HA/A$ for any $H\leq R$. Then $|\overline{\boldsymbol{T}_R(\lambda)}\cap\bar{K}|=|\overline{\boldsymbol{T}_K(\lambda)}|=4$ and thus $|\overline{\boldsymbol{T}_R(\lambda)}\cdot\bar{K}|=|\overline{\boldsymbol{T}_R(\lambda)}|\cdot|\bar{K}|\,/\,|\overline{\boldsymbol{T}_R(\lambda)}\cap\bar{K}|=3^m\cdot 60=|\bar{R}|$. This shows $\overline{\boldsymbol{T}_R(\lambda)}\nsubseteq\overline{\boldsymbol{C}_R(A)}\cdot\bar{K}<\bar{R}$. Hence there exists an element $\bar{s}\cdot\bar{k}\in\overline{\boldsymbol{T}_R(\lambda)}$, where $\bar{s}\in\bar{S}\backslash\overline{\boldsymbol{C}_R(A)}$, $\bar{k}\in\bar{K}$ and the order of $\bar{s}\cdot\bar{k}$ a power of 3. As $S/\boldsymbol{C}_S(A)$ operates fixedpointfreely on A (namely by multiplication with an element$\neq 1$ of $\mathrm{GF}(4)^{\times}$), $\bar{s}$ does not stabilize any character$\neq 1$ of A. Hence $\bar{k}\neq 1$, so $\bar{k}^3=1$ and $\bar{k}$ does not centralize any Sylow-2-subgroup of $\bar{K}$. As $\overline{\boldsymbol{T}_R(\lambda)}$ contains $\bar{s}\cdot\bar{k}$ and a Sylow-2-subgroup of $\bar{K}$, $\overline{\boldsymbol{T}_R(\lambda)}$ is not abelian, a contradiction. This shows finally $\boldsymbol{C}_N(A)=N$.

b) We can assume that N is an abelian 2-group and $N=N_1\times A$ for some subgroup N_1 of N:

As $N'\leq A\leq\boldsymbol{Z}(N)$ (*a*) and lemma 9), N is nilpotent. Neglecting abelian direct factors of G, we hence can assume that N is a 2-group. As $N\trianglelefteq G$, we have $N'\trianglelefteq G$ and $N'\leq A$. But $N'=A$ is impossible, for then a 5-element of T would operate trivially on N/N', but non-trivially on N'. By the same argument, $A\nsubseteq\Phi(N)$, hence $A\cap\Phi(N)=1$. This implies $N=N_1\times A$ for some N_1.

Now we show that all the groups described in lemma 11 have indeed $\omega(G)=2$.

Lemma 12. *Suppose G has the structure described in lemma* 11, *namely*



with an abelian 2-group N. Then G has the character degrees 1, 3, 4, 5, 15, *so* $\omega(G)=2$.

Proof. *a*) Let S be a Sylow-2-subgroup of T. We can assume that S operates on $A\cong\mathrm{GF}(4)^{(2)}$ by matrices of the form

$$\begin{pmatrix}1 & 0\\ k & 1\end{pmatrix}\qquad (k\in\mathrm{GF}(4))\,.$$

This shows that for any $1\neq s\in S$

$$\boldsymbol{C}_A(S)=\boldsymbol{C}_A(s)=\langle a^{s-1}\,|\,a\in A\rangle\,.$$

If $f\in\boldsymbol{Z}^1(T,A)$, then $f(s)\cdot f(s)^s=f(s^2)=f(1)=1$.

Hence $f(s)\in\boldsymbol{C}_A(s)=\boldsymbol{C}_A(S)$.

b) Let $\lambda\in\mathrm{Irr}(N)$. As $N=N_1\times A$, we can write $\lambda=\gamma\,\alpha$, where $\gamma\in\mathrm{Irr}(N_1)$ and $\alpha\in\mathrm{Irr}(A)$. As $\boldsymbol{T}(\lambda)$ splits over N, there exists an extension $\hat{\lambda}$ of λ to $\boldsymbol{T}(\lambda)$ and

$$\lambda^G = (\sum_j \hat{\lambda}\, \psi_j(1)\, \psi_j)^G \quad (\psi_j \in \mathrm{Irr}(\boldsymbol{T}(\lambda)/N)) .$$

