FINE POTENTIAL THEORY IN DIRICHLET SPACES

Torbjörn KOLSRUD[1]

Introduction. This article is an extended and revised version of a part of my thesis [17, §§ 6–8], and it should be seen as part two of [19]. It continues the study of potential theory with an emphasis on fine-topological questions from [18, 19], in the framework of (symmetric) Dirichlet spaces. In [19], we treated some questions on capacitary integrals and related matters. Here we present a rather general theory for superharmonic functions in Dirichlet spaces. (Some results were proved in [18].) As background serves a problem on bounded point evaluations for BLD-functions, harmonic on a certain set, see [14], and Fuglede's work on finely superharmonic functions [9], in particular their relations to certain functions in the space BLD, the archetype for all Dirichlet spaces, treated in [10]. (An application of the theory developed here is given in [21].)

Dirichlet spaces were originally introduced in the late 'fifties by Beurling and Deny. At about the same time, Hunt prepared the way for a general probabilistic potential theory. In the translation invariant case, i.e. in the case of Markov processes with stationary, independent increments, it is not hard to establish a one-to-one correspondence between (sufficiently smooth) symmetric Markov processes and (sufficiently smooth) Dirichlet spaces with translation invariant norm. Fukushima realised that if one looked upon the semi-groups involved as operators on L^2 —and not, as had been customary, on some class of continuous functions— a similar result was valid in general: under some mild smoothness assumptions there is a right-continuous strong Markov process (in fact, a Hunt process) to every Dirichlet space, and vice versa. (Here the smoothness assumptions are put directly on the Dirichlet space, while usually in potential theory one assumes that the semi-group (or the Green operator) is smooth in that it produces smooth functions.) We will use this correspondence whenever convenient and refer to Fukushima's book [11] for details.

The Dirichlet space approach to harmonic functions is by orthogonal projections, Dirichlet's principle from a modern point of view. The difference

---

with our setting as compared to the classical one is, that the operator replacing the Laplacian is no longer local, generally. It may depend on the global behaviour of functions. In particular, we have to make direct arguments and cannot use Fuglede's theory for finely harmonic functions or theory built on harmonic spaces, since these are of a local nature. Among articles prior to this one where harmonicity in non-local situations are treated, let us mention Hansen [12] and M. Itô [13].

Generally speaking the operators involved cannot be (differential or pseudo-) operators of order higher than two. As an example, consider the Sobolev space $W^{s,2}(\boldsymbol{R}^d)$, where $s \geq 0$ and the inner product is $\int (1-\Delta)^s uv\,dx$. This is a Dirichlet space if $0 \leq s \leq 1$, but not if $s > 1$. (See e.g. [20].) Consequently, only when $0 \leq s \leq 1$ are our results applicable to the Sobolev space $W^{s,2}(\boldsymbol{R}^d)$. Nevertheless, they seem to be new also in this case.

Now to the content of this article. In §1 we sketch the underlying theory. M. Fukushima observed that one of the basic assumptions from [19] (lower semicontinuity of excessive functions) was not needed in connection with the useful quasi Lindelöf and Choquet properties. It turns out that this assumption is superflous for the material treated below, and we show how to avoid it. Moreover we prove that also in the transient case, the excessive members of W form a hereditary subcone of the excessive functions. (Here W denotes our Dirichlet space.) This result makes possible the extension of results in [11] to the transient case as well. In §2, we introduce the class of "test functions" $W_0(E) = \{u \in W : u = 0$ q.e. off $E\}$ on a set E in M, the underlying topological space. Guided by the classical, that is, Newtonian, case, we define the class of functions in W which are weakly harmonic on E, $H(E)$, as the orthogonal complement of $W_0(E)$. We also introduce the weakly superharmonic functions on E, $S(E)$, as the cone dual to $W_0(E)^+$. The dependence on E and certain continuity properties for $W_0(\cdot)$ and the other spaces are examined.

Section 3 is devoted to harmonic measures and related concepts. There are some preliminary results on harmonic measures as well as results on the relations between the balayage operation and projection onto $H(E)$. We use the quasi Lindelöf property to show that every element in the dual, W', of W, has a fine support and that its relation to weak harmonicity is what it should be. We also consider an unsolved problem from Fukushima [11, §5.5], and prove that the fine supports of a measure of finite energy and its associated positive continuous additive functional coincide.

We also prove a variant of the so-called fine minimum principle, originating from [9], under the (necessary) additional assumption that W is local. In section 4, we follow Fuglede [9] and introduce finely superharmonic functions through harmonic measures. An identification of $S(E)$ with the functions in W which are finely superharmonic q.e. in the fine interior of E is established.

Since our investigations are pursued in a more general situation than those of Fuglede's (where the framework is that of harmonic spaces), we have to demand more of the functions in question and adopt a more restrictive notion of fine superharmonicity (the fine-topological version of Dynkin's definition of superharmonicity). It remains an open question if one can work with the more general definition that Fuglede employed in [9].

In the fifth and last section we return to bounded point evaluations and deduce results similar to those proved by the author in [14]. We also touch upon removable singularities for finely harmonic functions and their connection with bounded point evaluations.

It is a pleasure to acknowledge help and support from Lars Inge Hedberg and Bent Fuglede. Many thanks also to Peter Sjögren for some penetrating remarks. Finally, I want to thank M. Fukushima whose comments and ideas have —as I see it— simplified and improved this work.

1. Preliminaries and basic assumptions

1.1. In [19] we established the setting of this paper. Here we will only give a brief sketch of the prerequisites and refer to [19] and Fukushima [11] for further details. The Dirichlet space involved, W, is real and constructed from a transient semi-group $(p_t)_{t>0}$ of the form

$$(1.1) \qquad p_t f(x) = \int_M p(t, x, y) f(y) dm(y) .$$

Here M is the underlying topological space, assumed to be locally compact, Hausdorff and second countable; $m \geq 0$ is a Radon measure with support equal to M, and the semi-group is symmetric: $p(t, x, y) = p(t, y, x)$. We denote by $G = \int_0^\infty p_t dt$ the Green operator, and by $(.|.)$ the inner product of W. The condition for transience employed here is the following:

$$(1.2) \qquad K \text{ compact} \Rightarrow G1_K(x) = \int_0^\infty p_t 1_K(x) dt < +\infty , \qquad \forall x \in M .$$

The class of positive Radon measures of finite energy $I(\mu) = \int G\mu d\mu$ is denoted $\mathcal{E}$. For such measures μ, $G\mu \in W$ and

$$(u|G\mu) = \int u d\mu , \qquad u \in W, \mu \in \mathcal{E} .$$

The construction of W is such that $\{G\mu, \mu \in \mathcal{E} - \mathcal{E}\}$ is dense in W. We assume that W is regular, meaning that there are sufficiently many smooth (continuous and vanishing at infinity) functions in W to guarantee that they are dense in W as well as among the smooth functions. We denote by $\mathcal{S}$ the excessive func-

tions. The infimum operation in the lattice $\mathcal{S}$ is denoted $\wedge$, and the balayage functional $\hat{R}$. The fine topology is the topology generated by $\mathcal{S}$.

