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CONJUGATE SETS OF GAUSSIAN RANDOM FIELDS

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1. Introduction

In [3], [4], [5], [6] and [7], P. Lévy has introduced the notion of the conjugate sets associated with Gaussian random fields (G.r.f.'s) and studied the properties of these sets. Recently, in [1] and [2], we have also shown that this notion is effective to discuss the independence structures of G.r.f.'s. In this paper, we shall be concerned with the characterization of G.r.f.'s with parameter space R^d in terms of the conjugate sets associated with them.

Let $\boldsymbol{S}$ be the class of all the functions on $[0, \infty)$ expressed in the form

$$r(t) = ct^2 + \int_0^\infty (1-e^{-t^2u})u^{-1}d\gamma(u) \qquad (t \geqslant 0), \tag{1.1}$$

where c is a non-negative constant and γ denotes a measure on $(0, \infty)$ such that

$$\int_0^\infty (1+u)^{-1}d\gamma(u) < \infty \quad \text{and} \quad r(1) = 1.$$

An important subclass of $\boldsymbol{S}$ is given by

$$\boldsymbol{L} = \{r(t) = t^\alpha;\ 0<\alpha\leqslant 2\}. \tag{1.2}$$

Then it is well known that for every $r(t) \in \boldsymbol{S}$ and every $d \geqslant 1$ there exists a mean zero G.r.f. $\boldsymbol{X} = \{X(\boldsymbol{x});\ \boldsymbol{x} \in R^d\}$ with homogeneous and isotropic increments that is determined by the *structure function* $r(t)$, i.e.,

$$E[(X(\boldsymbol{x})-X(\boldsymbol{y}))^2] = r(|\boldsymbol{x}-\boldsymbol{y}|) \qquad \text{for every} \quad \boldsymbol{x}, \boldsymbol{y} \in R^d$$

and

$$E[X(\boldsymbol{x})] = 0 \qquad \text{for every} \quad \boldsymbol{x} \in R^d.$$

We can determine this G.r.f. $\boldsymbol{X}$ uniquely except for additional Gaussian random variables with mean zero. We may identify two G.r.f.'s on R^d which are determined by the same structure function, because such G.r.f.'s have the same probabilistic structure related to conditional dependence. From this point of view, we often use the notation $(\boldsymbol{X}, r(t))$ instead of $\boldsymbol{X}$. For details of these G.r.f.'s, see [2], [8], [9], [13] and Remark 2 in Section 2.

We now consider a G.r.f. $(\boldsymbol{X}, r(t))$ on R^d. For every $E\subset R^d$ $(E\neq\phi)$, the symbol $\mu_r(\boldsymbol{x}|E)$ denotes the conditional expectation of $X(\boldsymbol{x})$ conditioned by $\{X(\boldsymbol{z});\ \boldsymbol{z}\in E\}$ in the sense of [6]. In other words, choosing $\boldsymbol{z}_0\in E$ arbitrarily, we set

$$\mu_r(\boldsymbol{x}|E)=X(\boldsymbol{z}_0)+E[X(\boldsymbol{x})-X(\boldsymbol{z}_0)\,|\,X(\boldsymbol{z})-X(\boldsymbol{z}_0);\ \boldsymbol{z}\in E]\qquad(\boldsymbol{x}\in R^d)\,.$$

The conditional covariance function of $(\boldsymbol{X}, r(t))$ is defined by

$$R_r(\boldsymbol{x}, \boldsymbol{y}|E)=E[(X(\boldsymbol{x})-\mu_r(\boldsymbol{x}|E))(X(\boldsymbol{y})-\mu_r(\boldsymbol{y}|E))]\qquad(\boldsymbol{x}, \boldsymbol{y}\in R^d)\,.$$

We can now define, after P. Lévy, the *maximal conjugate set* $\mathcal{F}_X(\boldsymbol{x}|E)$ *of* $\boldsymbol{x}$ *relative to* E as follows:

$$\mathcal{F}_X(\boldsymbol{x}|E)=\{\boldsymbol{y}\in R^d;\ R_r(\boldsymbol{x}, \boldsymbol{y}|E)=0\}\,. \tag{1.3}$$

Since $(\boldsymbol{X}, r(t))$ is Gaussian, the set $\mathcal{F}_X(\boldsymbol{x}|E)$ proves to be the locus of $\boldsymbol{y}\in R^d$ for which $X(\boldsymbol{x})$ and $X(\boldsymbol{y})$ are conditionally independent under the conditioning by $\{X(\boldsymbol{z});\ \boldsymbol{z}\in E\}$. Throughout this paper the phrase "*conjugate set*" means the phrase "maximal conjugate set". We also use the notation S_t to indicate the similar transformation on R^d defined by $S_t\boldsymbol{x}=t\boldsymbol{x}$ $(t>0,\ \boldsymbol{x}\in R^d)$. We are now in a position to state our problems:

PROBLEM 1. *Let* $(\boldsymbol{X}, r(t))$ *be a G.r.f. on* R^d. *Suppose that, given another G.r.f.* $(\boldsymbol{X}_1, r_1(t))$ *on* R^d, *the relation*

$$\mathcal{F}_X(\boldsymbol{x}|E)\subset\mathcal{F}_{X_1}(\boldsymbol{x}|E) \tag{1.4}$$

holds for certain pairs $\{\boldsymbol{x}, E\}$, $\boldsymbol{x}\in R^d$, $E\subset R^d$. *Then is it true that* $r_1(t)=r(t)$?

PROBLEM 2. *Let* $(\boldsymbol{X}, r(t))$ *be a G.r.f. on* R^d. *Suppose that the relation*

$$\mathcal{F}_X(S_t\boldsymbol{x}|S_tE)=S_t\mathcal{F}_X(\boldsymbol{x}|E)\qquad \textit{for every}\quad t>0 \tag{1.5}$$

holds for certain pairs $\{\boldsymbol{x}, E\}$, $\boldsymbol{x}\in R^d$, $E\subset R^d$. *Then is it true that* $r(t)\in\boldsymbol{L}$?

Formerly we studied the special case that E contains at most two points ([1], [2]). The main purpose of this paper is to give affirmative answers to these problems for more general finite sets E under certain reasonable conditions (see Section 2). Generally speaking, if E is finite, Problems 1 and 2 will be reduced to solve some functional equations for $f(x)=r_1(r^{-1}(x))$ and $r(t)$ respectively (see Section 5). Here we shall illustrate the intuitive meanings of our problems. The inclusion (1.4) tells us the following: If a random variable $X(\boldsymbol{y})$ is conditionally independent of $X(\boldsymbol{x})$ under the conditioning by $\{X(\boldsymbol{z});\ \boldsymbol{z}\in E\}$ in the G.r.f. $(\boldsymbol{X}, r(t))$, the same statement holds also for the corresponding random variables in the G.r.f. $(\boldsymbol{X}_1, r_1(t))$. Therefore, if Problem 1 is solved affirmatively, the family $\{\mathcal{F}_X(\boldsymbol{x}|E)\}$ is thought of as a characteristic of the G.r.f. $(\boldsymbol{X}, r(t))$, so far as the conditional independence is concerned. On the other hand, if Problem 2 is solved affirmatively, we can claim that the scale invariance of $(\boldsymbol{X}, r(t))$ in the sense of [9] (also see Remark 2 in Section 2) is derived from the invariance property (1.5) of the family $\{\mathcal{F}_X(\boldsymbol{x}\,|\,E)\}$.

The organization of this paper is as follows. Our main results will be stated in Section 2. In Section 3 we shall discuss the non-degeneracy of $\mathcal{F}_X(\boldsymbol{x}\,|\,E)$, which is guaranteed by the condition (R) mentioned in Section 2. Next we prepare, in Section 4, several lemmas necessary for the proofs of the results mentioned above. By using these lemmas, we shall prove our main results in Section 5. Section 6 is devoted to the proofs of all the propositions stated in Section 3. Finally, in Section 7, we shall give some remarks about Problems 1 and 2.

2. Main results

Let $(\boldsymbol{X}, r(t))$ be a G.r.f. on R^d and E be a non-empty subset of R^d. Throughout this paper we promise that the parameter space R^d is equipped with the following orthogonal decomposition into subspaces $\boldsymbol{G}$ and $\boldsymbol{H}$:

$$R^d = \boldsymbol{G}\oplus\boldsymbol{H}, \quad d\geqslant 3 \quad \text{and} \quad \dim \boldsymbol{H} = 2\,.$$

We always assume that E is finite and expressed as follows:

$$E = \{\boldsymbol{a}_k\}_{1<k<n} \quad \text{and} \quad n=\#E\geqslant 1\,, \tag{2.1}$$

where $\#E$ denotes the cardinal number of E. Then the conditional expectation $\mu_r(\boldsymbol{x}\,|\,E)$ can be expressed in the form

$$\mu_r(\boldsymbol{x}\,|\,E) = \sum_{k=1}^{n} X(\boldsymbol{a}_k)\gamma_r^k(\boldsymbol{x}\,|\,E) \qquad (\boldsymbol{x}\in R^d) \tag{2.2}$$

with certain real numbers $\gamma_r^k(\boldsymbol{x}\,|\,E)$ $(1\leqslant k\leqslant n)$ satisfying the equation $\sum_{k=1}^{n}\gamma_r^k(\boldsymbol{x}\,|\,E) =1$. We are interested in the case that E satisfies one of the following conditions:

(A.1) The points of E are *independent*, i.e., $\#E=1$, or else the vectors $\boldsymbol{a}_k-\boldsymbol{a}_1$ $(2\leqslant k\leqslant n)$ are linearly independent; and

(A.2) The points of E are *symmetric*, i.e., the set $\{|\boldsymbol{a}_j-\boldsymbol{a}_k|\}_{1<k<n}$ is independent of j $(1\leqslant j\leqslant n)$, including the multiplicities.

Further we shall direct our attention to the case that E is contained in a sphere $S(l)=\{\boldsymbol{x}\in R^d;\ |\boldsymbol{x}|=l\}$ $(l\geqslant 0)$. Now we can give answers to Problems 1 and 2 simultaneously.

Theorem 1. *Let* $(\boldsymbol{X}, r(t))$ *be a G.r.f. on* R^d *rigged with* $\{\boldsymbol{a}, E\}$, *where*

$r(t)\in \boldsymbol{S}$, $\boldsymbol{a}\in \boldsymbol{H}$ *and* $E\subset \boldsymbol{G}$. *Suppose that* $\{\boldsymbol{a}, E, r(t)\}$ *satisfies the conditions* (A.1) *and*

(R) $\quad \boldsymbol{a}\neq \boldsymbol{0}$ *and* $R_r(\boldsymbol{a}, -\boldsymbol{a}\,|\,E)<0$; *and further*

(2.3) $\quad \sharp E\geqq 2$ *and* $\gamma_r^j(\boldsymbol{a}\,|\,E)\gamma_r^k(\boldsymbol{a}\,|\,E)\neq 0$ *for some* j, k $(j\neq k)$.