If $\alpha=1$, then $\boldsymbol{T}(\lambda)=G$ and we obtain only irreducible characters $\hat{\lambda}\,\psi_j$ of G of degrees 1, 3, 4, 5.
Suppose $\alpha \neq 1$. Then $|\boldsymbol{T}(\alpha)/N|=4$. We show that for any $s \in \boldsymbol{T}(\alpha)$ also $\lambda^s = \lambda$: Obviously, for $a \in A$

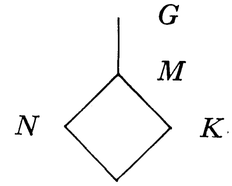
$$\lambda^s(a) = \lambda(a^s) = \alpha^s(a) = \alpha(a) = \lambda(a) .$$

If $n \in N_1$ and $n^s = n \cdot f(s)$ with $f(s) \in A$, then $\lambda^s(n) = \lambda(n^s) = \gamma(n)\, \alpha(f(s))$. But $\alpha^s = \alpha$ implies by *a*) that $f(s) \in \boldsymbol{C}_A(S) = \langle a^{s-1} \mid a \in A \rangle = \mathrm{Ker}\, \alpha$. So $\lambda^s(n) = \gamma(n) = \lambda(n)$. This shows $|\boldsymbol{T}(\lambda)/N|=4$. Then $(\hat{\lambda}\,\psi_j)^G(1) = |G: \boldsymbol{T}(\lambda)| = 15$.

The lemmas 10–12 complete our proof for the case $A \cong M_2$. There remain by lemma 9 the cases $A=1$ or $A \cong M_0$ (=trivial module).

Lemma 13. *We suppose the conditions of lemma* 9 *with* $A=1$ *or* $A \cong M_0$. *Then* G/K *is abelian and* $N \leq \boldsymbol{Z}(G)$.

Proof. *a*) We show that G/K is abelian: As N/A is abelian (lemma 9), we may assume that $A=1$ and $|G/M|=2$.



Let $1 \neq \lambda \in \mathrm{Irr}(N)$ and $\chi \in \mathrm{Irr}(K)$, such that $\chi(1)=4$. If $\boldsymbol{T}(\lambda)=M$, then $\boldsymbol{T}(\lambda\chi)=M$, so $(\lambda\chi)^G \in \mathrm{Irr}(G)$ and $(\lambda\chi)^G(1)=8$, a contradiction. Hence G fixes all characters of N, so also the elements of N. Therefore $N \leq \boldsymbol{Z}(G)$ and so G/K is abelian. From now on we may also assume $A \cong M_0$, hence $K \cong \mathrm{SL}(2,5)$.
b) We show at first that $[N, K]=1$: If $x \in K$ and $n \in N$, then $n^x \cdot n^{-1} \in A$. Hence the automorphism induced by x on N has at most order 2. As $K \cong \mathrm{SL}(2,5)$ has no non-trivial 2-factor-group, K centralizes N. Hence $M=NK$ is an epimorphic image of $N \times K$.
c) Suppose now that N is non-abelian, hence $N'=A$. As $K \cong \mathrm{SL}(2,5)$, K has an irreducible character χ with $\chi(1)=4$ and $\chi(a)=-4$ for $1 \neq a \in A$. On the other hand, N has an irreducible character φ with $\varphi(1)>1$ and $\varphi(a)=-\varphi(1)$, because $A=N' \leq \boldsymbol{Z}(N)$. Now $\varphi\chi \in \mathrm{Irr}(N \times K)$ and $\varphi\chi((a, a))=\varphi\chi(1)$. Hence the kernel $\langle (a, a) \rangle$ of the epimorphism of $N \times K$ onto NK lies in the kernel of $\varphi\chi$. So $\varphi\chi \in \mathrm{Irr}(NK)$ and $\varphi\chi(1)=4 \cdot \varphi(1)$. This contradicts $\omega(NK) \leq \omega(G)=2$.
d) Suppose $N \not\leq \boldsymbol{Z}(G)$, hence $|G/M|=2$. Let $G=M\langle t \rangle$. There exists an $\lambda \in \mathrm{Irr}(N)$ such that $\lambda^t \neq \lambda$ and then $A \not\leq \mathrm{Ker}\, \lambda$. Also, $K \cong \mathrm{SL}(2,5)$ has an irreducible character χ with $\chi(1)=4$, $A \not\leq \mathrm{Ker}\, \chi$. Hence $\lambda\chi$ is a character of

$NK=M$. As t permutes at most the two classes of elements of K of order 5 resp. 10 and χ takes on these classes the values 1, 1 resp. -1, -1, so $\chi^t=\chi$ (cf. Dornhoff [1] page 228). Also $(\lambda\chi)^t=\lambda^t\chi\neq\lambda\chi$. This shows that $\boldsymbol{T}(\lambda\chi)=M$. Hence $(\lambda\chi)^G\in\mathrm{Irr}(G)$ and $(\lambda\chi)^G(1)=8$, contradicting $\omega(G)=2$.