In [19] one of the basic assumptions was that the excessive functions were l.s.c. (lower semi-continuous), and we used this to deduce several fundamental results. Below we will show that this assumption —(A 10) in [19]— can be dispensed with altogether, so in this article *excessive functions are not assumed to be l.s.c.* (More about this later!) With the notation of [19], our assumptions are (A1)–(A9).

We redefine the operation $\wedge$ as follows (*cf.* [19, § 3.4]): for any function f, let $\hat{f}(x)=$fine lim inf $f(y)$ as $y \to x$, and $\wedge_i u_i = (\inf_i u_i)\hat{\,}$, where $(u_i) \subset \mathcal{S}$.

We will write cap E for the capacity of a set E. "Quasi everywhere", meaning "except for a set of capacity zero" will be abbreviated "q.e.", and a set of capacity zero is called "polar."

We will also need some probabilistic concepts. There is a Hunt process $X_t = X(t)$ with probability laws P^x, $x \in M$, such that $p_t f = E^\cdot f(X_t) = \int f(X_t(\omega)) \cdot P^\cdot(d\omega)$. If $A \subset M$ is sufficiently measurable, we will use the notation T_A for the corresponding hitting time

$$T_A = \inf\{t > 0 : X_t \in A\}.$$

Throughout this paper we will use the convention that all functions (or all sets, identifying sets with their indicators) on M are extended to the compactified space $M_\partial = M \cup \{\partial\}$ with value zero at the point of infinity, ∂.

1.2. In connection with fine-topological questions, two results of fundamental importance are the following:

The quasi Lindelöf property (Doob). *If $(V_i, i \in I)$ is any family of finely open sets, one can extract a sequence $(i_n)_{n \geq 1} \subset I$ such that*

$$\bigcup_i V_i \setminus \bigcup_n V_{i_n} \qquad \text{is polar}.$$

The Choquet property. *Finely open sets are quasi-open. That is, given V, finely open, we can find a set ω, open in the habitual topology, such that $V \subset \omega$ and* cap $(\omega \setminus V)$ *is as small as we wish.*

(Another, equivalent formulation was used in [19].)

In [19], these results were deduced using the above-mentioned assumption that excessive functions are l.s.c. M. Fukushima observed that they follow from (1.1). It is well known (see [2] or [4]) that the quasi Lindelöf property follows from Meyer's remarkably simple condition that there is a *representing measure*, i.e. a measure, λ say, (positive and Radon) such that for any $E \in \mathscr{B}^*$ (the universally measurable sets) we have

(1.3) $G1_E \equiv 0 \Leftrightarrow \lambda(E) = 0$.

Now m is representing if (1.1) holds. To see this, assume $m(E)=0$. (The other implication is obvious.) Then $G1_E(x)=0$ for m-a.e. x, according to [11, Lemma 4.2.1]. Therefore (1.1) gives, for any $x \in M$ and $t>0$,

$$0 = \int_M p(t, x, y) G1_E(y) m(dy) = \int_t^\infty p_s 1_E(x) ds \uparrow G1_E(x) , \qquad t \downarrow 0 ,$$

and (1.3) follows for $\lambda = m$.

In order to get a better understanding of the relations between the above properties we state the following theorem, the rest of which is proved in Appendix A.1.

Theorem. (i) *Under condition* (1.1), *m is representing. Consequently the quasi Lindelöf property holds.*
(ii) *The quasi Lindelöf property and the Choquet property are equivalent.*

1.3. In [19] we stated that $\mathcal{S} \cap W$ is a hereditary subcone of $\mathcal{S}$. In other words,

$$(1.4) \qquad [u, v \in \mathcal{S}, u \leq v, v \in W] \Rightarrow [u \in W, ||u|| \leq ||v||] .$$

This is obvious if $v \in L^2$, because then also u is in L^2 and we can use the spectral calculus. In the general case one needs another argument however. Now this result was used repeatedly in [19] without proof. Moreover, it is important for the "transient" theory: under (1.4) statements in [11] referring to the one-order form $\mathcal{E}_1$ carries over to the zero-order form $\mathcal{E}$ (we use Fukushima's notation [11]). Let us therefore state this result explicitly.

Theorem. *For a regular Dirichlet space W, (1.4) follows from the transience condition* (1.2).

Proof. It is known (see Chung [4, Th. 2, p. 126]) that (1.2) implies the existence of a function $g \in \mathcal{B}^*$ such that

$$(1.5) \qquad 0 < Gg < +\infty \quad everywhere .$$

(Here we use that X_t is a Hunt process.) We know that v, being a member of $\mathcal{S} \cap W$, is of the form $v = Gv$, for some $\nu \in \mathcal{E}$. By a well-known approximation procedure (see e.g. Blumenthal-Getoor [2, Prop. (5.11), p. 132]), it follows from (1.5) that

$$u = \lim_{n \to \infty} \uparrow Gf_n , \qquad f_n \in \mathcal{B}_+^* .$$

Clearly $Gf_n \in W$, because $Gf_n \leq v$ and then

$$(1.6) \qquad \int Gf_n \cdot f_n dm \leq \int v f_n dm = \int G\nu \cdot f_n dm = \int Gf_n d\nu \leq \int v d\nu = ||v||^2 .$$

From (1.6) we see that $||Gf_n||$ has a limit, because it increases with n. But

then (Gf_n) is Cauchy, because $||Gf_n - Gf_k||^2 \leq ||Gf_n||^2 - ||Gf_k||^2$ if $n \geq k$. But then $u \in W$, as stated, and therefore $u = G\mu$ for some $\mu \in \mathcal{E}$. The argument in (1.6) gives

$$||u||^2 = \int G\mu \, d\mu \leq \int G\nu \, d\nu = ||v||^2 ,$$

and the theorem follows.

2. Weakly superharmonic functions

2.1. If $E \subset M$ is any set, we define

$$W_0(E) = \{u \in W : u = 0 \text{ q.e off } E\} ,$$

and $H(E) = W_0(E)^\perp$, so that

$$W = W_0(E) \oplus H(E) .$$

We also introduce

$$S(E) = \{u \in W : (u \,|\, \phi) \geq 0, \; \forall \phi \in W_0(E)^+\} .$$

Then $S(E)$ is a closed convex (positive) cone.
Finally, we let $P(E)$ denote the subcone

$$P(E) = S(E) \cap W_0(E) ,$$

so that (with obvious meaning)

$$S(E) = P(E) \oplus H(E) .$$

2.2. It is obvious from the definitions, yet worth noting, that none of the classes defined changes if the underlying set is changed by a polar set. For instance is $W_0(E_1) = W_0(E_2)$ if the symmetric difference $E_1 \triangle E_2$ is polar.

2.3. The following result displays the dependence on the set E. As in [10] we will denote by E' the *fine interior* (that is, the interior with respect to the fine topology) of E.

Proposition. *For any set E, $W_0(E) = W_0(E')$.*

Corollary. $H(E) = H(E')$, $S(E) = S(E')$, *and* $P(E) = P(E')$.

Proof. If $u \in W_0(E')$, then $u \in W_0(E)$ since $E' \subset E$. If $u \in W_0(E)$, choose a polar set e such that $u \,|\, M \backslash e$ is finely continuous in $M \backslash e$ (in the relative fine topology). Then $\{u > 0\} \backslash e$ is finely open in $M \backslash e$, hence contained in $E' \backslash e$, since E' is the largest finely open subset of E. Similarly $\{u < 0\} \backslash e \subset E' \backslash e$, so $u = 0$ q.e. on $M \backslash E'$, proving the proposition, from which the corollary easily

follows.