(i) *For another G.r.f.* $(\boldsymbol{X}_1, r_1(t))$ *on* R^d *with* $r_1(t)\in \boldsymbol{S}$, *the identity* $r_1(t)=r(t)$ *holds if and only if*

$$\mathcal{F}_{\boldsymbol{X}}(\boldsymbol{a}\,|\,E)\subset \mathcal{F}_{\boldsymbol{X}_1}(\boldsymbol{a}\,|\,E)\,. \tag{2.4}$$

(ii) *It holds that* $r(t)\in \boldsymbol{L}$ *if and only if*

$$\mathcal{F}_{\boldsymbol{X}}(S_t\boldsymbol{a}\,|\,S_tE)=S_t\mathcal{F}_{\boldsymbol{X}}(\boldsymbol{a}\,|\,E) \qquad \text{for any} \quad t>0\,. \tag{2.5}$$

Theorem 2. *Let* $(\boldsymbol{X}, r(t))$ *be a G.r.f. on* R^d *rigged with* $\{\boldsymbol{a}, E\}$, *where* $r(t)\in \boldsymbol{S}$, $\boldsymbol{a}\in \boldsymbol{H}$ *and* $E\subset S(l)\cap \boldsymbol{G}$. *Suppose that* $\{\boldsymbol{a}, E, r(t)\}$ *satisfies the condition* (R).
(i) *For another G.r.f.* $(\boldsymbol{X}_1, r_1(t))$ *on* R^d *with* $r_1(t)\in \boldsymbol{S}$, *the identity* $r_1(t)=r(t)$ *holds if and only if there exists an open interval* (t_1, t_2) $(t_1<1<t_2)$ *such that*

$$\mathcal{F}_{\boldsymbol{X}}(S_t\boldsymbol{a}\,|\,E)\cap \boldsymbol{H}\subset \mathcal{F}_{\boldsymbol{X}_1}(S_t\boldsymbol{a}\,|\,E)\cap \boldsymbol{H} \qquad \text{for any} \quad t\in(t_1, t_2)\,. \tag{2.6}$$

(ii) *It holds that* $r(t)\in \boldsymbol{L}$ *if and only if*

$$\mathcal{F}_{\boldsymbol{X}}(S_t\boldsymbol{x}\,|\,S_tE)\cap \boldsymbol{H}=(S_t\mathcal{F}_{\boldsymbol{X}}(\boldsymbol{x}\,|\,E))\cap \boldsymbol{H} \qquad \text{for any } \boldsymbol{x}\in \boldsymbol{H} \text{ and any } t>0\,. \tag{2.7}$$

It is meaningful to restate the second parts of the above theorems by using the notion of the projective invariance of G.r.f.'s in the sense of [8] (see Remark 2). We denote by $\mathcal{I}(R^d; E)$ the set of transformations on R^d which consists of all translations, orthogonal transformations, similar transformations and inversions with respect to spheres with centers contained in E. Then we can easily obtain the following corollaries.

Corollary 1. *Let* $(\boldsymbol{X}, r(t))$ *with* $\{\boldsymbol{a}, E\}$ *be a G.r.f. on* R^d *satisfying the same conditions stated in Theorem* 1. *Then it holds that* $r(t)\in \boldsymbol{L}$ *if and only if*

$$\mathcal{F}_{\boldsymbol{X}}(T\boldsymbol{a}\,|\,TE)=T\mathcal{F}_{\boldsymbol{X}}(\boldsymbol{a}\,|\,E) \qquad \text{for any} \quad T\in \mathcal{I}(R^d; E)\,. \tag{2.8}$$

Corollary 2. *Let* $(\boldsymbol{X}, r(t))$ *with* $\{\boldsymbol{a}, E\}$ *be a G.r.f. on* R^d *satisfying the same conditions stated in Theorem* 2. *Then it holds that* $r(t)\in \boldsymbol{L}$ *if and only if*

$$\mathcal{F}_{\boldsymbol{X}}(T\boldsymbol{x}\,|\,TE)\cap \boldsymbol{H}=(T\mathcal{F}_{\boldsymbol{X}}(\boldsymbol{x}\,|\,E))\cap \boldsymbol{H} \tag{2.9}$$
$$\text{for any } \boldsymbol{x}\in \boldsymbol{H} \text{ and any } T\in \mathcal{I}(R^d; E)\,.$$

As for the answer to Problem 1, we have also the following

Theorem 3. *Let* $(\boldsymbol{X}, r(t))$ *be a G.r.f. on* R^d *rigged with* $\{\boldsymbol{a}, E\}$, *where* $r(t)\in \boldsymbol{L}$, $\boldsymbol{a}\in S(l)\cap \boldsymbol{H}$ *and* $E\subset S(l)\cap \boldsymbol{G}$. *Suppose that* $\{\boldsymbol{a}, E, r(t)\}$ *satisfies the con-*

ditions (A.2) *and* (R). *Then, for another G.r.f.* $(\boldsymbol{X}_1, r_1(t))$ *on* R^d *with* $r_1(t)\in\boldsymbol{S}$, *the identity* $r_1(t)=r(t)$ *holds if and only if there exists an open interval* (t_1, t_2) *such that*

$$(2.10)\qquad \mathcal{F}_X(S_t\boldsymbol{a}\,|\,S_tE)\cap\boldsymbol{H}\subset\mathcal{F}_{X_1}(S_t\boldsymbol{a}\,|\,S_tE)\cap\boldsymbol{H}\qquad \textit{for any}\quad t\in(t_1, t_2)\,.$$

REMARK 1. As was stated above, our results are given under the assumption that E is finite. But we can also show that Theorem 2 holds even if E is infinite.

REMARK 2. We denote by $\boldsymbol{S}_d$ the class of all the functions on $[0, \infty)$ expressed in the form

$$(2.11)\qquad r(t)=c_dt^2+\int_0^\infty\{1-Y_d(tu)\}\,dL_d(u)\qquad(t\geqslant 0)\,,$$

$$(2.12)\qquad Y_d(t)=\Gamma(d/2)(2/t)^{(d-2)/2}J_{(d-2)/2}(t)\qquad(t\geqslant 0)\,,$$

where $J_\nu(t)$ is the Bessel function of order ν and c_d is a non-negative constant and further L_d denotes a measure on $(0, \infty)$ such that

$$\int_0^\infty u^2(1+u^2)^{-1}dL_d(u)<\infty\qquad\text{and}\quad r(1)=1\,.$$

Then there exists a one-to-one correspondence between the class $\boldsymbol{S}_d$ and the class of those G.r.f.'s $(\boldsymbol{X}, r(t))$ $(r(1)=1)$ on R^d which are continuous in quadratic mean ([10], [13]). The class $\boldsymbol{S}$ defined by (1.1) is also characterized by the relation $\boldsymbol{S}=\bigcap_{d\geqslant 1}\boldsymbol{S}_d$. As for the class $\boldsymbol{L}$, we note that a G.r.f. $(\boldsymbol{X}, r(t))$ is scale invariant in the sense of [9] (and also projective invariant in the sense of [8]) if and only if $r(t)\in\boldsymbol{L}$.

3. The non-degeneracy of $\mathcal{F}_X(x\,|\,E)$ and the classes of structure functions

In the preceding section we have considered G.r.f.'s $(\boldsymbol{X}, r(t))$ on R^d rigged with $\{\boldsymbol{a}, E\}$, for which $\{\boldsymbol{a}, E, r(t)\}$ satisfies the condition (R) stated in Theorem 1. This assumption plays an important role in our discussion about the non-degeneracy of the conjugate sets $\mathcal{F}_X(\boldsymbol{x}\,|\,E)$ concerned. Precisely speaking, the non-degeneracy of these sets is guaranteed by the following two propositions.

Proposition 1. *Let* $(\boldsymbol{X}, r(t))$ *be a G.r.f. on* R^d *rigged with* $\{\boldsymbol{a}, E\}$, *where* $r(t)\in\boldsymbol{S}$, $\boldsymbol{a}\in\boldsymbol{H}$ *and* $E\subset\boldsymbol{G}$. *Suppose that* $\{\boldsymbol{a}, E, r(t)\}$ *satisfies the conditions* (A.1) *and* (R). *Then there exists a sequence* $\{I_k\}_{1\leqslant k\leqslant n}$ *of open intervals such that*

$$(3.1)\qquad \Phi_E(\boldsymbol{a})\in\prod_{k=1}^n I_k\subset\prod_{k=1}^n(|\boldsymbol{a}_k|, \infty)\quad\textit{and}$$

$$\prod_{k=1}^{n} I_k \subset \Phi_E(\mathcal{F}_X(\boldsymbol{a}\,|\,E))\,, \tag{3.2}$$

where we set $\Phi_E(\boldsymbol{x})=(|\boldsymbol{x}-\boldsymbol{a}_1|,\cdots,|\boldsymbol{x}-\boldsymbol{a}_n|)$ *for* $\boldsymbol{x}\in R^d$.

Proposition 2. *Let* $(X, r(t))$ *be a G.r.f. on* R^d *rigged with* $\{\boldsymbol{a}, E\}$, *where* $r(t)\in \boldsymbol{S}$, $\boldsymbol{a}\in \boldsymbol{H}$ *and* $E\subset S(l)\cap \boldsymbol{G}$. *Suppose that* $\{\boldsymbol{a}, E, r(t)\}$ *satisfies the condition* (R). *Then there exists an open interval I such that*

$$\Psi_E(\boldsymbol{a})\in I\subset (l, \infty) \quad and \tag{3.3}$$

$$I\subset \Psi_E(\mathcal{F}_X(\boldsymbol{x}\,|\,E)\cap \boldsymbol{H}) \qquad for\ any \quad \boldsymbol{x}\in \Psi_E^{-1}(I)\cap \boldsymbol{H}\,, \tag{3.4}$$

where we set $\Psi_E(\boldsymbol{x})=|\boldsymbol{x}-\boldsymbol{a}_1|$ *for* $\boldsymbol{x}\in R^d$.

In what follows we shall give some examples of $\{\boldsymbol{a}, E, r(t)\}$ satisfying the condition (R). As for the case $E=\{0\}$, we have the following

Proposition 3. (i) *Suppose that* $r(t)\in \boldsymbol{S}$ *is strictly convex on* $(0, t_0)$ *for some* t_0 $(0<t_0<\infty)$. *Then* $\{\boldsymbol{a}, \{0\}, r(t)\}$ *satisfies the condition* (R) *for any* $\boldsymbol{a}\in \boldsymbol{H}$ *with sufficiently small* $|\boldsymbol{a}|>0$.
(ii) *Suppose that* $r(t)\in \boldsymbol{S}$ *is strictly concave on* $(0, t_0)$, *strictly convex on* (t_0, ∞) *for some* t_0 $(0<t_0<\infty)$ *and* $r'(+0)\leqslant r'(\infty)$. *Then* $\{\boldsymbol{a}, \{0\}, r(t)\}$ *satisfies the condition* (R) *for any* $\boldsymbol{a}\in \boldsymbol{H}$ *with sufficiently large* $|\boldsymbol{a}|>0$.

We now proceed to the more general case of finite sets E with $\sharp E\geqslant 2$. Let $\{\boldsymbol{e}_i\}_{1\leqslant i\leqslant d}$ be the canonical orthonormal basis of R^d and assume that the subspace $\boldsymbol{G}$ is spanned by $\{\boldsymbol{e}_i\}_{1\leqslant i\leqslant d-2}$. Let us introduce the sets $E_n^j(l)$ $(l>0,\ n\geqslant 2,\ 1\leqslant j\leqslant 4)$ defined as follows:

$$E_n^1(l) = \{\boldsymbol{a}_k = l\sqrt{n/(n-1)}(\boldsymbol{e}_k-\frac{1}{n}\sum_{j=1}^{n}\boldsymbol{e}_j);\ 1\leqslant k\leqslant n\}\,; \tag{3.5}$$

$$E_n^2(l) = \{\boldsymbol{a}_k = l(-1)^k\boldsymbol{e}_{[(k+1)/2]};\ 1\leqslant k\leqslant n\} \quad (n\text{: even}); \tag{3.6}$$

$$E_n^3(l) = \{\boldsymbol{a}(I) = \sum_{k=1}^{m}(l/\sqrt{m})(-1)^{\chi(k|I)}\boldsymbol{e}_k;\ I\subset\{1, 2, \cdots, m\}\} \quad (n=2^m); \tag{3.7}$$

$$E_n^4(l) = \{\boldsymbol{a}_k = (l\cos 2k\pi/n)\boldsymbol{e}_1+(l\sin 2k\pi/n)\boldsymbol{e}_2;\ 1\leqslant k\leqslant n\}\,, \tag{3.8}$$

where we set $\chi(k\,|\,I)=1$ for $k\in I$ and $\chi(k\,|\,I)=0$ otherwise. We note that each set $E_n^j(l)$ given above is contained in $S(l)$ and satisfies the condition (A.2). Moreover the set $E_n^1(l)$ satisfies the condition (A.1). Since each set $E_n^j(l)$ $(1\leqslant j\leqslant 3)$ consists of all the vertices of a high-dimensional regular polyhedron, the number $n=\sharp E_n^j(l)$ should be dominated by some constant related to the dimension d of R^d. In particular, when $E_n^j(l)$ $(1\leqslant j\leqslant 3)$ is contained in $\boldsymbol{G}$, we must assume the following:

$$n = \sharp E_n^j(l)\leqslant \begin{cases} d-2 & \text{for } j=1\,, \\ 2(d-2) & \text{for } j=2\,, \\ 2^{d-2} & \text{for } j=3\,. \end{cases} \tag{3.9}$$

By using the sets $E_n^j(l)$ given above, we can describe the condition (R) for any $r(t)\in \boldsymbol{S}$.