Lemma 14. *We again suppose the conditions of lemma* 9 *and* $A=1$ *or* $A\cong M_0$. *Neglecting abelian direct factors, we have one of the following cases:*
(*1*) $A=1$, $|G/M|=1$: *Then* $G\cong A_5$.
(*2*) $A=1$, $|G/M|=2$: *Then* $G/N\cong S_5$ *and* G/K *is an abelian* 2*-group.*
(*3*) $A\cong M_0$, $|G/M|=1$: *Now* G *is a central product of* $K\cong SL(2,5)$ *with the abelian* 2*-group* N.
(*4*) $A\cong M_0$, $|G/M|=2$: *Now* M *is a central product of* $K\cong SL(2,5)$ *with an abelian* 2*-group* N *and* G/K *is abelian. Also* $N\leq\boldsymbol{Z}(G)$.

Proof. (*1*) If $A=1$ and $|G/M|=1$, then $G=N\times K$, where N is abelian and $K\cong A_5$.
(*2*) Let $A=1$ and $|G/M|=2$. By lemma 13, N is central in G. Hence the 2-complement of N is a direct summand of G. As $N=\boldsymbol{C}_G(K)$, G/N is a group of automorphisms of K, hence $G/N\cong S_5$.
(*3*) Suppose $A\cong M_0$ and $|G/M|=1$. Then G is a central product of $K\cong$ SL(2,5) (lemma 9) with the abelian group N (lemma 13). Obviously, we can assume that N is a 2-group.
(*4*) Finally suppose $A\cong M_0$ and $|G/M|=2$. Then $M=NK$ has the structure described in (*3*), and by lemma 13, G/K is abelian and $N\leq\boldsymbol{Z}(G)$.

Lemma 15. *All the groups* G *described in lemma* 14 *have* $\omega(G)=2$.

Proof. (*1*) Clearly, $\omega(A_5)=2$.
(*2*) Now N is central in G (lemma 13). Let $\lambda\chi\in\mathrm{Irr}(N\times K)$, where $\lambda\in\mathrm{Irr}(N)$ and $\chi\in\mathrm{Irr}(K)$. The behaviour of the irreducible characters of $K\cong A_5$ under the automorphism induced by $G/N\cong S_5$ shows that $\boldsymbol{T}(\chi)=G$, if $\chi(1)=4$. In this case $\chi\lambda$ has an extension to G, as G/M is cyclic. In the other cases $\omega(\chi(1))\leq 1$, hence the irreducible components ψ of $(\lambda\chi)^G$ have $\omega(\psi(1))\leq 2$.
(*3*) As N is abelian and $K\cong$SL(2,5), we have $\omega(N\times K)=\omega(K)=2$. (By Dornhoff [1] page 228, the character degrees of SL(2,5) are 1, 2, 3, 4, 5, 6, without multiplicities.) As $G=NK$ is an epimorphic image of $N\times K$, so also $\omega(G)=2$.
(*4*) As $M=NK$ is a central product, any character of M is of the form $\lambda\chi$, where $\lambda\in\mathrm{Irr}(N)$ and $\chi\in\mathrm{Irr}(K)$. As $N\leq\boldsymbol{Z}(G)$, so $\boldsymbol{T}(\lambda)=G$. If $\chi(1)=4$ or 6, then inspection of the character table of SL(2,5) shows that χ is stable under any automorphism of SL(2,5). Hence $\boldsymbol{T}(\lambda\chi)=G$ in this case and $\lambda\chi$ can be extended to G. Otherwise, $\chi(1)\in\{1, 2, 3, 5\}$ and then all irreducible components ψ of $(\lambda\chi)^G$ have $\omega(\psi(1))\leq 2$.

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