Let us remark that functions in W are finely continuous q.e. simply because functions of the form $G\mu$, $|\mu| \in \mathcal{E}$, form a dense subset of W, and every potential is finely continuous, of course.

2.4.   The preceding proposition makes clear that it suffices to study $W_0(E)$ (and $H(E)$ or $S(E)$) for finely open sets E. We will therefore do so in what follows. As in Fuglede [10, Proposition 8] one can use Choquet's capacitability theorem to deduce certain results on the continuity of $W_0(\cdot)$:

   Proposition.   *If E is finely open, then $W_0(E) = \bigcap_\omega W_0(\omega)$, where ω ranges over all open supersets of E.*

   Proof.   Suppose that u vanishes q.e. on $\mathcal{C}\omega$ for every open set ω containing E. Choose e, polar, as in § 2.3, and define

$$A = \{x \in (M \backslash E) \backslash e : u(x) \neq 0\} .$$

By the Choquet property A is quasi Borel, hence capacitable, so $\operatorname{cap} A = \sup\{\operatorname{cap} F : F \subset A, F \text{ closed}\}$. But the assumption yields $\operatorname{cap} F = 0$ for all such F, so $\operatorname{cap} A = 0$. This proves the inclusion $\supset$, while the converse is obvious. $\square$

   Corollary. $H(E) = \overline{\bigcup_\omega H(\omega)}$, *and* $S(E) = \overline{\bigcup_\omega S(\omega)}$, *with ω as above.*

(For $P(E)$ there is no "limit theorem" of this kind because two different limits are involved. One may prove however, that the projection of $u \in W$ onto (the closed convex cone) $P(\omega)$, ω open, $\supset E$, E finely open, converges to the projection of u onto $P(E)$ as $\omega \downarrow E$.)

2.5.   The following result, known as "spectral synthesis" in the classical, i.e. Newtonian, case is another limit theorem. It was proved in [18], but included here too, for later use, and in order to get a more complete picture.

   Proposition.   *For any finely open set E,*

$$W_0(E) = \overline{\bigcup \{W_0(K) : K \subset E, K \text{ compact}\}} .$$

The proposition motivates that $W_0(E)$ should be looked upon as the test functions on E.—In the classical case, $W_0(E)$ is the closure of $C_0^\infty(E)$ when E is open.

By mere definition it is clear that $H(E)$ ($S(E)$) should be looked upon as the functions in W which are weakly harmonic (superharmonic) in E.—We recall that in the classical situation, when E is open, weak harmonicity and harmonicity are identical concepts (Weyl's lemma), and $H(E)$ consists of all functions in W which are harmonic in E. (See [14].)

Moreover, $P(E)$ is, in this case, the class of potentials (of positive measures)

of finite energy w.r.t. the Green function of E. In [18] we used the assumption that excessive functions are l.s.c. to prove Proposition 2.5. From [18] we see that one only needs the following result, which holds without the l.s.c.-assumption:

If $\mu \in \mathcal{E}$, there is an increasing sequence (K_n) of compact subsets of E such that

$$(2.1) \qquad \pi_{H(K_n)} G\mu \to \pi_{H(E)} G\mu \qquad \text{q.e.,} \quad n \to \infty.$$

(π stands for orthogonal projection.) The proof of (2.1) is given in Appendix A.2.

2.6. In [18] we proved the following "localisation theorem." We will use it in §§ 3–4 below. The assumption on l.s.c. was used only when referring to Proposition 2.5, so we do not need that assumption here neither.

Theorem.[2] *Let* $(V_i, \; i \in I)$ *be any family of finely open sets in* M. *Then*

$$H(\bigcup_i V_i) = \bigcap_i H(V_i) \quad and \quad S(\bigcup_i V_i) = \bigcap_i S(V_i).$$

For later use we note the following result, valid for finely open sets U and V. It follows from Theorem 2.6 by duality.

$$(2.2) \qquad W_0(U \cup V) = \overline{W_0(U) + W_0(V)}.$$

(This identity holds for the whole family (V_i): $W_0(\bigcup_i V_i) = \overline{\sum_i W_0(V_i)}$.)

3. Harmonic measures, projections and fine supports

3.1. The balayage operation $\hat{R}_u^A = \bigwedge \{v \in \mathcal{S} : v \geq u \text{ on } A\}$, $u \in \mathcal{S}$, was introduced in [19]. It suffices to consider finely closed sets (and we will henceforth do so without special mention), because $\hat{R}_u^A = \hat{R}_u^{\bar{A}}$, where $\bar{A}$ denotes the fine closure of A. (*Cf.* the situation with $W_0(\cdot)$ in § 2.3.)

Let $f \in \mathcal{S} \cap W$, denote by u_1 the projection of f onto $H(CA)$, and let u_2 denote the balayage of f onto A: $u_1 = \pi_{H(CA)} f$ and $u_2 = \hat{R}_f^A$.

As in [19, § 5.2] one shows that u_2 is the unique solution of an obstacle problem with obstacle $f \cdot 1_A$. In other words, u_2 has minimal norm amongst all functions in W that majorises f on A (q.e. on A is enough). Clearly $\|u_2\| \leq \|u_1\|$, because $u_1 = f$ q.e. on A. The latter holds also for u_2, so $u_2 - f \in W_0(CA)$, and $\|u_2\| \geq \|\pi_{H(CA)} f\| = \|u_1\|$. By uniqueness, $u_1 = u_2$. Since $\hat{R}_f^A \in \mathcal{S} \cap W$, it must be the potential of a measure of finite energy: $\hat{R}_f^A = G\mu^A$, $\mu^A \in \mathcal{E}$. Now this measure can have no mass on CA, because $\int \varphi \, d\mu^A = (\varphi \mid G\mu^A) = (\varphi \mid \pi_{H(CA)} f) = 0$

---

[2] We take this opportunity to adjust the proof of Lemma A in [18]. First of all the set Ω_δ should be defined as $(\omega_1 \cup \omega_2)^\delta$. Secondly, the function $\chi_0 \in C_0$ is chosen such, that $\chi_0 = 1$ on $K \backslash \omega_2$, and $\chi_0 = 0$ on $C(\omega_1 \cup \omega_2)$. From this point on, there are no changes needed.

for all functions φ vanishing on A. Thus $\mu^A \in \mathcal{E}(A) = \{\mu \in \mathcal{E}: \mu(CA) = 0\}$. Let us record this as

$$(3.1) \qquad \pi_{H(CA)} G\mu = G\mu^A = \hat{R}^A_{G\mu}, \qquad \mu \in \mathcal{E}.$$

Since $\{G\mu: |\mu| \in \mathcal{E}\}$ is dense in W we get

> **Proposition.** *Let E be finely open. Then*
> (a) $\{G\mu: |\mu| \in \mathcal{E}(CE)\}$ *is dense in $H(E)$;*
> (b) $\{G\mu_1 - G\mu_2: \mu_1 \in \mathcal{E}, \mu_2 \in \mathcal{E}(CE)\}$ *is dense in $S(E)$.*

REMARKS. 1. By spectral synthesis (Proposition 2.5) we may require in (a) that supp $\mu \subset CE$, and not only that μ is carried by CE, that is $|\mu|(E) = 0$. 2. If $\mu \in \mathcal{E}(CE)$, then $\mu^{CE} = \mu$, so functions of the form $G\mu - G\nu^{CE}$ with μ and ν in $\mathcal{E}$ are dense in $S(E)$.