Proposition 4. *Let $\boldsymbol{a}\in S(l)\cap \boldsymbol{H}$ and $r(t)\in \boldsymbol{S}$ be given arbitrarily. Then, for each j $(1\leqslant j\leqslant 3)$, $\{\boldsymbol{a}, E_n^j(l), r(t)\}$ satisfies the condition* (R) *provided that n is chosen to be sufficiently large under the restriction* (3.9).

Before stating the results on the class $\boldsymbol{L}$, we shall introduce here the real number $p[E]$ which corresponds to each set $E\subset S(l)$ $(l>0,\ \sharp E\geqslant 2)$. When we set

$$F(\alpha)=F(\alpha;\,E)=2(\sqrt{2})^{\alpha}-2^{\alpha}-\frac{1}{n}\sum_{k=1}^{n}(|\boldsymbol{a}_k-\boldsymbol{a}_1|/l)^{\alpha}\qquad(0<\alpha\leqslant 2)\,, \tag{3.10}$$

we see that the function $F(\alpha)$ is strictly concave on $(0, 2]$ and satisfies the inequalities $F(+0)=\dfrac{1}{n}>0>F(2)$. Then the real number $p[E]$ is defined as the unique solution of $F(\alpha)=0$ in $(0, 2)$. Obviously the equality $p[S_tE]=p[E]$ holds for each $t>0$. Further we see that $F(\alpha)>0$ on $(0, p[E])$ and $F(\alpha)<0$ on $(p[E], 2]$. Thus setting

$$\boldsymbol{L}(\beta)=\{r(t)=t^{\alpha};\ \beta<\alpha\leqslant 2\}\qquad(0<\beta<2)\,, \tag{3.11}$$

we have the following

Proposition 5. *Let $\boldsymbol{a}\in S(l)\cap \boldsymbol{H}$, $E\subset S(l)\cap \boldsymbol{G}$ and $r(t)\in \boldsymbol{L}$ be given. Suppose that E satisfies the conditions* (A.2) *and $\sharp E\geqslant 2$. Then $\{\boldsymbol{a}, E, r(t)\}$ satisfies the condition* (R) *if and only if $r(t)\in \boldsymbol{L}(p[E])$.*

We can extend this result to the case of regularly varying functions, which correspond to G.r.f.'s with non-degenerate scaling limits (see [9] and [11]). In general, a function $r(t)$ is called a *regularly varying function with exponent α* (*r.v.f.* (α)) for some $\alpha>0$ if $r(t)$ is a positive continuous function defined on some interval $(0, t_0)$ and satisfies the equality

$$\lim_{t\to+0} r(xt)/r(t)=x^{\alpha}\qquad\text{for any}\quad x>0\,. \tag{3.12}$$

We denote by $\bar{\boldsymbol{L}}$ the class of r.v.f.'s $r(t)\in \boldsymbol{S}$ with exponent α for some $\alpha\in(0, 2]$. Obviously we have $\boldsymbol{L}\subset\bar{\boldsymbol{L}}$. More general examples of subclasses of $\bar{\boldsymbol{L}}$ will be given in the next section. Now setting

$$\bar{\boldsymbol{L}}(\beta)=\{r(t)\in \boldsymbol{S};\ r(t)\text{ is a r.v.f. }(\alpha)\text{ for some }\alpha\in(\beta, 2]\}\quad(0<\beta<2), \tag{3.13}$$

we have the following

Proposition 6. *Let $\boldsymbol{a}\in S(l)\cap \boldsymbol{H}$, $E\subset S(l)\cap \boldsymbol{G}$ and $r(t)\in\bar{\boldsymbol{L}}(p[E])$ be given. Suppose that E satisfies the conditions* (A.2) *and $\sharp E\geqslant 2$. Then $\{S_\rho\boldsymbol{a},\ S_\rho E,\ r(t)\}$*

satisfies the condition (R) *for sufficiently small* $\rho>0$.

Consequently, we can describe the condition (R) for the classes $\boldsymbol{L}$ and $\bar{\boldsymbol{L}}$ by using Propositions 5, 6 and the following

Proposition 7. *When we set* $\alpha_{nj}=p[E_n^j(l)]$ $(n\geqslant 2,\ 1\leqslant j\leqslant 4)$, *we have*

$$\lim_{n\to\infty}\alpha_{nj}=0\ \ (1\leqslant j\leqslant 4);\ \textit{and so} \tag{3.14}$$

$$\boldsymbol{L}=\bigcup_n \boldsymbol{L}(\alpha_{nj})\quad \textit{and}\quad \bar{\boldsymbol{L}}=\bigcup_n \bar{\boldsymbol{L}}(\alpha_{nj})\qquad (1\leqslant j\leqslant 4). \tag{3.15}$$

It is difficult in general to describe the value of $p[E]$ explicitly. In the special case of $E=E_n^1(l)$, however, we can find an analogue α_n of $\alpha_{n1}=p[E_n^1(l)]$ defined by

$$\alpha_n=\frac{\log((n+1)/n)^2}{\log(2(n+1)/n)}\qquad (n\geqslant 2). \tag{3.16}$$

Proposition 8. *Suppose that* $\boldsymbol{a}\in S(\sqrt{(n+1)/(n-1)}\,l)\cap \boldsymbol{H}$, $l>0$ *and* $2\leqslant n\leqslant d-2$. *Then the following assertions hold:*

(i) *Given* $r(t)\in \boldsymbol{L}$, $\{\boldsymbol{a}, E_n^1(l), r(t)\}$ *satisfies the condition* (R) *if and only if* $r(t)\in \boldsymbol{L}(\alpha_n)$.

(ii) *Given* $r(t)\in \bar{\boldsymbol{L}}(\alpha_n)$, $\{S_\rho \boldsymbol{a}, S_\rho E_n^1(l), r(t)\}$ *satisfies the condition* (R) *for sufficiently small* $\rho>0$.

Obviously we see that $\lim_{n\to\infty}\alpha_n=0$ and so

$$\boldsymbol{L}=\bigcup_{n\geqslant 2}\boldsymbol{L}(\alpha_n)\quad\text{and}\quad \bar{\boldsymbol{L}}=\bigcup_{n\geqslant 2}\bar{\boldsymbol{L}}(\alpha_n).$$

We also note that, inspired by the defining condition (3.12) of r.v.f. (α), we can similarly discuss the case that $r(t)\in \boldsymbol{S}$ satisfies the equality $\lim_{t\to\infty} r(xt)/r(t)=x^{\alpha}$ for any $x>0$. All the propositions stated in this section will be proved in Section 6.

4. Lemmas

In this section, we shall provide some preliminary lemmas. Let $(\boldsymbol{X}, r(t))$ be a G.r.f. on R^d and E be a subset of R^d given by (2.1). First we see that $\mu_r(\boldsymbol{x}\,|\,\boldsymbol{z})=X(\boldsymbol{z})$ and so

$$R_r(\boldsymbol{x},\boldsymbol{y}\,|\,\boldsymbol{z})=\{r(|\boldsymbol{x}-\boldsymbol{z}|)+r(|\boldsymbol{y}-\boldsymbol{z}|)-r(|\boldsymbol{x}-\boldsymbol{y}|)\}/2\qquad (\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in R^d).$$

In general, we can employ the expression (2.2) of $\mu_r(\boldsymbol{x}\,|\,E)$ $(\sharp E\geqslant 2)$. Strictly speaking, the coefficients $\gamma_k=\gamma_r^k(\boldsymbol{x}\,|\,E)$ $(1\leqslant k\leqslant n)$ satisfy the following equations:

$$(4.1)\qquad \begin{cases} \sum_{k=1}^{n} \gamma_k = 1 \\ \sum_{k=1}^{n} R_r(\boldsymbol{a}_j, \boldsymbol{a}_k \mid \boldsymbol{a}_1)\gamma_k = R_r(\boldsymbol{a}_j, \boldsymbol{x} \mid \boldsymbol{a}_1) \qquad (2 \leqslant j \leqslant n)\,. \end{cases}$$

Moreover, if we assume that $r(t)\in \boldsymbol{S}$ ($r(t)\not\equiv t^2$), the solution of these equations can be determined uniquely on account of the property (iv) of Lemma 5. It is convenient to introduce the following notations:

$$(4.2)\qquad \Lambda_r(\boldsymbol{x}, \boldsymbol{y} \mid E) = \sum_{k=1}^{n} r(|\boldsymbol{y}-\boldsymbol{a}_k|)\gamma_r^k(\boldsymbol{x} \mid E) \qquad (\boldsymbol{x}, \boldsymbol{y}\in R^d) \quad \text{and}$$

$$(4.3)\qquad \Lambda(r;\, E) = \Lambda_r(0, \boldsymbol{a}_1 \mid E) = \sum_{k=1}^{n} r(|\boldsymbol{a}_1-\boldsymbol{a}_k|)\gamma_r^k(0 \mid E)\,.$$

Then we immediately obtain the following expression: For any $\boldsymbol{x}, \boldsymbol{y}\in R^d$,

$$(4.4)\qquad 2R_r(\boldsymbol{x}, \boldsymbol{y} \mid E) = r(|\boldsymbol{x}-\boldsymbol{a}_1|)+\Lambda_r(\boldsymbol{x}, \boldsymbol{y} \mid E)-r(|\boldsymbol{x}-\boldsymbol{y}|)-\Lambda_r(\boldsymbol{x}, \boldsymbol{a}_1 \mid E)\,.$$

Lemma 1. *Let $(X, r(t))$ be a G.r.f. on R^d and let $E\subset S(l)\cap \boldsymbol{G}$ be given arbitrarily. Then the coefficients $\gamma_r^k(\boldsymbol{x} \mid E)$ $(1\leqslant k\leqslant n)$ in the expression* (2.2) *may be chosen to satisfy the relation*

$$(4.5)\qquad \gamma_r^k(\boldsymbol{x} \mid E) = \gamma_r^k(0 \mid E) \qquad (\boldsymbol{x}\in \boldsymbol{H},\ 1\leqslant k\leqslant n)\,.$$

Moreover $R_r(\boldsymbol{x}, \boldsymbol{y} \mid E)$ has the following expression: For any $\boldsymbol{x}, \boldsymbol{y}\in \boldsymbol{H}$,

$$(4.6)\qquad 2R_r(\boldsymbol{x}, \boldsymbol{y} \mid E) = r(|\boldsymbol{x}-\boldsymbol{a}_1|)+r(|\boldsymbol{y}-\boldsymbol{a}_1|)-r(|\boldsymbol{x}-\boldsymbol{y}|)-\Lambda(r;\, E); \quad \textit{and}$$

$$(4.7)\qquad 0\leqslant \Lambda(r;\, E)\leqslant 2r(l)\,.$$

Proof. It follows from the assumption on E that

$$R_r(\boldsymbol{a}_j, \boldsymbol{x} \mid \boldsymbol{a}_1) = r(|\boldsymbol{a}_j-\boldsymbol{a}_1|)/2 \qquad \text{for any} \quad \boldsymbol{x}\in \boldsymbol{H} \quad (2\leqslant j\leqslant n)\,.$$

Therefore the solution $\gamma_k=\gamma_r^k(\boldsymbol{x} \mid E)$ $(1\leqslant k\leqslant n)$ of (4.1) for each $\boldsymbol{x}\in \boldsymbol{H}$ depends only on E, which implies the relation (4.5). The expression (4.6) immediately follows from (4.4). The inequalities (4.7) are derived from the following: $\Lambda(r;\, E)=2\{r(l)-R_r(0, 0 \mid E)\}$ and $0\leqslant R_r(0, 0 \mid E)\leqslant R_r(0, 0 \mid \boldsymbol{a}_1)=r(l)$. The proof is thus completed.

Now we shall consider the roles of the conditions (A.1) and (A.2) to be imposed on E. For the sake of convenience, we assume that the space R^d is realized by row vectors. Then we shall employ the expression $\boldsymbol{a}_k=(a_{k1}, \cdots, a_{kd})$ $(1\leqslant k\leqslant n)$ and assume that $1\leqslant n\leqslant d$. On the other hand, given $\boldsymbol{a}=(a_1, \cdots, a_d)\in R^d$ and $I=(i_1, \cdots, i_n)$ $(1\leqslant i_1<\cdots<i_n\leqslant d)$, we set $\boldsymbol{a}_I=(a_{i_1}, \cdots, a_{i_n})$ and

$$\boldsymbol{a}_I[\boldsymbol{y}] = (a_1, \cdots, \underset{\overset{\cup}{i_1}}{y_1}, \cdots, \underset{\overset{\cup}{i_n}}{y_n}, \cdots, a_d) \quad \text{for every } \boldsymbol{y} = (y_1, \cdots, y_n)\in R^n\,.$$

Further we shall introduce the following notations: For every $\boldsymbol{y}=(y_1, \cdots, y_n)\in R^n$, we set

$$F_{I,a,E}(y)=(|a_I[y]-a_1|,\cdots,|a_I[y]-a_n|)$$

and

$$f_I^E(y)=\begin{vmatrix} y_1-a_{1i_1},\cdots,y_n-a_{1i_n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ y_1-a_{ni_1},\cdots,y_n-a_{ni_n}\end{vmatrix}.$$

Given $y\in R^n$ and $\delta>0$, we denote by $V_\delta(y)$ the open ball in R^n with center y and radius δ. Then we see that the Jacobian of the mapping $F_{I,a,E}$: $R^n\to R^n$ for each y ($a_I[y]\notin E$) is given by

$$(4.8)\qquad (\mathcal{J}F_{I,a,E})(y)=(\prod_{k=1}^{n}|a_I[y]-a_k|)^{-1}f_I^E(y).$$

By using this relation, we can discuss the regularity of $F_{I,a,E}$ under the assumption (A.1) on E.

Lemma 2. *Let $a\in R^d$ and $E\subset R^d$ ($1\leqslant\#E\leqslant d$) be given such that the points a, $a_1,\cdots,a_n$ are independent, i.e., the vectors $a-a_k$ ($1\leqslant k\leqslant n$) are linearly independent. Then there exist $I=(i_1,\cdots,i_n)$ ($1\leqslant i_1<\cdots<i_n\leqslant d$) and $\delta>0$ such that the mapping*

$$(4.9)\qquad F_{I,a,E}\colon V_\delta(a_I)\to U_\delta(I,a,E)$$

provides a homeomorphism, where we set $U_\delta(I,a,E)=F_{I,a,E}(V_\delta(a_I))$.

Proof. Because of the assumption on the arrangement of a and E, there exists $I=(i_1,\cdots,i_n)$ ($1\leqslant i_1<\cdots<i_n\leqslant d$) such that $f_I^E(a_I)\neq 0$. This implies that there exists $\delta>0$ such that

$$(4.10)\qquad f_I^E(y)\neq 0 \quad\text{and}\quad a_I[y]\notin E \qquad \text{for any}\quad y\in V_\delta(a_I).$$

It follows from (4.8) and (4.10) that the mapping $F_{I,a,E}$ is regular on $V_\delta(a_I)$. Thus we see by using the inverse mapping theorem that the mapping (4.9) provides a homeomorphism for a sufficiently small δ.

It is notable that, if $E\subset S(l)\cap G$ satisfies the condition (A.2), we may choose the real numbers $\gamma_r^k(x|E)$ ($1\leqslant k\leqslant n$) as follows:

$$\gamma_r^k(x|E)=\frac{1}{n}\qquad (x\in H,\ 1\leqslant k\leqslant n).$$

In the preceding section we have introduced several sets E which satisfy the conditions (A.2) and $E\subset S(l)$ for some $l>0$. We note that such sets will be also constructed by using the following lemma.

Lemma 3. *Let E_i ($i=1, 2$) be two finite subsets of R^d satisfying the conditions (A.2) and $E_i\subset S(l_i)$ ($i=1, 2$). Suppose that $(x, y)=0$ for any $x\in E_1$ and* *any* $\boldsymbol{y}\in E_2$. *Then the set*

$$E=\{\boldsymbol{x}+\boldsymbol{y};\ \boldsymbol{x}\in E_1,\ \boldsymbol{y}\in E_2\}$$

satisfies the conditions (A.2) *and* $E\subset S(\sqrt{l_1^2+l_2^2})$.

The proof is elementary, and so is omitted. We shall now discuss the properties of functions in the class $\boldsymbol{S}$.

Lemma 4. *Suppose that* $r(t)\in\boldsymbol{S}$ *is given by* (1.1). *Then* $r(t)$ *admits the following expression*:

$$r(t)=ct^2+\int_0^\infty\{1-Y_d(tu)\}f_d^\gamma(u)u^{d-1}du \qquad (t\geqslant 0), \tag{4.11}$$

where $f_d^\gamma(u)$ *is defined by*

$$f_d^\gamma(u)=[2^{d-1}\Gamma(d/2)]^{-1}\int_0^\infty s^{-d/2-1}e^{-u^2/4s}d\gamma(s) \qquad (u>0).$$

Proof. Let us introduce the formula

$$e^{-t^2s}=\int_0^\infty Y_d(tu)[2^{d-1}\Gamma(d/2)]^{-1}s^{-d/2}u^{d-1}e^{-u^2/4s}du \qquad (t\geqslant 0,\ s>0). \tag{4.12}$$

This will be easily shown by using the following alternative expression of $Y_d(t)$:

$$Y_d(t)=\int_{S^{d-1}}e^{i(\boldsymbol{x},\boldsymbol{z})}d\sigma_d(\boldsymbol{z}) \qquad (t=|\boldsymbol{x}|,\ \boldsymbol{x}\in R^d),$$

where σ_d is the uniform probability measure on the unit sphere $S^{d-1}=\{\boldsymbol{z}\in R^d;\ |\boldsymbol{z}|=1\}$. We now immediately obtain the desired expression (4.11) by combining (1.1) and (4.12).

Lemma 5. *Each function* $r(t)\in\boldsymbol{S}$ *satisfies the following properties*:
(i) $r(t)$ *is strictly increasing and analytic on* $(0,\infty)$;
(ii) $r^{-1}(x)$ *is strictly increasing and analytic on* $(0, r(\infty))$;
(iii) $r(\sqrt{t})$ *is strictly concave on* $(0,\infty)$ *except the case* $r(t)=t^2$;
(iv) *For any* n *distinct points* $\boldsymbol{x}_k\in R^d\setminus\{0\}$ $(d\geqslant 1,\ n\geqslant 1,\ 1\leqslant k\leqslant n)$, *the positive definit quadratic form*

$$Q_r(\Xi)=\sum_{j,k=1}^n R_r(\boldsymbol{x}_j,\boldsymbol{x}_k\,|\,0)\xi_j\xi_k, \qquad \Xi=(\xi_1,\cdots,\xi_n)\in R^n,$$

is non-degenerate except the case $r(t)=t^2$. *In other words,* $Q_r(\Xi)=0$ *implies that* $\Xi=\boldsymbol{0}$.

Proof. Suppose that $r(t)$ is given by (1.1). Then we have

$$r'(t)=2t\{c+\int_0^\infty e^{-t^2u}d\gamma(u)\}>0 \qquad \text{for any} \quad t>0.$$

It follows that $r(t)$ is strictly increasing on $(0, \infty)$. Further we can extend this function analytically to the function $r(z)$ on the complex domain $\{z\in \boldsymbol{C};\ |\arg z|<\pi/4\}$ ([12]). Therefore we obtain the assertions (i) and (ii). The assertion (iii) will be seen by the following fact: If $r(t)\not\equiv t^2$, we have

$$\frac{d^2}{dt^2}r(\sqrt{t}) = -\int_0^\infty e^{-tu}u d\gamma(u)<0 \qquad \text{for any} \quad t>0 .$$

We shall now proceed to the proof of the assertion (iv). On account of the expression (4.11) of $r(t)$, we see that

$$r(|\boldsymbol{x}|) = c|\boldsymbol{x}|^2+\int_{R^d} |e^{i(\boldsymbol{x},\boldsymbol{z})}-1|^2(2\omega_d)^{-1}f_d^\gamma(|\boldsymbol{z}|)d\boldsymbol{z} \qquad (\boldsymbol{x}\in R^d),$$

where we set $\omega_d=2\pi^{d/2}/\Gamma(d/2)$. Further we have, for any $\boldsymbol{x}, \boldsymbol{y}\in R^d$,

$$R_r(\boldsymbol{x}, \boldsymbol{y}|0) = c(\boldsymbol{x}, \boldsymbol{y})+\int_{R^d}(e^{i(\boldsymbol{x},\boldsymbol{z})}-1)(e^{-i(\boldsymbol{y},\boldsymbol{z})}-1)(2\omega_d)^{-1}f_d^\gamma(|\boldsymbol{z}|)d\boldsymbol{z} .$$

Thus we obtain the following representation of $Q_r(\Xi)$:

$$Q_r(\Xi) = c|\sum_{k=1}^n \xi_k\boldsymbol{x}_k|^2+\int_{R^d} |\sum_{k=1}^n \xi_k(e^{i(\boldsymbol{x}_k,\boldsymbol{z})}-1)|^2(2\omega_d)^{-1}f_d^\gamma(|\boldsymbol{z}|)d\boldsymbol{z} .$$

Now we assume that $r(t)\not\equiv t^2$ and $Q_r(\Xi)=0$. Then we have $\gamma((0, \infty))>0$ and so the function $f_d^\gamma(u)$ is positive and continuous on $(0, \infty)$. This implies that

$$\sum_{k=1}^n \xi_k(e^{i(\boldsymbol{x}_k,\boldsymbol{z})}-1) = 0 \qquad \text{for any} \quad \boldsymbol{z}\in R^d . \tag{4.13}$$

Now we set

$$V = \{\boldsymbol{z}\in R^d;\ (\boldsymbol{x}_k, \boldsymbol{z})\neq 0\ (1\leqslant k\leqslant n) \quad \text{and} \quad (\boldsymbol{x}_j, \boldsymbol{z})\neq(\boldsymbol{x}_k, \boldsymbol{z}) \quad \text{for any} \quad j, k\ (1\leqslant j<k\leqslant n)\} .$$

It is easy to see that V is a non-empty open subset of R^d and satisfies the relation $S_tV=V$ for any $t>0$. Let us choose a point $\boldsymbol{z}_0\in V\cap S^{d-1}$ arbitrarily and set $c_k=i(\boldsymbol{x}_k, \boldsymbol{z}_0)$ $(1\leqslant k\leqslant n)$. Then, setting $\boldsymbol{z}=t\boldsymbol{z}_0$ in (4.13), we have the equality

$$\sum_{k=1}^n \xi_k(e^{c_kt}-1) = 0 \qquad \text{for any} \quad t>0 .$$

Further differentiating in t, we have

$$\sum_{k=1}^n \xi_kc_ke^{c_kt} = 0 \qquad \text{for any} \quad t>0 .$$

By the way, the constants c_k $(1\leqslant k\leqslant n)$ satisfy the following conditions: $c_k\neq 0$ $(1\leqslant k\leqslant n)$ and $c_j\neq c_k$ for any j, k $(1\leqslant j<k\leqslant n)$. Therefore we can easily show that $\xi_k=0$ $(1\leqslant k\leqslant n)$, which completes the proof.

Before stating the next lemma, we shall introduce some notations. For every $r(t)\in \boldsymbol{S}_d$ and every probability measure λ on $(0, 1]$, we set $s_*[\lambda]=\inf\{\text{support of } \lambda\}$ and further

$$r^{\lambda}(t)=\int_0^1 r(t)^p d\lambda(p) \quad \text{and} \quad r_{\lambda}(t)=\int_0^1 r(t^p)d\lambda(p) \qquad (t\geqslant 0)\,.$$

The following lemma provides various examples of r.v.f.'s $r(t)\in \boldsymbol{S}$.

Lemma 6. *Let* λ *be a probability measure on* $(0, 1]$.
(i) *Assume that* $s_*[\lambda]>0$ *and* $r(t)\in \boldsymbol{S}_d$ *is a r.v.f.* (α) *for some* $\alpha>0$. *Then* $r^{\lambda}(t)$ *and* $r_{\lambda}(t)$ *are r.v.f.'s* $(s_*[\lambda]\alpha)$.
(ii) *For every* $r(t)\in \boldsymbol{S}$, *it holds that* $r^{\lambda}(t)\in \boldsymbol{S}$ *and* $r_{\lambda}(t)\in \boldsymbol{S}$.

The details of the proof are omitted. We can obtain the assertion (i) by elementary calculation. As for the assertion (ii), we may employ the theory of the inner transformations of completely monotone functions ([10]). We shall next consider an interesting functional equation related to Problem 2.