3.2. **Theorem.** (a) *If $\mu \geq 0$ is a Radon measure which carries no mass on polar sets, then there is a smallest finely closed set that carries μ, called the fine support of μ, and denoted by $\mathrm{supp}_f \mu$. Moreover $\mathrm{supp}_f \mu$ is everywhere regular, i.e. a base, and therefore a Borel set.*
(b) *If $\mu \in \mathcal{E}$, and $V = M \backslash \mathrm{supp}_f \mu$, then V is the largest finely open set for which $G\mu \in H(V)$.*

REMARK. This may be carried out also for signed measures if we define $\mathrm{supp}_f \mu = \mathrm{supp}_f(|\mu|)$ in this case.

Proof. (a) We define the fine support of μ as follows. A point $x \in M$ is in the complement of $\mathrm{supp}_f \mu$ if there is a fine neighbourhood V_x of x such that $\mu(V_x) = 0$.[3] Thus $\mathrm{supp}_f \mu$ is the complement of the finely open set $V = \cup_x V_x$. By the quasi Lindelöf property, we may write $V = (\cup_{n \geq 1} V_{x_n}) \cup e$, where e is polar. Consequently $\mu(V) \leq \sum_n \mu(V_{x_n}) + \mu(e) = 0$. By definition is V the largest finely open set of μ-measure zero. In other words is $\mathrm{supp}_f \mu$ the smallest finely closed set that carries μ.
Write $\Sigma = \mathrm{supp}_f \mu = CV$. The base of Σ the finely closed set $b(\Sigma) = \{x \in M: \Sigma$ is non-thin at $x\} \subset \Sigma$. Now $\mu(Cb(\Sigma)) = 0$ as well, because $\Sigma \backslash b(\Sigma)$ is the set $\{x \in \Sigma: \Sigma$ is thin at $x\}$ which is polar according to the Kellogg property ([19, § 3.8]). The base of a set is always Borel (Blumenthal-Getoor [2, proof of V.1.14]), so (a) follows.

---

[3] Strictly speaking we should write $\mu^*(V_x) = 0$ (outer measure). However, by the Choquet property [19] there is a G_δ superset E of V_x such that $E \backslash V_x$ is polar. We may then define $\mu(V_x) = \mu(E)$, since μ does not carry any mass on polar sets.—Alternatively, we could enlarge the Borel σ-algebra $\mathcal{B}(M)$ so as to include the null sets common to all measure of finite energy.
It is known, see [2, V. 1.18], that any set in the *fine* Borel σ-algebra is the union of a Borel set and a semi-polar ($=$ polar under the present hypotheses, see [19]) set.

For (b) it suffices to note that if $u=G\mu$, $\mu\in\mathcal{E}$, then

$$u\in H(V)\Leftrightarrow(u\,|\,\varphi)=0,\ \forall\varphi\in W_0(V)\Leftrightarrow\int\varphi\,d\mu=0,\ \forall\varphi\in W_0(V)\Leftrightarrow\mu(V)=0\,.$$

3.3.   By Theorem 3.2 (b) it is clear that $\mathrm{supp}_f\,\mu$ is also what we could call the harmonic support of $G\mu$.   Formally we could write this

$$(3.2)\qquad\qquad\qquad\mathrm{supp}_f\,\mu=\mathrm{supp}_h\,G\mu\,.$$

For an arbitrary element, θ, in W' (dual space) there is no obvious way to define its fine support.   Consider instead its dual element $u\in W$, uniquely determined by $\theta(v)=(u\,|\,v)$, for any v in W.   We want to define the harmonic support of u—to be denoted by $\mathrm{supp}_h\,u$—by requiring that $V\equiv\mathcal{C}\,\mathrm{supp}_h\,u$ be the largest finely open set for which $u\in H(V)$.   The argument in Theorem 3.2 may be used to show that if $\mathrm{supp}_h\,u$ exists, it has to be a base.
Consider all points x in M for which there are finely open neighbourhoods V_x of x such that $u\in H(V_x)$, and let V denote the union of these sets.   Then V is finely open and $u\in H(V)$, according to Theorem 2.6.   The complement of V is called the *harmonic support* of u: $\mathrm{supp}_h\,u=\mathcal{C}V$.   We may then imitate (3.2) and *define*

$$\mathrm{supp}_f\,\theta\equiv\mathrm{supp}_h\,u,\ \theta\in W'\,.$$

   Theorem.   *Each element $u\in W$ has a unique harmonic support, denoted by* $\mathrm{supp}_h\,u$.   *The harmonic support is a base,* $u\in H(M\setminus\mathrm{supp}_h\,u)$, *and* $\mathrm{supp}_h\,u$ *is the smallest finely closed set with this property.*

   REMARK.   The harmonic support is the fine-topological counterpart of the "spectrum" used by Fukushima in [11, pp. 79–81].

   Corollary.   *If A is a finely closed set, then*

$$H(\mathcal{C}A)'=\{\theta\in W':\mathrm{supp}_f\,\theta\subset A\}\,.$$

3.4.   The balayage operation $u\to\hat{R}_u^A$ or, equivalently (see § 3.2), $\mu\to\mu^A$ gives rise to a family of measures δ_x^A, by the formula

$$(3.3)\qquad G\mu^A(x)=\int_M G\mu^A d\delta_x=\int_M G\mu\,d\delta_x^A,\qquad x\in M,\ \mu\in\mathcal{E}\,.$$

One can use the identity (3.1) together with a limiting argument to produce a measure δ_x^A satisfying (3.3).   We will not carry this argument out here.   Instead, we will use the connection to probability theory.   (See § 1 for definitions.) Let u be excessive and define $u^A(x)=E^x[u(X(T_A))]$.
Then u^A is excessive, $u^A\leq u$, and $u^A=u$ q.e. on A, because $T_A=T_A^x=0$ a.s P^x for q.e. $x\in A$.   (By the Kellogg property, see [19].)   Using this, it is easily

seen that $u^A = \hat{R}_u^A$, see e.g. Fukushima [11, Lemma 4.4.1]. (This gives a proof of the celebrated theorem of Hunt [2, Chapter III.6] in the much simpler symmetric case considered here.) Hence

$$(3.4) \qquad \pi_{H(CA)} u = \hat{R}_u^A = \delta_{\bullet}^A(u) = E^{\bullet}\{u(X(T_A))\}, \qquad u \in \mathcal{S} \cap W,$$

so we get

$$\delta_x^A(J) = P^x\{X(T_A) \in J\}, \qquad J \subset M.$$

When $T_A = 0$ a.s. P^x, $E^x[f(X(T_A))] = f(x)$, and we *define* $\delta_x^A = \delta_x$ in this case. Note that $P^x[X(T_A) \in \{x\}] = 0$ when $\{x\}$ is polar.