Lemma 7. *Let* $p(t)$, $q(t)$, $f(t)$ *and* $g(t)$ *be functions on* $(0, \infty)$ *such that* $p(t)\not\equiv 0$ *and* $q(t)\not\equiv 0$, *and let* $h(u, v)$ *be a positive function on* $I\times J$, *where* I *and* J *are open intervals contained in* $(0, \infty)$. *Assume that these functions satisfy the functional equation*

$$f(th(u, v))=p(t)f(tu)+q(t)f(tv)+g(t) \tag{4.14}$$

for any $(t, u, v)\in(0, \infty)\times I\times J$, *and further assume that* $f(t)$ *is twice differentiable and strictly monotone on* $(0, \infty)$. *Then* $f(t)$ *admits the following expression*:

$$f(t)=C_1t^{\alpha}+C_2 \quad \text{or} \quad f(t)=\beta\log t+C_3 \qquad (t>0)\,, \tag{4.15}$$

where α, β *and* C_i $(1\leqslant i\leqslant 3)$ *are arbitrary real constants* $(\alpha C_1\neq 0,\ \beta\neq 0)$.

Proof. First we can show by using the equation (4.14) and the assumption on $f(t)$ that $h(u, v)$ is twice differentiable on $I\times J$. By differentiating the both sides of (4.14) in u or v, we have the following two equations: For any $(t, u, v)\in(0, \infty)\times I\times J$,

$$f'(th(u, v))\frac{\partial h}{\partial u}(u, v)=p(t)f'(tu) \quad \text{and}$$
$$f'(th(u, v))\frac{\partial h}{\partial v}(u, v)=q(t)f'(tv)\,.$$

Therefore we see that $\frac{\partial h}{\partial u}(u, v)\frac{\partial h}{\partial v}(u, v)\neq 0$ for any $(u, v)\in I\times J$, because there exist $t_1, t_2\in(0, \infty)$ such that $p(t_1)q(t_2)\neq 0$. Now differentiating the both sides of the last equation in u, we have the following: For any $(t, u, v)\in(0, \infty)\times I\times J$,

$$f''(th(u, v))t\frac{\partial h}{\partial u}(u, v)\frac{\partial h}{\partial v}(u, v)+f'(th(u, v))\frac{\partial^2 h}{\partial u\partial v}(u, v) = 0$$

and further

$$\frac{f''(th(u, v))t}{f'(th(u, v))} = \frac{f''(h(u, v))}{f'(h(u, v))}.$$

By the change of variables this equation can be replaced by the following:

$$\frac{f''(t)t}{f'(t)} = \frac{f''(h(u, v))h(u, v)}{f'(h(u, v))} \quad \text{for any} \quad (t, u, v)\in(0, \infty)\times I\times J.$$

Therefore the both sides of this equation are identically equal to a certain real constant a which is independent of the variables t, u and v. It follows that, for any $t>0$,

$$\frac{f''(t)}{f'(t)} = \frac{a}{t} \quad \text{or equivalently} \quad \frac{d}{dt}\log f'(t) = \frac{a}{t}.$$

Then we have the expression $f'(t)=bt^a$ $(b\neq 0)$ and further

$$f(t) = \{b/(a+1)\}t^{a+1}+C \quad \text{or} \quad f(t) = b\log t+C' \qquad (t>0)$$

according as $a\neq -1$ or $a=-1$ respectively. Thus we obtain the desired expression (4.15).

REMARK 3. As for the assumptions on $f(t)$ in Lemma 7, the phrase "*strictly monotone*" may be replaced by the phrase "*non-constant*" provided that $I\cap J\neq\phi$ and $p(t)q(t)\neq 0$ for each $t>0$.

5. Proofs of main results

Proof of Theorem 1. Without loss of generality, we may assume that $\gamma_r^1(\boldsymbol{a}|E)\gamma_r^2(\boldsymbol{a}|E)\neq 0$. It follows from Proposition 1 that there exists a sequence $\{I_k\}_{1<k<n}$ of open intervals, for which the conditions (3.1) and (3.2) hold. Therefore, for each $\boldsymbol{u}=(u_1, \cdots, u_n)\in\prod_{k=1}^n I_k$, there exists $\boldsymbol{y}[\boldsymbol{u}]\in\mathcal{F}_X(\boldsymbol{a}|E)$ such that

$$\Phi_E(\boldsymbol{y}[\boldsymbol{u}]) = \boldsymbol{u} \quad \text{or equivalently} \quad |\boldsymbol{y}[\boldsymbol{u}]-\boldsymbol{a}_k| = u_k \qquad (1\leqslant k\leqslant n).$$

In order to show the part (i), it suffices to prove the "*if*" part. We note that the sets $r(I_k)$ $(1\leqslant k\leqslant n)$ are non-empty open intervals contained in $(0, \infty)$ and we set $\Delta_r(\boldsymbol{p})=(r^{-1}(p_1), \cdots, r^{-1}(p_n))$ for every $\boldsymbol{p}=(p_1, \cdots, p_n)\in\prod_{k=1}^n r(I_k)$. Then we see that, for every $\boldsymbol{p}\in\prod_{k=1}^n r(I_k)$, we have $\Delta_r(\boldsymbol{p})\in\prod_{k=1}^n I_k$ and

$$\begin{cases} \boldsymbol{y}[\Delta_r(\boldsymbol{p})] \in \mathcal{F}_X(\boldsymbol{a}\,|\,E) \subset \mathcal{F}_{X_1}(\boldsymbol{a}\,|\,E)\,, \\ |\boldsymbol{y}[\Delta_r(\boldsymbol{p})] - \boldsymbol{a}_k| = r^{-1}(p_k) \qquad (1 \leqslant k \leqslant n)\,. \end{cases}$$

Therefore we can show by (4.4) the following two equations: For every $\boldsymbol{p} \in \prod_{k=1}^{n} r(I_k)$,

$$\text{(5.1)} \qquad \begin{cases} r(|\boldsymbol{y}[\Delta_r(\boldsymbol{p})] - \boldsymbol{a}|) = \sum_{k=1}^{n} p_k \gamma_k + M\,, \\ r_1(|\boldsymbol{y}[\Delta_r(\boldsymbol{p})] - \boldsymbol{a}|) = \sum_{k=1}^{n} r_1(r^{-1}(p_k))\gamma_k^1 + M_1\,, \end{cases}$$

where we set $\gamma_k = \gamma_r^k(\boldsymbol{a}\,|\,E)$, $\gamma_k^1 = \gamma_{r_1}^k(\boldsymbol{a}\,|\,E)$ $(1 \leqslant k \leqslant n)$, $M = r(|\boldsymbol{a} - \boldsymbol{a}_1|) - \Lambda_r(\boldsymbol{a}, \boldsymbol{a}_1\,|\,E)$ and $M_1 = r_1(|\boldsymbol{a} - \boldsymbol{a}_1|) - \Lambda_{r_1}(\boldsymbol{a}, \boldsymbol{a}_1\,|\,E)$. Let $f(x)$ be the function on $[0, r(\infty))$ defined by $f(x) = r_1(r^{-1}(x))$. Then we obtain from (5.1) the following functional equation: For any $(p_1, \cdots, p_n) \in \prod_{k=1}^{n} r(I_k)$,

$$\text{(5.2)} \qquad f\Big(\sum_{k=1}^{n} p_k \gamma_k + M\Big) = \sum_{k=1}^{n} f(p_k)\gamma_r^1 + M_1\,.$$

We note that $f(x)$ is analytic on $(0, r(\infty))$ and the range of the function

$$x = \sum_{k=1}^{n} p_k \gamma_k + M \qquad \Big((p_1, \cdots, p_n) \in \prod_{k=1}^{n} r(I_k)\Big)$$

contains an interior point because of the assumption $\gamma_1\gamma_2 \neq 0$. Now differentiating the both sides of (5.2) in p_1 and p_2 successively, we have the following:

$$f''\Big(\sum_{k=1}^{n} p_k \gamma_k + M\Big) = 0 \qquad \text{for any} \quad (p_1, \cdots, p_n) \in \prod_{k=1}^{n} r(I_k)\,.$$

Therefore we see by the analyticity of $f''(x)$ that $f''(x) \equiv 0$ on $(0, r(\infty))$. Further we have $f(x) = x$ on $[0, r(\infty))$ by using the conditions $f(0) = 0$ and $f(1) = 1$. This implies that, for any $t \geqslant 0$, $r_1(t) = r_1(r^{-1}(r(t))) = f(r(t)) = r(t)$. The proof of the part (i) is thus completed.

We now proceed to the proof of the part (ii). It suffices to prove the "*if*" part. Let us again use the notation $\boldsymbol{y}[\boldsymbol{u}]$ for every $\boldsymbol{u} \in \prod_{k=1}^{n} I_k$, which was introduced above. Then we see by the assumption (2.5) that

$$\text{(5.3)} \qquad S_t \boldsymbol{y}[\boldsymbol{u}] \in \mathcal{F}_X(S_t \boldsymbol{a}\,|\,S_t E) \qquad \text{for any} \quad (t, \boldsymbol{u}) \in (0, \infty) \times \prod_{k=1}^{n} I_k\,.$$

For the sake of convenience, we shall introduce the following functions:

$$\begin{cases} p_k(t) = \gamma_r^k(S_t \boldsymbol{a}\,|\,S_t E) \qquad (t > 0,\ 1 \leqslant k \leqslant n)\,, \\ g(t) = r(t\,|\boldsymbol{a} - \boldsymbol{a}_1|) - \sum_{k=1}^{n} r(t\,|\boldsymbol{a}_1 - \boldsymbol{a}_k|) p_k(t) \qquad (t > 0) \quad \text{and} \\ h(u_1, \cdots, u_n) = r^{-1}\Big(\sum_{k=1}^{n} r(u_k) p_k(1) + g(1)\Big) \qquad \Big((u_1, \cdots, u_n) \in \prod_{k=1}^{n} I_k\Big)\,. \end{cases}$$

Then we can derive from (5.3) the following functional equation: For any $(t, u_1, \cdots, u_n) \in (0, \infty) \times \prod_{k=1}^{n} I_k$,

$$r(th(u_1, \cdots, u_n)) = \sum_{k=1}^{n} r(tu_k)p_k(t) + g(t)\,. \tag{5.4}$$

It should be noted that $p_1(1)p_2(1) \neq 0$ and $h(u_1, \cdots, u_n) > 0$ on $\prod_{k=1}^{n} I_k$. By applying Lemma 7 to the equation (5.4), we see that $r(t)$ can be expressed in the form

$$r(t) = C_1 t^{\alpha} + C_2 \quad \text{or} \quad r(t) = \beta \log t + C_3 \qquad (t>0)\,,$$

where α, β and C_i $(1 \leqslant i \leqslant 3)$ are real constants $(\alpha C_1 \neq 0, \beta \neq 0)$. Therefore we obtain the desired expression $r(t) = t^{\alpha}$ $(0 < \alpha \leqslant 2)$ by using the conditions $r(0) = 0$ and $r(1) = 1$ and also the concavity of $r(\sqrt{t})$. The proof of the part (ii) is thus completed.

Proof of Theorem 2. It follows from Proposition 2 that there exists an open interval I such that the conditions (3.3) and (3.4) hold. If we set $a = |\boldsymbol{a}|$ and $\boldsymbol{a}(u) = (\sqrt{u^2 - l^2}/a)\boldsymbol{a}$ $(u > l)$, we have $\boldsymbol{a}(u) \in \Psi_E^{-1}(I) \cap \boldsymbol{H}$ for any $u \in I$. Therefore, for any $u, v \in I$, there exists $\boldsymbol{b}(u, v) \in \mathcal{F}_X(\boldsymbol{a}(u) \mid E) \cap \boldsymbol{H}$ such that

$$\Psi_E(\boldsymbol{b}(u, v)) = v \quad \text{or equivalently} \quad |\boldsymbol{b}(u, v) - \boldsymbol{a}_1| = v\,.$$

In order to show the part (i), it suffices to prove the "*if*" part. Without loss of generality, we may assume that $I \subset (\sqrt{t_1^2 a^2 + l^2}, \sqrt{t_2^2 a^2 + l^2})$. Then $r(I)$ is a non-empty open interval contained in $(0, \infty)$. Further we set, for every $p, q \in r(I)$,

$$\boldsymbol{a}[p] = \boldsymbol{a}(r^{-1}(p)) \quad \text{and} \quad \boldsymbol{b}[p, q] = \boldsymbol{b}(r^{-1}(p), r^{-1}(q))\,.$$

Then it follows from the assumption (2.6) that, for any $p, q \in r(I)$,

$$\boldsymbol{b}[p, q] \in \mathcal{F}_X(\boldsymbol{a}[p] \mid E) \cap \boldsymbol{H} \subset \mathcal{F}_{X_1}(\boldsymbol{a}[p] \mid E) \cap \boldsymbol{H}\,.$$

Therefore we can show by (4.6) the following two equations: For every $p, q \in r(I)$,

$$\begin{cases} r(|\boldsymbol{a}[p] - \boldsymbol{b}[p, q]|) = p + q - \Lambda\,, \\ r_1(|\boldsymbol{a}[p] - \boldsymbol{b}[p, q]|) = r_1(r^{-1}(p)) + r_1(r^{-1}(q)) - \Lambda_1\,, \end{cases}$$

where we set $\Lambda = \Lambda(r; E)$ and $\Lambda_1 = \Lambda(r_1; E)$. Now setting $f(x) = r_1(r^{-1}(x))$ for $x \in [0, r(\infty))$, we can derive from the above equations the following functional equation:

$$f(p + q - \Lambda) = f(p) + f(q) - \Lambda_1 \qquad \text{for any} \quad p, q \in r(I)\,.$$

Therefore we see that $f(x)=x$ on $[0, r(\infty))$ by using the analyticity of $f(x)$ and the conditions $f(0)=0$ and $f(1)=1$. Thus we have the desired identity $r_1(t)=r(t)$.

We now proceed to the proof of the part (ii). It suffices to prove the "*if*" part. Let us again use the notations $\boldsymbol{a}(u)$ and $\boldsymbol{b}(u, v)$ for every $u, v\in I$, which were introduced above. Then we see by the assumption (2.7) that

$$(5.5)\quad S_t\boldsymbol{b}(u, v)\in\mathcal{F}_X(S_t\boldsymbol{a}(u)\,|\,S_tE)\cap\boldsymbol{H} \qquad \text{for any} \quad (t, u, v)\in(0, \infty)\times I\times I\,.$$

Therefore setting

$$\begin{cases} g(t) = -\Lambda(r;\, S_tE) & (t>0)\,, \\ h(u, v) = r^{-1}(r(u)+r(v)+g(1)) & (u, v\in I)\,, \end{cases}$$

we can derive from (5.5) the following functional equation: For any $(t, u, v)\in(0, \infty)\times I\times I$,

$$r(th(u, v)) = r(tu)+r(tv)+g(t)\,.$$

Thus we obtain the desired expression $r(t)=t^{\alpha}$ $(0<\alpha\leqslant 2)$ by the same discussion as the proof of Theorem 1.

Proof of Theorem 3. It suffices to prove the "*if*" part. We see by Proposition 2 that there exists an open interval I such that $\Psi_E(\boldsymbol{a})\in I\subset(l, \infty)$ and $I\subset\Psi_E(\mathcal{F}_X(\boldsymbol{a}\,|\,E)\cap\boldsymbol{H})$. Therefore, for any $u\in I$, there exists $\boldsymbol{b}[u]\in\mathcal{F}_X(\boldsymbol{a}\,|\,E)\cap\boldsymbol{H}$ such that

$$\Psi_E(\boldsymbol{b}[u]) = u \qquad \text{or equivalently} \quad |\boldsymbol{b}[u]-\boldsymbol{a}_1| = u\,.$$

By using the assertion (ii) of Theorem 2 and the assumption (2.10), we have the following: For any $(u, t)\in I\times J$,

$$S_t\boldsymbol{b}[u]\in\mathcal{F}_X(S_t\boldsymbol{a}\,|\,S_tE)\cap\boldsymbol{H}\subset\mathcal{F}_{X_1}(S_t\boldsymbol{a}\,|\,S_tE)\cap\boldsymbol{H}\,,$$

where we set $J=(t_1, t_2)$. Therefore we obtain by (4.6) the following two equations: For every $(u, t)\in I\times J$,

$$(5.6)\qquad \begin{cases} r(t\,|\boldsymbol{a}-\boldsymbol{b}[u]|) = r(ut)+r(\sqrt{2}\,lt)-\Lambda(r;\, S_tE)\,, \\ r_1(t\,|\boldsymbol{a}-\boldsymbol{b}[u]|) = r_1(ut)+r_1(\sqrt{2}\,lt)-\Lambda(r_1;\, S_tE)\,. \end{cases}$$

For the sake of convenience, we set $\Lambda_0=r(\sqrt{2}\,l)$ and $\Lambda_k=r(|\boldsymbol{a}_1-\boldsymbol{a}_k|)$ $(1\leqslant k\leqslant n)$. Then we have

$$(5.7)\qquad \Lambda(r;\, S_tE) = \frac{1}{n}\sum_{k=1}^{n} r(|t\boldsymbol{a}_1-t\boldsymbol{a}_k|) = \frac{r(t)}{n}\sum_{k=1}^{n}\Lambda_k\,,$$

because the set S_tE satisfies the condition (A.2). Now setting $f(x)=r_1(r^{-1}(x))$

for $x\in[0, r(\infty))$, we can derive from (5.6) and (5.7) the following functional equation: For any $(s, q)\in r(I)\times r(J)$,

$$f(sq+\Lambda q)=f(sq)+f(\Lambda_0 q)-\frac{1}{n}\sum_{k=1}^{n}f(\Lambda_k q),$$

where $\Lambda=\Lambda_0-\frac{1}{n}\sum_{k=1}^{n}\Lambda_k$. Further setting $p=sq$, this equation can be replaced by the following: For any $(p, q)\in U$,

$$f(p+\Lambda q)=f(p)+f(\Lambda_0 q)-\frac{1}{n}\sum_{k=1}^{n}f(\Lambda_k q), \tag{5.8}$$

where U denotes a domain of R^2 defined by

$$U=\{(p, q)\in R^2;\ q\in r(J),\ p/q\in r(I)\}.$$

On the other hand, by using the property (iii) of Lemma 5 and (4.7), we have

$$\begin{aligned}\Lambda(r; E)&=\frac{1}{n}\sum_{k=1}^{n}r(\sqrt{|\boldsymbol{a}_1-\boldsymbol{a}_k|^2})\\&\leqslant r\left(\sqrt{\frac{1}{n}\sum_{k=1}^{n}|\boldsymbol{a}_1-\boldsymbol{a}_k|^2}\right)\\&\leqslant r(\sqrt{2l^2})=\Lambda_0.\end{aligned} \tag{5.9}$$

Then combining (5.7) and (5.9), we have $\Lambda\geqslant 0$. In the special case $\Lambda=0$, we have $r(t)=t^2$ and $\frac{1}{n}\sum_{k=1}^{n}\Lambda_k=\Lambda_0$. Therefore, if we assume that $r_1(t)\not\equiv r(t)$, the function $f(x)=r_1(\sqrt{x})$ is strictly concave. Thus we see by (5.8) that

$$f(\Lambda_0 q)=\frac{1}{n}\sum_{k=1}^{n}f(\Lambda_k q)<f\left(\frac{1}{n}\sum_{k=1}^{n}\Lambda_k q\right)=f(\Lambda_0 q).$$

Consequently we have the desired identity $r_1(t)=r(t)$ by contradiction. As for the case $\Lambda>0$, we can easily obtain the expression $f(x)=x$ on $[0, r(\infty))$ from (5.8) by using the analyticity of $f(x)$ and the conditions $f(0)=0$ and $f(1)=1$. Thus we have the identity $r_1(t)=r(t)$, which completes the proof.

6. Proofs of Propositions in Section 3

In Section 3, we have assumed that $\{\boldsymbol{e}_i\}_{1\leqslant i\leqslant d}$ is the canonical orthonormal basis of R^d and further the subspaces $\boldsymbol{G}$ and $\boldsymbol{H}$ are spanned by $\{\boldsymbol{e}_i\}_{1\leqslant i\leqslant d-2}$ and $\{\boldsymbol{e}_{d-1}, \boldsymbol{e}_d\}$ respectively. In order to prove Proposition 1 we shall introduce the family $\{T_\theta;\ 0\leqslant\theta\leqslant\pi\}$ of orthogonal transformations on R^d defined as follows:

$$\begin{cases}T_\theta\boldsymbol{e}_i=\boldsymbol{e}_i & (1\leqslant i\leqslant d-2),\\ T_\theta\boldsymbol{e}_{d-1}=\boldsymbol{e}_{d-1}\cos\theta+\boldsymbol{e}_d\sin\theta,\\ T_\theta\boldsymbol{e}_d=-\boldsymbol{e}_{d-1}\sin\theta+\boldsymbol{e}_d\cos\theta.\end{cases}$$

Proof of Proposition 1. It immediately follows from the assumption that the points $\boldsymbol{a}, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ are independent and $1\leqslant n\leqslant d-1$. Therefore we see by Lemma 2 that there exist $I=(i_1, \cdots, i_n)$ $(1\leqslant i_1<\cdots<i_n\leqslant d)$, $\delta>0$ and a domain $U_\delta(I, \boldsymbol{a}, E)$ of R^n such that the mapping

$$F_{I,\boldsymbol{a},E}\colon V_\delta(\boldsymbol{a}_I)\to U_\delta(I, \boldsymbol{a}, E)$$

provides a homeomorphism. In particular, we have $\Phi_E(\boldsymbol{a})=F_{I,\boldsymbol{a},E}(\boldsymbol{a}_I)\in U_\delta(I, \boldsymbol{a}, E)$. Therefore there exists a sequence $\{I_k\}_{1\leqslant k\leqslant n}$ of open intervals satisfying the conditions (3.1) and $\prod_{k=1}^{n} I_k\subset U_\delta(I, \boldsymbol{a}, E)$. We denote by $\hat{V}_\delta(\boldsymbol{a}_I)$ the inverse image of $\prod_{k=1}^{n} I_k$ by the mapping $F_{I,\boldsymbol{a},E}$. Then the mapping

$$F_{I,\boldsymbol{a},E}\colon \hat{V}_\delta(\boldsymbol{a}_I)\to \prod_{k=1}^{n} I_k$$

is also a homeomorphism. Now we shall introduce a continuous function $f(\theta, \boldsymbol{x})$ on $[0, \pi]\times R^d$ defined by

$$f(\theta, \boldsymbol{x}) = R_r(\boldsymbol{a}, T_\theta \boldsymbol{x}\,|\,E) \qquad ((\theta, \boldsymbol{x})\in[0, \pi]\times R^d)\,.$$

Because of the property (iv) of Lemma 5 and the assumption (R), we have

$$f(0, \boldsymbol{a}) = R_r(\boldsymbol{a}, \boldsymbol{a}\,|\,E)>0 \quad\text{and}\quad f(\pi, \boldsymbol{a}) = R_r(\boldsymbol{a}, -\boldsymbol{a}\,|\,E)<0\,.$$

Further by choosing the above-mentioned δ sufficiently small, we have the following: For any $\boldsymbol{y}\in\hat{V}_\delta(\boldsymbol{a}_I)$,

$$f(0, \boldsymbol{a}_I[\boldsymbol{y}])>0 \quad\text{and}\quad f(\pi, \boldsymbol{a}_I[\boldsymbol{y}])<0\,.$$

Therefore we see by using the intermediate value theorem that, for any $\boldsymbol{y}\in\hat{V}_\delta(\boldsymbol{a}_I)$, there exists $\Theta(\boldsymbol{y})\in(0, \pi)$ such that $f(\Theta(\boldsymbol{y}), \boldsymbol{a}_I[\boldsymbol{y}])=0$. In other words, we have

$$T_{\Theta(\boldsymbol{y})}\boldsymbol{a}_I[\boldsymbol{y}]\in\mathscr{F}_X(\boldsymbol{a}\,|\,E) \qquad \text{for any} \quad \boldsymbol{y}\in\hat{V}_\delta(\boldsymbol{a}_I)\,.$$

Then we see that, for any $\boldsymbol{y}\in\hat{V}_\delta(\boldsymbol{a}_I)$,

$$F_{I,\boldsymbol{a},E}(\boldsymbol{y}) = \Phi_E(\boldsymbol{a}_I[\boldsymbol{y}]) = \Phi_E(T_{\Theta(\boldsymbol{y})}\boldsymbol{a}_I[\boldsymbol{y}])\in\Phi_E(\mathscr{F}_X(\boldsymbol{a}\,|\,E))\,,$$

and so we have $F_{I,\boldsymbol{a},E}(\hat{V}_\delta(\boldsymbol{a}_I))\subset\Phi_E(\mathscr{F}_X(\boldsymbol{a}\,|\,E))$. Thus we obtain the relation (3.2), which completes the proof.