REMARK. With the assumption—stronger ┆than the regularity assumption used here: $C_\infty(M) \cap W$ is dense in W and $C_\infty(M)$ (the continuous functions $M \to \mathbf{R}$ that vanish at infinity)—that potentials of signed measures of finite energy are uniformly dense in $C_\infty(M)$, an alternative proof of the existence of δ_x^A goes as follows. Take two measures, μ and ν, from $\mathcal{E}$, such that $0 \le G\mu - G\nu \le 1$, i.e. $G\nu \le G\mu \le G\nu + 1$. From the definition of $\hat{R}.$, we immediately get $G\nu^A(x) \le G\mu^A(x) \le G\nu^A(x) + \hat{R}_1^A(x) \le G\nu^A(x) + 1$; that is

$$0 \le G\mu^A(x) - G\nu^A(x) \le 1.$$

Hence the map, densely defined on $C_\infty(M)$,

$$G\mu - G\nu \to G\mu^A(x) - G\nu^A(x)$$

is positive and bounded by one in the supremum norm. Accordingly, it has an extension to $C_\infty(M)$ with the same properties, and the Riesz representation theorem provides us with a measure satisfying (3.3).

3.5. We will now return to fine supports, introduced in § 3.2. Let $u = G\mu$, where $\mu \in \mathcal{E}$. Theorem 1.3 implies that Fukushima's results on additive functionals are valid also for the zero-order form, that is in the transient situation considered here. Hence by [11, Th. 5.1.1] there is a positive continuous additive functional $A = (A_t, t \ge 0)$ such that

$$u(x) = E^x[A_\infty] = E^x\left[\int_0^\infty dA_t\right].$$

(First we get equality q.e., and therefore also m-a.e. ([11]), then everywhere, since m is representing and u and $E^{\bullet}[A_\infty]$ are both excessive.) Define the *fine support* of A as the set

$$\Sigma = \{x: R = 0 \ P^x\text{-a.s.}\},$$

where

$$R = \inf \{t > 0: A_t > 0\}.$$

Then $\sum$ is a base and $R = T_{\Sigma}$ by [2, Ch. V.3.5–6]. By the strong Markov property (as in Dynkin's formula: *cf.* [2, II.1.2]), we get, for any stopping time T

$$u(x) = E^x[u(X_T)] + E^x[A_T] .$$

Since $E^x[A_R] \equiv 0$, we get

$$u(x) = E^x[u(X_R)] = E^x[u(X(T_{\Sigma}))] .$$

By (3.4) $u \in H(\mathcal{C}\Sigma)$, so $\mathcal{C}\Sigma \subset \mathcal{C}\,\mathrm{supp}_f\,\mu$ by Theorem 3.2. Consequently, with τ denoting the hitting time of $\mathrm{supp}_f\,\mu$, we have

$$\tau \geq T_{\Sigma} \quad \text{a.s.}$$

By definition of R, it is clear that R is *maximal* in the following sense: If T is a stopping time such that $u = E^{\cdot}u(X_T)$, then $T \leq R$ a.s. Hence

$$\tau \leq T_{\Sigma} \quad \text{a.s.,}$$

and therefore

$$\tau = T_{\Sigma} \quad \text{a.s.,}$$

so

$$\mathrm{supp}_f\,\mu = b(\mathrm{supp}_f\,\mu) = \{x \colon \tau = 0,\ P^x\text{-a.s.}\}$$
$$= \{x \colon T_{\Sigma} = 0,\ P^x\text{-a.s.}\} = b(\Sigma) = \Sigma .$$

We have proved

Theorem. *For any measure $\mu \in \mathcal{E}$, the fine support of μ coincide with that of its associated positive continuous additive functional.*

REMARK. This gives an answer to the problem treated by Fukushima in [11, § 5.5]. The point is that we use the *fine* topology, whereas Fukushima considered the usual support of the measure in question. It illustrates the importance of the quasi Lindelöf property.

3.6. Let us now return to the measures δ_x^A from § 3.4. When $A = CV$, and V is finely open, δ_x^{CV} is the *harmonic measure* for the set V at the point x.—From (3.4) it seems clear that in some sense $\delta_{\cdot}^{CV}$ represents projection onto $H(V)$. In general δ_x^{CV} is not of finite energy (see § 5), so one has to proceed with some care.

To study the relations between the harmonic measures and the spaces $H(\cdot)$, we need the following result.

3.6 **Lemma.** *If $u_n \to u$ strongly in W, there is a polar set e and a subsequence $(u_{n'})$ such that*

$$\int_M |u - u_{n'}| \, d\delta_x^{CV} \to 0$$

for any finely open set $V \subset M \backslash e$, and any point x in V.

Proof. By Doob's theorem [2, Chapter V.1],

$$|u_n - u| \leq \hat{R}_{|u_n - u|} \quad \text{q.e.}$$

because we always have $\hat{R}_f = R_f$ q.e. If V is any finely open set, $x \in V$, and B is polar, then $\delta_x^{CV}(B) = P^x[X(T_{CV}) \in B] \leq P^x[T_B < \infty] = 0$. Consequently

$$(3.5) \qquad \int_M |u_n - u| \, d\delta_x^{CV} \leq \int_M \hat{R}_{|u_n - u|} \, d\delta_x^{CV} = \hat{R}_{|u_n - u|}^{CV}(x) \leq \hat{R}_{|u_n - u|}(x), \qquad x \in V.$$

Now $\hat{R}_{|u_n - u|} \to 0$ in W if $u_n \to u$ in W, because in general $\hat{R}_f$ is of minimal norm among all (if any) functions in W majorising f q.e. In particular is $\|\hat{R}_f\| \leq \|f\|$ if $f \in W$; see [19, Lemma 5.2].—A subsequence of $(u_n)_{n \geq 1}$ will converge pointwise to zero outside some polar set e:

$$\hat{R}_{|u_{n'} - u|}(x) \to 0, \, n' \to \infty, \qquad x \in M \backslash e.$$

This together with (3.5) proves the lemma.

3.7. We will use the following lemma in § 4.

Lemma. *Suppose that $u \in W$ satisfies*

$$(3.6) \qquad u(x) = \int u \, d\delta_x^{CV}, \qquad x \in V,$$

for some finely open set V. Then $u \in H(V)$. In fact

$$(3.7) \qquad \mu^{CV} = \int \delta_x^{CV}(\cdot) \, d\mu(x), \qquad \mu \in \mathcal{E}.$$

Proof. Let $\mu \in \mathcal{E}$ and consider the measure $\nu \colon E \to \int \delta_x^{CV}(E) \, d\mu(x)$. If φ is a function of the form $\varphi = G\lambda_1 - G\lambda_2$, where $\lambda_i \in \mathcal{E}$, then

$$\begin{aligned}
\nu(\varphi) &= \int \delta_x^{CV}(\varphi) \, d\mu(x) = \int \left\{ \int (G\lambda_1 - G\lambda_2) \, d\delta_x^{CV} \right\} d\mu(x) \\
&= \int (G\lambda_1^{CV} - G\lambda_2^{CV})(x) \, d\mu(x) = \int (G\lambda_1 - G\lambda_2)(x) \, d\mu^{CV}(x) \\
&= \int \varphi(x) \, d\mu^{CV}(x) = \mu^{CV}(\varphi).
\end{aligned}$$

(Here we used properties of projections in Hilbert spaces together with the fact that $\pi_{H(V)} \varphi = \delta_{\cdot}^{CV}(\varphi)$ for φ as above; see (3.4).)

Now let $f \in W$ be a given compactly supported and continuous function. By
the regularity assumption, mentioned in § 1, it suffices to prove that $\nu(f) = \mu^{CV}(f)$
for such f's. This is however immediate since we can write $f = (f - \varphi) + \varphi$ with
φ as above, and with the norm of $f - \varphi$ as small as we please. Then $|\nu(f - \varphi)|$
$\leq \|G\nu\| \cdot \|f - \varphi\|$, and similarly for μ^{CV}. Since $\nu(\varphi) = \mu^{CV}(\varphi)$, this must hold
for f too, and (3.7) follows.

Now we know that for q.e. $x \in CV$, $\delta_x^{CV} = \delta_x$ (cf. the discussion in § 3.4 after
(3.4)). Hence the assumption (3.6) implies

$$u(x) = \int_M u \, d\delta_x^{CV}, \qquad \text{q.e.} \quad x \in M,$$

and therefore, by (3.7)

$$\int_M u(x) \, d\mu(x) = \int_M u(x) \, d\mu^{CV}(x), \qquad \mu \in \mathcal{E},$$

by a limiting argument using the regularity assumption again.—To justify
the use of Fubini's theorem we can dominate u by the potential $\hat{R}_{|u|} \in \mathcal{S} \cap W$.
Finally, the density results in § 3.1 show that $u \in H(V)$. □

3.8. We will now prove a variant of Fuglede's fine minimum principle [9,
Th. 9.1]. (In this connexion, see also Theorem 4.6 below.) Let us say that
W is *local* if for any u, $v \in W$ it holds that

$$uv = 0 \quad \text{q.e.} \Rightarrow (u \mid v) = 0.$$

By spectral synthesis (Prop. 2.5), this is equivalent to requiring that functions
with disjoint supports be orthogonal as in [11].

For a finely open set U we denote by $\partial_f U$ its *fine boundary*, i.e. the set $\tilde{U} \setminus U$
(where $\sim$ denotes fine closure).

Theorem. *Suppose that W is local, and that $u \in S(U)$ fulfils*

(3.8) $$\text{fine} \lim_{U \ni y \to x} \inf u(y) \geq 0, \qquad \text{q.e.} \quad x \in \partial_f U.$$

Then $u \geq 0$ q.e. on U. In particular, if $u \in H(U)$ and

(3.9) $$\text{fine} \lim_{U \ni y \to x} u(y) = 0, \qquad \text{q.e.} \quad x \in \partial_f U,$$

then $u = 0$ q.e. on U.

Proof. We start with the second statement, and therefore assume that
(3.9) holds for $u \in H(U)$. There is a polar set e such that u is finely continuous
on $M \setminus e$, and we may include in e the subset of $\partial_f U$ where (3.9) fails. From
the fine continuity of u it follows that $u = 0$ on $\partial_f U \setminus e$. Hence $u \in W_0(C\partial_f U)$,

because e is polar. Now $C\partial_f U = U \cup C\tilde{U}$ (where $C\tilde{U} = C(\tilde{U})$), so by (2.2) u belongs to the closure of $W_0(U) + W_0(C\tilde{U})$. Thus we can find $u_n \in W_0(U)$ and $\tilde{u}_n \in W_0(C\tilde{U})$, such that $u = \lim_n(u_n + \tilde{u}_n)$. By the assumption that W is local, $W_0(U)$ and $W_0(C\tilde{U})$ are orthogonal. It follows that

$$u = \pi_{W_0(U)}u + \pi_{W_0(C\tilde{U})}u = \pi_{W_0(C\tilde{U})}u \,,$$

because $u \in H(U) \perp W_0(U)$. Hence $u = 0$ q.e. on $C(C\tilde{U})$, in particular this holds on U. Suppose now that $u \in S(U)$ and that (3.8) holds. By Proposition 3.1, or rather the comments following it, there are measures μ_n and ν_n in $\mathcal{E}$ such that

$$u = \lim_n(G\mu_n - G\nu_n^{CU}), \quad \text{in } W \text{ and q.e.}$$

Consequently

(3.10) $\pi_{W_0(U)}u = \lim_n \pi_{W_0(U)}\{G\mu_n - G\nu_n^{CU}\} = \lim_n(G\mu_n - G\mu_n^{CU}) \geq 0 \quad q.e.$

By (3.8) u^- satisfies (3.9), so by the first part of the proof,

(3.11) $\pi_{H(U)}(u^-) = 0 \qquad q.e. \text{ on } U.$

Since $u = \pi_{W_0(U)}u + \pi_{H(U)}(u^+) - \pi_{H(U)}(u^-)$, the assertion follows from (3.10) and (3.11). $\square$

The first result in this direction (fine-topological) seems to be Brelot [3, Lemma 1]. Let us also mention Feyel and de La Pradelle [8, Th. 15], and the author's article [15].

4. Finely superharmonic functions

In this section we will consider functions in W which are (super-) "harmonic" in the sense that they have the (super-)mean value property w.r.t. a suitable class of harmonic measures. Our aim is to establish an identification between such functions and the spaces $(S(\cdot))$ $H(\cdot)$, as in Fuglede's article [10]. Since the process X_t in general is only right-continuous, the harmonic measures are in general carried by the fine *exteriors* of the sets in question, and not by their fine *boundaries*, as in [9]. (For more information on this, see [11, p. 113 ff.].) Therefore, when studying "fine (super-)harmonicity" in our setting, we are forced to consider globally defined functions. This is the main difference between our situation and Fuglede's.

We start with the following definition, to be compared with [9, pp. 67–68].

4.1. DEFINITION. Let $U \subset M$ be a finely open set. A function $u: M \to [-\infty, +\infty]$ is called *finely harmonic* in U, if u is finely continuous and finite on U, and if, for all finely open sets $V \in C\mathcal{V}$, where $C\mathcal{V}$ is a base for the fine topology in U, we have

(4.1) $u(x) = \delta_x^{CV}(u) , \qquad x \in V .$

(We use the functional notation $\mu(u) = \mu u = \int u \, d\mu .$)

Let us recall that for any set A (with fine closure $\tilde{A}$), we have $\delta_x^A = \delta_x^{\tilde{A}}$. This shows that the finely open sets are the relevant class of sets to consider in any definition of "harmonicity" in the spirit of 4.1; in other words, the finely open sets give us all conceivable generality. (*Cf.* § 2.3.)

4.2.   DEFINITION   A function u, numerically valued and defined q.e. on M, is *finely harmonic q.e.* in a finely open set U, if there is a polar set e such that u is finely harmonic in $U \backslash e$. The class of such functions will be denoted by $H_q(U)$.

We note that $H_q(U)$, just as $H(U)$, depends on U only modulo polar sets. Note also that every polar set is finely closed, so $U \backslash e$ above is finely open. Below we will prove that $H(U) = H_q(U) \cap W$, as in [10]. In order to obtain a similar result for $S(U)$, we introduce the following concept.