Before we proceed to the proof of Proposition 2, we shall introduce an alternative expression of $R_r(\boldsymbol{x}, \boldsymbol{y}\,|\,E)$ restricted to $\boldsymbol{H}\times\boldsymbol{H}$. Suppose that $E\subset S(l)\cap\boldsymbol{G}$. Then we have, for any $\boldsymbol{x}, \boldsymbol{y}\in\boldsymbol{H}$,

(6.1) $$2R_r(\boldsymbol{x}, \boldsymbol{y}\,|\,E) = f_r^E(|\boldsymbol{x}|, |\boldsymbol{y}|, \angle(\boldsymbol{x}, \boldsymbol{y}))\,,$$

where $\angle(\boldsymbol{x}, \boldsymbol{y})$ denotes the angle between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ $(0 \leqslant \angle(\boldsymbol{x}, \boldsymbol{y}) \leqslant \pi)$ and $f_r^E(\xi, \eta, \theta)$ denotes a continuous function on $[0, \infty)\times[0, \infty)\times[0, \pi]$ defined by

$$f_r^E(\xi, \eta, \theta) = r(\sqrt{\xi^2+l^2})+r(\sqrt{\eta^2+l^2})-r(\sqrt{\xi^2+\eta^2-2\xi\eta\cos\theta})-\Lambda(r; E)\,.$$

Proof of Proposition 2. Because of the property (iv) of Lemma 5 and the assumption (R), we have

$$f_r^E(a, a, 0) = 2R_r(\boldsymbol{a}, \boldsymbol{a}\,|\,E)>0 \quad \text{and} \quad f_r^E(a, a, \pi) = 2R_r(\boldsymbol{a}, -\boldsymbol{a}\,|\,E)<0\,,$$

where we set $a=|\boldsymbol{a}|$. Further there exists an open interval (t_1, t_2) $(0<t_1<1<t_2)$ such that we have, for any $s, t\in(t_1, t_2)$,

$$f_r^E(as, at, 0)>0 \quad \text{and} \quad f_r^E(as, at, \pi)<0\,.$$

Then we see by using the intermediate value theorem that, for any $s, t\in(t_1, t_2)$, there exists $\Theta(s, t)\in(0, \pi)$ such that $f_r^E(as, at, \Theta(s, t))=0$. It follows that, given $\boldsymbol{x}(s)\in\boldsymbol{H}$ satisfying $|\boldsymbol{x}(s)|=as$, there exists $\boldsymbol{y}(s, t)\in\mathcal{F}_X(\boldsymbol{x}(s)\,|\,E)\cap\boldsymbol{H}$ such that

$$(6.2) \qquad |\boldsymbol{y}(s, t)| = at \quad \text{and} \quad \angle(\boldsymbol{x}(s), \boldsymbol{y}(s, t)) = \Theta(s, t)\,.$$

If we set $I=(\sqrt{a^2t_1^2+l^2}, \sqrt{a^2t_2^2+l^2})$, we immediately obtain (3.3): $\Psi_E(\boldsymbol{a})=\sqrt{a^2+l^2}\in I\subset(l, \infty)$. Therefore we have only to prove (3.4). Given $\boldsymbol{x}\in\Psi_E^{-1}(I)\cap\boldsymbol{H}$ and $u\in I$, we set $s=\sqrt{\Psi_E(\boldsymbol{x})^2-l^2}/a$ and $t=\sqrt{u^2-l^2}/a$. Then we have $s, t\in(t_1, t_2)$ and $|\boldsymbol{x}|=as$. Therefore setting $\boldsymbol{x}(s)=\boldsymbol{x}$, there exists $\boldsymbol{y}(s, t)\in\mathcal{F}_X(\boldsymbol{x}(s)\,|\,E)\cap\boldsymbol{H}$ such that the condition (6.2) holds. Then we see that $u=\Psi_E(\boldsymbol{y}(s, t))\in\Psi_E(\mathcal{F}_X(\boldsymbol{x}\,|\,E)\cap\boldsymbol{H})$, which implies the property (3.4).

Proof of Proposition 3. Let us consider the function $g(t)$ on $[0, \infty)$ defined by $g(t)=2r(t)-r(2t)$. Then we have

$$2R_r(\boldsymbol{x}, -\boldsymbol{x}\,|\,0) = g(|\boldsymbol{x}|) \qquad \text{for any} \quad \boldsymbol{x}\in R^d\,.$$

Therefore, given $\boldsymbol{a}\in\boldsymbol{H}$ $(\boldsymbol{a}\neq 0)$, $\{\boldsymbol{a}, \{0\}, r(t)\}$ satisfies the condition (R) if and only if $g(a)<0$, where we set $a=|\boldsymbol{a}|$. Under the assumption stated in the part (i), we have $g(0)=0$ and $g'(t)<0$ on $(0, t_0/2)$. Then we have $g(a)<0$ for any $a\in(0, t_0/2)$, which completes the proof of the part (i). We now proceed to the proof of the part (ii). We may assume that $a>t_0$. Because of the strict convexity of $r(t)$ on (t_0, ∞) we see that $G(t)=r(t+a)-r(t)$ is strictly increasing on (t_0, ∞). It follows that $G(t_0)<G(a)$ and so we have

$$g(a)<-r(t_0+a)+r(a)+r(t_0) = -t_0r'(a+\theta t_0)+r(t_0)$$

for some θ $(0<\theta<1)$. On the other hand, $r'(t)$ and $g(t)$ are strictly decreasing on $(0, t_0)$ and on (t_0, ∞) respectively. Therefore we see that

$$\lim_{a\to\infty} g(a) \leqslant -t_0 r'(\infty) + r(t_0) \leqslant -t_0 r'(+0) + r(t_0)$$
$$= \int_0^{t_0} \{r'(t) - r'(+0)\}\, dt < 0\,.$$

Consequently we have $g(a)<0$ for a sufficiently large a, which completes the proof of the part (ii).

Proof of Proposition 4. First we note that we have

$$\gamma_r^k(\boldsymbol{x} \mid E_n^j(l)) = \frac{1}{n} \qquad (\boldsymbol{x} \in \boldsymbol{H},\ 1 \leqslant k \leqslant n)\,,$$

because of the property (A.2) of $E_n^j(l)$ $(1 \leqslant j \leqslant 3)$. Then we obtain the following equalities:

$$(6.3) \qquad 2R_r(\boldsymbol{a}, -\boldsymbol{a} \mid E_n^j(l)) = 2r(\sqrt{2}\,l) - r(2l) - \Lambda(r; E_n^j(l)) \qquad (1 \leqslant j \leqslant 3);$$
$$\Lambda(r; E_n^j(l)) = \begin{cases} r(\sqrt{2n/(n-1)}\,l)\,(n-1)/n & (j=1)\,, \\ \{r(2l) + (n-2) r(\sqrt{2}\,l)\}/n & (j=2)\,, \\ \sum_{k=1}^{m} r(2\sqrt{k/m}\,l) \binom{m}{k} / 2^m & (j=3,\ n=2^m); \end{cases}$$

$$(6.4) \qquad \lim_{n\to\infty} \Lambda(r; E_n^j(l)) = r(\sqrt{2}\,l) \qquad (1 \leqslant j \leqslant 3)\,.$$

As for the equalities (6.4), the cases $j=1$ and $j=2$ are obvious. We can show the case $j=3$ by using the following formula derived from the de Moivre-Laplace theorem: For any p, q $(0 \leqslant p < q \leqslant 1)$,

$$\lim_{m\to\infty} \sum_{p<k/m\leqslant q} \binom{m}{k} / 2^m = \begin{cases} 1 & \text{if } \ 1/2 \in (p, q)\,, \\ 1/2 & \text{if } \ p = 1/2 \text{ or } q = 1/2\,, \\ 0 & \text{if } \ 1/2 \notin [p, q]\,. \end{cases}$$

Therefore we obtain from (6.3) and (6.4) that

$$\lim_{n\to\infty} 2R_r(\boldsymbol{a}, -\boldsymbol{a} \mid E_n^j(l)) = r(\sqrt{2}\,l) - r(2l) < 0 \qquad (1 \leqslant j \leqslant 3)\,.$$

Consequently we have the inequalities $R_r(\boldsymbol{a}, -\boldsymbol{a} \mid E_n^j(l)) < 0$ $(1 \leqslant j \leqslant 3)$ if n is chosen to be sufficiently large under the restriction (3.9).

Proof of Proposition 5. In what follows we shall employ the notation $r_\alpha(t) = t^\alpha$ $(0 < \alpha \leqslant 2)$. Then we see by (3.10) and (6.1) that

$$2R_{r_\alpha}(\boldsymbol{a}, -\boldsymbol{a} \mid E) = f_{r_\alpha}^E(l, l, \pi) = l^\alpha F(\alpha; E) \qquad (0 < \alpha \leqslant 2)\,.$$

Therefore we see by using the property of $p[E]$ that the inequality $R_{r_\alpha}(\boldsymbol{a}, -\boldsymbol{a} \mid E) < 0$ holds if and only if $p[E] < \alpha \leqslant 2$, which means the assertion of Proposition 5.

Proof of Proposition 6. Assume that $r(t)$ is a r.v.f. (α) for some $\alpha\in(p[E], 2]$. Then we see by (3.10) and (6.1) that

$$\begin{aligned}&\lim_{\rho\to+0} 2R_r(S_\rho \boldsymbol{a}, -S_\rho \boldsymbol{a}\,|\,S_\rho E)/r(\rho)\\&=\lim_{\rho\to+0}\{2r(\sqrt{2}\,l\rho)/r(\rho)-r(2l\rho)/r(\rho)-\frac{1}{n}\sum_{k=1}^{n} r(|\boldsymbol{a}_k-\boldsymbol{a}_1|\rho)/r(\rho)\}\\&=l^\alpha F(\alpha; E)<0\,.\end{aligned}$$

Therefore the inequality $R_r(S_\rho \boldsymbol{a}, -S_\rho \boldsymbol{a}\,|\,S_\rho E)<0$ holds for sufficiently small $\rho>0$, which means the assertion of Proposition 6.

Proof of Proposition 7. We see by (3.10) and (4.3) that

$$F(\alpha; E_n^j(l)) = 2(\sqrt{2})^\alpha-2^\alpha-\Lambda(r_\alpha; E_n^j(l))/l^\alpha \qquad (0<\alpha\leqslant 2,\ 1\leqslant j\leqslant 4)\,.$$

Therefore by using the property (6.4) and the approximation property of the definite integral $\int_0^{\pi/2}\sin^\alpha x dx$, we can show that, for each j $(1\leqslant j\leqslant 4)$, the functions $F_{nj}(\alpha)=F(\alpha; E_n^j(l))$ $(n\geqslant 2)$ converge to a certain function $F_j(\alpha)$ on $(0, 2]$ as $n\to\infty$. In fact, we have

$$F_j(\alpha)=\begin{cases}(\sqrt{2})^\alpha\{1-(\sqrt{2})^\alpha\} & (1\leqslant j\leqslant 3)\,,\\ -\{(\sqrt{2})^\alpha-1\}^2-\left\{\dfrac{2^\alpha\Gamma((\alpha+1)/2)}{\sqrt{\pi}\,\Gamma((\alpha+2)/2)}-1\right\} & (j=4)\,.\end{cases}$$

Since we have $F_j(\alpha)<0$ on $(0, 2]$ $(1\leqslant j\leqslant 4)$, we immediately obtain the equalities $\lim_{n\to\infty}\alpha_{nj}=0$ $(1\leqslant j\leqslant 4)$ and also the relations in (3.15).

Proof of Proposition 8. First we have, for any $\alpha\in(0, 2]$,

$$2R_{r_\alpha}(\boldsymbol{a}, -\boldsymbol{a}\,|\,E_n^1(l))=l^\alpha(\sqrt{2n/(n-1)})^\alpha\{(n+1)/n-(\sqrt{2(n+1)/n})^\alpha\}\,.$$

Therefore the inequality $R_{r_\alpha}(\boldsymbol{a}, -\boldsymbol{a}\,|\,E_n^1(l))<0$ holds if and only if $\alpha_n<\alpha\leqslant 2$, which means the assertion (i). In order to prove the part (ii) we assume that $r(t)\in \boldsymbol{S}$ is a r.v.f. (α) for some $\alpha\in(\alpha_n, 2]$. Then we see by using the assertion (i) that

$$\lim_{\rho\to+0} R_r(S_\rho \boldsymbol{a}, -S_\rho \boldsymbol{a}\,|\,S_\rho E_n^1(l))/r(\rho)=R_{r_\alpha}(\boldsymbol{a}, -\boldsymbol{a}\,|\,E_n^1(l))<0\,.$$

Therefore the inequality $R_r(S_\rho \boldsymbol{a}, -S_\rho \boldsymbol{a}\,|\,S_\rho E_n^1(l))<0$ holds for sufficiently small $\rho>0$, which means the assertion (ii).

7. Additional results

In this section we shall be concerned with certain modifications of our problems stated in Section 1. Let $(\boldsymbol{X}, r(r))$ be an arbitrary G.r.f. on R^d. In

[2], we have introduced a family of subsets of R^d defined as follows: For every $\boldsymbol{a}, \boldsymbol{b}\in R^d$ and every $q\in R$, we set

$$C_X(\boldsymbol{a}, \boldsymbol{b}; q) = \{\boldsymbol{x}\in R^d;\ \mu_r(\boldsymbol{x}\,|\,\boldsymbol{a}, \boldsymbol{b}) = X(\boldsymbol{a})(1-q)/2+X(\boldsymbol{b})(1+q)/2\}\,.$$

It is interesting to see that the sets $C_X(\boldsymbol{a}, \boldsymbol{b}; q)$ and $\mathcal{F}_X(\boldsymbol{x}\,|\,E)$ have some properties in common. In particular, the increments $X(\boldsymbol{x})-X(\boldsymbol{y})$ and $X(\boldsymbol{a})-X(\boldsymbol{b})$ are mutually independent if and only if there exists $q\in R$ such that $\boldsymbol{x}, \boldsymbol{y}\in C_X(\boldsymbol{a}, \boldsymbol{b}; q)$. It is obvious to see that $C_X(\boldsymbol{a}, \boldsymbol{b}; 1)=\mathcal{F}_X(\boldsymbol{a}\,|\,\boldsymbol{b})$, $C_X(\boldsymbol{a}, \boldsymbol{b}; q)=C_X(\boldsymbol{b},\boldsymbol{a}\ ; -q)$ and

$$\begin{aligned}(7.1)\qquad C_X(\boldsymbol{a}, \boldsymbol{b}; q) &= \{\boldsymbol{x}\in R^d;\ r(|\boldsymbol{x}-\boldsymbol{a}|) = r(|\boldsymbol{x}-\boldsymbol{b}|)+qr(|\boldsymbol{a}-\boldsymbol{b}|)\} \\ &= \{\boldsymbol{x}\in R^d;\ R_r(\boldsymbol{x}, \boldsymbol{a}\,|\,\boldsymbol{b}) = r(|\boldsymbol{a}-\boldsymbol{b}|)(1-q)/2\}\,.\end{aligned}$$

Therefore the set $C_X(\boldsymbol{a}, \boldsymbol{b}; q)$ proves to be a solid of revolution with axis containing $\boldsymbol{a}$ and $\boldsymbol{b}$. For this reason, we shall consider the set $C_X(\boldsymbol{a}, \boldsymbol{b}; q)$ under the following restriction: $\boldsymbol{a}, \boldsymbol{b}\in \boldsymbol{H}$ and $q>0$. Now setting, for every $r(t)\in \boldsymbol{S}$ and every $q>0$,

$$D_r^q = \{(u, v)\in R^2;\ u>0,\ v>0,\ r(|u-v|)\leqslant r(u)+qr(v)\leqslant r(u+v)\}\,,$$

we have the following results.

Theorem 4. *Let $(\boldsymbol{X}, r(t))$ be a G.r.f. on R^d, where $r(t)\in \boldsymbol{S}$. Suppose that D_r^q contains an interior point for some $q>0$.*
(i) *For another G.r.f. $(\boldsymbol{X}_1, r_1(t))$ on R^d with $r_1(t)\in \boldsymbol{S}$, the identity $r_1(t)=r(t)$ holds if and only if there exists $q_1>0$ such that*

$$(7.2)\qquad C_X(\boldsymbol{a}, \boldsymbol{b}; q)\cap \boldsymbol{H}\subset C_{X_1}(\boldsymbol{a}, \boldsymbol{b}; q_1)\cap \boldsymbol{H} \qquad \text{for any}\quad \boldsymbol{a}, \boldsymbol{b}\in \boldsymbol{H}\,.$$

(ii) *It holds that $r(t)\in \boldsymbol{L}$ if and only if*

$$(7.3)\qquad C_X(S_t\boldsymbol{a}, S_t\boldsymbol{b}; q)\cap \boldsymbol{H} = (S_tC_X(\boldsymbol{a}, \boldsymbol{b}; q))\cap \boldsymbol{H} \qquad \text{for any } \boldsymbol{a}, \boldsymbol{b}\in \boldsymbol{H} \text{ and any } t>0\,.$$

In order to prove this theorem we shall employ the following lemma.

Lemma 8. *Let $(\boldsymbol{X}, r(t))$ be a G.r.f. on R^d, where $r(t)\in \boldsymbol{S}$. Suppose that D_r^q contains an interior point for some $q>0$. Then there exist open intervals I and J contained in $(0, \infty)$, for which the following statement holds: For any $(u, v)\in I\times J$ and any $\boldsymbol{a}, \boldsymbol{b}\in \boldsymbol{H}$ satisfying $|\boldsymbol{a}-\boldsymbol{b}|=v$, there exists $\boldsymbol{x}[u, v]\in C_X(\boldsymbol{a}, \boldsymbol{b}; q)\cap \boldsymbol{H}$ such that $|\boldsymbol{x}[u, v]-\boldsymbol{b}|=u$.*

Proof. We see by the assumption that there exist open intervals I and J contained in $(0, \infty)$ such that $I\times J\subset D_r^q$. Let us choose $(u, v)\in I\times J$ and $\boldsymbol{a}, \boldsymbol{b}\in H$ $(|\boldsymbol{a}-\boldsymbol{b}|=v)$ arbitrarily. Then we have $r(u)+qr(v)\in I_{u,v}$, where we set $I_{u,v}=[r(|u-v|), r(u+v)]$. If $\boldsymbol{x}\in \boldsymbol{H}$ runs over the circle with center $\boldsymbol{b}$ and radius u, the range of the function $r(|\boldsymbol{x}-\boldsymbol{a}|)$ coincides with the interval $I_{u,v}$. It follows from the intermediate value theorem that there exists $\boldsymbol{x}[u, v]\in\boldsymbol{H}$ such that $|\boldsymbol{x}[u, v]-\boldsymbol{b}|=u$ and $r(|\boldsymbol{x}[u, v]-\boldsymbol{a}|)=r(u)+qr(v)$. In other words, we see by (7.1) that $\boldsymbol{x}[u, v]\in\mathcal{C}_X(\boldsymbol{a}, \boldsymbol{b}; q)$, which completes the proof.

Proof of Theorem 4. Let I and J be the open intervals stated in Lemma 8. Then, for any $(u, v)\in I\times J$ and any $\boldsymbol{a}\in\boldsymbol{H}$ $(|\boldsymbol{a}|=v)$, there exists $\boldsymbol{x}[u, v]\in \mathcal{C}_X(\boldsymbol{a}, \boldsymbol{0}; q)\cap\boldsymbol{H}$ such that $|\boldsymbol{x}[u, v]|=u$. In order to show the part (i), it suffices to prove the "*if*" part. To this end, setting $f(x)=r_1(r^{-1}(x))$ for $x\in[0, r(\infty))$, we have only to prove that $f(x)=x$ on $[0, r(\infty))$. We can show by (7.1) and (7.2) the following two equations: For any $(u, v)\in I\times J$ and any $\boldsymbol{a}\in\boldsymbol{H}$ $(|\boldsymbol{a}|=v)$,

$$\begin{cases} r(|\boldsymbol{x}[u, v]-\boldsymbol{a}|) = r(u)+qr(v)\,, \\ r_1(|\boldsymbol{x}[u, v]-\boldsymbol{a}|) = r_1(u)+q_1r_1(v)\,. \end{cases}$$

Then we obtain from these equations the following functional equation:

$$f(x+qy) = f(x)+q_1f(y) \qquad \text{for any} \quad (x, y)\in r(I)\times r(J)\,.$$

Thus we can easily show that $f(x)=x$ on $[0, r(\infty))$ in the same way as the proof of Theorem 1. In order to show the part (ii), it suffices to prove the "*if*" part. We see by (7.3) that

$$S_t\boldsymbol{x}[u, v]\in\mathcal{C}_X(S_t\boldsymbol{a}, S_t\boldsymbol{0}; q)\cap\boldsymbol{H} \qquad \text{for any} \quad (t, u, v)\in(0, \infty)\times I\times J\,.$$

Therefore, setting $h(u, v)=r^{-1}(r(u)+qr(v))$ for $(u, v)\in I\times J$, we obtain by (7.1) the following functional equation:

$$r(th(u, v)) = r(tu)+qr(tv) \qquad \text{for any} \quad (t, u, v)\in(0, \infty)\times I\times J\,.$$

Consequently we obtain the desired expression $r(t)=t^{\alpha}$ $(0<\alpha\leqslant 2)$ by using Lemma 7.

Finally we shall give a result about the existence of interior points in D_r^q.

Proposition 9. *Assume that $r(t)\in\boldsymbol{S}$ and $q>0$. In order that there exists an interior point in D_r^q, it is sufficient that the pair $\{r(t), q\}$ satisfies one of the following four conditions*:

(i) $0<q<1$ *and $r(t)$ is arbitrary*;

(ii) $q\geqslant 1$ *and $r(t)$ is strictly convex on $(0, t_0)$ with $qr'(+0)<r'(t_0/2)$ for some t_0 $(0<t_0<\infty)$*;

(iii) $q\geqslant 1$ *and $r(t)$ is strictly convex on (t_0, ∞) with $q\bar{r}'(+0)<r'(\infty)$ for some t_0 $(0<t_0<\infty)$, where we set $\bar{r}'(+0)=\overline{\lim}_{t\to+0} r'(t)$; and*

(iv) $q=1$ *and $r(t)=t$.*

Proof. Because of the continuity of $r(t)$, it suffices to prove that there exist $u>0$ and $v>0$ such that

$$r(|u-v|)<r(u)+qr(v)<r(u+v)\,. \tag{7.4}$$

Let us consider the case (i). For any $v>0$, we have $0<(1-q)r(v)<r(v)$. Then there exists u such that $0<u<v$ and $r(u)=(1-q)r(v)$ or equivalently $r(v)=r(u)+qr(v)$, from which we obtain (7.4). By the way, the first inequality of (7.4) holds for each $q\geqslant 1$, $u>0$ and $v>0$, since we have $r(|u-v|)<\max\{r(u), r(v)\}<r(u)+qr(v)$. Therefore, in the case $q\geqslant 1$, we have only to show the second inequality of (7.4). We now proceed to the proof of the case (ii). Noting that $r'(t)$ is strictly increasing on $(0, t_0)$ and $qr'(+0)<r'(t_0/2)$, we can choose v satisfying $0<v<t_0/2$ and $r'(+0)<qr'(v)<r'(t_0/2)$. Then there exists u such that $0<u<t_0/2$ and $r'(u)=qr'(v)$. Therefore we have $qr'(t)<qr'(v)=r'(u)<r'(u+t)$ for any $t\in(0, v)$. It follows that

$$r(u+v)-r(u)-qr(v)=\int_0^v\{r'(u+t)-qr'(t)\}\,dt>0\,, \tag{7.5}$$

which completes the proof of the case (ii). In the case (iii), we can choose u satisfying $u>t_0$ and $q\bar{r}'(+0)<r'(u)$. Then there exists $v>0$ such that the inequalities $qr'(t)<r'(u)<r'(u+t)$ hold on $(0, v)$. Thus we again obtain (7.5), which completes the proof of the case (iii). The proof of the case (iv) is obvious, since we have $D_r^1=(0, \infty)\times(0, \infty)$.

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