4.3.   DEFINITION   A function $u \colon M \to [-\infty, +\infty]$ is *finely superharmonic in the strong sense* in the finely open set U, if u is finely l.s.c., finite q.e. and $> -\infty$ in U, and if for *every* finely open subset V of U we have

(4.2) $u(x) \geqq \delta_x^{CV}(u) , \qquad x \in V .$

4.4.   REMARKS.   1. In the above definitions integrability is understood. If one wants a less restrictive definition, the integral to the right in (4.2) should be replaced by an upper integral, as in [9].
2. The probabilistic analogue of Definition 4.3 goes back to Dynkin. See [6, Ch. 12]. However, only the case when U is open in the usual topology is considered; sometimes it is also assumed in [6] that the process is continuous. Dynkin requires—apart from fine l.s.c. etc.—that u satisfy

(4.2)' $u(x) \geqq \delta_x^{CV}(u) , \qquad x \in U ,$

for a certain class of sets $V \subset U$. Let us show that (4.2) may be replaced by (4.2)': $V_0 \subset U$ are finely open sets. If $x \in V_0$, then (4.2)' is trivially true. Define $V_1 = \{y \colon CV_0 \text{ is thin at } y\} = M \backslash b(CV_0)$ (see § 3.2 ff.). Then $V_1 \supset V_0$, and their difference is polar. Hence, if $x \in V_1 \backslash V_0$, then by (4.2), $u(x) \geqq \delta_x^{CV_1}(u) = \delta_x^{CV_0}(u)$, and if $x \in U \backslash V_1$, then $\delta_x^{CV_1} = \delta_x$, in which case (4.4)' certainly holds. Summing up, it does not matter if we employ Fuglede's definition (4.2) or Dynkin's (4.2)'. They are equivalent.
Dynkin proved ([6, Ch, 12]) that (in his case) functions which are superharmonic in a set U are excessive w.r.t. the relevant subprocess living in U. Below we will use Hunt's theory for multiplicative functionals, as developed in [2],

to deduce the corresponding result, from which the superharmonic analogue of the afore-mentioned result $H(U) = H_q(U) \cap W$ will follow easily. At this point, let us also mention the related Theorem 4.4.2 in Fukushima [11]. We will need the following concept:

4.5. DEFINITION   If u is finely superharmonic in the strong sense in U, non-negative there, and if for any v, finely harmonic in U, with $v \leq u$, we must have $v \leq 0$, then we say that u is a *fine potential in the strong sense* in U.

In this definition, finely harmonic minorants may be replaced by finely sub-harmonic ones. *Cf.* [9].

The definition of the class $S_q(U)$—the functions which are finely superharmonic in some finely open subset U_0 of U, with $U \setminus U_0$ polar—and similarly the class $P_q(U)$ are taken for granted in what follows.

We are now ready to prove the following result, inspired by Fuglede's [10, Th. 11].

4.6. **Theorem.**   *Let U be a finely open set.   Then*
   (a) $H(U) = H_q(U) \cap W$,
   (b) $S(U) = S_q(U) \cap W$,   *and*
   (c) $P(U) = P_q(U) \cap W$.

Proof.   (a)   Let $u \in H(U)$, and write $u = \lim_n u_n$, where $u_n = G\mu_n$ for some measures μ_n with $|\mu_n| \in \mathcal{E}(CU)$.   We may assume that $G|\mu_n|$ is bounded and that we have pointwise convergence outside some polar set e_1.   We may also assume that u is finely continuous on $M \setminus e_1$.

For any finely open set V in U, and for any point x in V we have

$$u_n(x) = \int u_n \, d\delta_x^{CV} = \delta_x^{CV}(u_n) \,.$$

Consequently

$$|u(x) - \delta_x^{CV}(u)| \leq |u(x) - u_n(x)| + |u_n(x) - \delta_x^{CV}(u_n)| + |\delta_x^{CV}(u_n) - \delta_x^{CV}(u)|$$
$$\leq |u(x) - u_n(x)| + \delta_x^{CV}(|u - u_n|) \,.$$

The first term to the right-hand side tends to zero as n approaches infinity if $x \notin e_1$.   By Lemma 3.6, there is a subsequence $(u_{n'}) \subset (u_n)$ such that the second term to the right tends to zero for $x \in V \subset Ce_2$, with e_2 polar.   Denoting by e the union of e_1 and e_2, we get a polar set such that u satisfies (4.1) for x in any finely open subset of $U \setminus e$.   By the choice of e_1 it is clear that u is finely continuous in $U \setminus e$, so $u \in H_q(U)$.

Now, suppose that (4.1) holds for $x \in V \in \mathcal{CV}$, with V finely open in $U \setminus e$, where e is polar and $\mathcal{CV}$ is a base for the fine topology in $U \setminus e$.   By Lemma 3.7 $u \in H(V)$

for all $V \in \mathcal{V}$. Since $\mathcal{V}$ is a base, the union over all sets V in $\mathcal{V}$ is equal to $U \backslash e$. Thus the localisation Theorem 2.6 together with the comment in 2.2 gives $u \in H(U)$, because

$$H(U) = H(U \backslash e) = H(\cup \{V: V \in \mathcal{V}\}) = \cap \{H(V): V \in \mathcal{V}\} \, .$$

This proves (a).

(b) Suppose that $u \in S(U)$, and choose $u_n = G\mu_n - G\nu_n$, where $\mu_n \in \mathcal{E}$ and $\nu_n \in \mathcal{E}(CU)$, such that $u_n \to u$ in W and q.e. We also assume, as we may, that u is finely continuous on the set where pointwise convergence hold, and that the potentials of μ_n and ν_n are bounded. It follows that

$$\delta_x^{CV}(u_n) = G\mu_n^{CV}(x) - G\nu_n^{CV}(x) = G\mu_n^{CV}(x) - G\nu_n(x)$$
$$\leq G\mu_n(x) - G\nu_n(x) = u_n(x) \, ,$$

for V finely open in U, and $x \in V$.

We can argue exactly as in (a) above to find a set e, polar, such that u satisfies (4.2) for $x \in V$, where V is any finely open subset of $U \backslash e$. This proves that $S(U) \subset S_q(U)$.

We now turn to the converse, and assume that $u \in S_q(U) \cap W$. Replacing, as we may, U by the finely open set $U \backslash e$ (e polar), and recalling that then $S(U) = S(U \backslash e)$, we assume that u is finite and satisfies Definition 4.3 on the whole of U. Moreover, we may assume that CU is a base (i.e. each point is regular). To see this, we note that each point x in U has a fine neighbourhood V_x of this kind. If $u \in S(V_x)$ for all $x \in U$, then the Localisation Theorem 2.6 shows that $u \in S(U)$.

We may write $u = (u - \delta_\cdot^{CU} u) + \delta_\cdot^{CU} u \equiv v + h$. Then $h \in H(U) \subset S(U)$, and v is, as one easily checks, finely superharmonic in the strong sense in U. Now $\delta_x^{CU} = \delta_x$ for $x \notin U$, because CU is a base. We may therefore also assume that $u \equiv 0$ off U; the extended function is then finely continuous throughout M, and it is a member of $W_0(U)^+$ (assuming that u is replaced by the obviously non-negative function v above).

Due to the transience condition we may also assume that $T = T_{CU}$ is finite a.s., simply by replacing U with a relatively compact subset with the same properties that the original U had.—This follows from the proof of [2, Prop. II.4.4].

Next, we introduce the resolvent $G_p^U f(x) = E^x \left[\int_0^T e^{-pt} f(X_t) dt \right]$. In the terminology of [2, Ch. III], this is the resolvent associated with the multiplicative functional $N_t = 1_{[0,T)}(t)$, which is *exact* by [2, Example after III. 4.8], because being a base, CU is (nearly) Borel.—For $p = 0$ we obtain the Green kernel for U, to be denoted by G^U. We note that $\{ \int G^U \mu \, d\mu \}^{1/2}$ is the norm in $W_0(U)$.

The next step is to show that—under the above assumptions—u is excessive

w.r.t. the resolvent $(G_p^U, p \geq 0)$. (This is Dynkin's theorem in the fine-topological case.) Since the argument in Meyer [22, Th. 11, pp. 11–12] carries over with minor modifications, we omit it. The conclusion is that u is super-mean valued w.r.t. G_p^U: for each $p \geq 0$ we have $pG_p^U u \leq u$. Since u is also finely continuous, it follows that u is excessive.

We will now use the assumption "$T < +\infty$ a.s." Since u is excessive it follows (because the hypothesis (D) in [2] is fulfilled; see the remark after (5.13) on p. 133 and Prop. III.5.11) that

$$u = \lim_n \uparrow G^U f_n, \qquad f_n \in \mathscr{B}_+^*,$$

where, for each n, $G^U f_n$ and f_n are bounded. Replacing, as we may, f_n by $f_n \cdot 1_{K_n}$ for an increasing sequence of compacts K_n, we assume that

$$\forall n: \int G^U f_n \cdot f_n \, dm < +\infty.$$

It follows that $G^U f_n \in W_0(U)$ for all n. (Write the integral as $\int G g_n \cdot g_n \, dm$, where $g_n = f_n - f_n^{CU}$.) A convexity argument of standard type shows that there are functions, h_n say, such that $h_n \in \mathscr{B}_+^*$ and $G^U h_n \to u$ in W and pointwise. Consequently

$$(u \mid \varphi) = \lim_n (G^U h_n \mid \varphi) = \lim_n \int \varphi h_n \, dm,$$

and the latter is non-negative if $\varphi \in W_0(U)^+$, so $u \in S(U)$ and (b) follows.

(c) Suppose that $u \in P(U)$. As in (b) we may assume that CU is a base, and that $u \equiv 0$ on U. Then $u \geq 0$, because $\delta_{\cdot}^{CU} u = 0$. If $v \leq u$ and v is finely harmonic on U, it follows that $v = \delta_{\cdot}^{CU} v \leq \delta_{\cdot}^{CU} u$ for all finely open sets V in U. Choosing $V = U$, we see that u is a fine potential in the strong sense in U.

Suppose now that $u \in P_q(U) \cap W$. Then $u \geq 0$, so $\delta_{\cdot}^{CU} u \geq 0$ as well. Moreover, $u \geq \delta_{\cdot}^{CU} u$, so $\delta_{\cdot}^{CU} u = 0$, since by assumption also $\delta_{\cdot}^{CU} u \leq 0$ holds. It follows that $u \in S_q(U) \cap W_0(U) = S(U) \cap W_0(U) = P(U)$.

4.7. REMARKS 1. Let us mention that an alternative approach to finely harmonic functions in this semi-group setting of ours, would be to lean on Bliedtner and Hansen [1, Theorem 5.2], where necessary and sufficient conditions are given for the class of excessive functions to form the positive hyperharmonic functions in a $\mathscr{B}$-harmonic space.—In particular this is a local case. From the Choquet property we know that all excessive functions are quasi continuous, which implies that the harmonic space in question satisfies the *domination axiom* (D) (Constantinescu and Cornea [5, p. 228]), and this is the situation in which Fuglede works; see [9].

2. We want to point out an important difference between the "defining relations" for $H(\cdot)$, $S(\cdot)$ and $H_q(\cdot)$ on the one hand, and $S_q(\cdot)$ on the other. For instance, $u \in S(U)$ implies $u \in S(V)$ for any finely open set $V \subset U$, whereas if $u \geq \delta_\bullet^{CU} u$, then it need not hold that $u \geq \delta_\bullet^{CU} u$ for $V \subset U$.

3. It remains an open problem if the qualifier "in the strong sense" can be removed in Theorem 4.6 (b) and (c). If it is known that u is a potential of a signed measure μ of finite energy, then one may argue as follows. (Incidentally, this leads one to ask if there is some simple and direct analytic proof of Th. 4.6 (b).)

Let (A, B) be the Hahn-Jordan decomposition of μ into positive and negative sets, i.e. $\mu^+ B = \mu^- A = 0$, $A \cap B = \emptyset$, $A \cup B = M$, where A and B are Borel sets. Choose V in the given family $\mathcal{V}$ (a base for the fine topology in U) with $V \subset (B \cap U)'$ (the fine interior of $B \cap U$), and let $p = G\mu - \delta_\bullet^{CU} G\mu = \pi_{W_0(V)} G\mu$, so that $p \geq 0$ q.e. Then $0 \leq \|p\|^2 = -\int p \, d\mu^- \leq 0$, so $G\mu \in H(V)$. Theorem 2.6 now gives $G\mu \in H(\cup \{V \in \mathcal{V}: V \subset (B \cap U)'\}) = H((B \cap U)') = H(B \cap U)$, where the last equality is by Corollary 2.3. From this follows that μ^- vanishes on $B \cap U$, hence that $G\mu \in S(U)$.

5. Bounded point evaluations

We have now arrived at the problem which —in a sense— has been the reason for these investigations. In the Newtonian case the results to follow were proved in [14] and partly in [10].

5.1. If $U \subset M$ is finely open, it is clear that the map $H(U) \ni f \to f(x)$, $x \in M$, can be densely defined by

$$(5.1) \qquad G\mu(x) = \int_M G\mu \, d\delta_x^{CU}, \qquad \mu \in \mathcal{F}(U),$$

where

$$\mathcal{F}(U) = \{\mu = \mu_1 - \mu_2 : |\mu| \in \mathcal{E}(CU), \ G\mu_i \text{ bounded}\}.$$

Let us say that $x \in M$ is a *bounded point evaluation*[4] *(BPE)* for $H(U)$ if the map $f \to f(x)$, $f \in H(U)$, densely defined according to (5.1), is bounded (in the norm of W).

The following results parallel those of [14].

5.2. **Theorem.** *For a finely open set U, the following are equivalent:*
  (a) *x is a BPE for $H(U)$;*
  (b) *$G\delta_x^{CU} \in H(U)$;*

---

[4] We are fully aware of the solecism. It is however in everyday language, at least in this part of Mathematics.

(c) $\delta_x^{CU} \in \mathcal{E}$;

(d) $\int_0^\infty \operatorname{cap} \{G\delta_x^{CU} > t\} \, dt^2 < \infty$;

(e) $\int_0^\infty \operatorname{cap} [\{G(x, \cdot) > t\} \setminus U] \, dt^2 < \infty$.

Proof. Suppose (a) holds. Then, for some $g \in H(U)$,

$$(5.2) \qquad (g \mid G\mu) = \int G\mu \, d\delta_x^{CU}, \qquad \mu \in \mathscr{F}(U).$$

Choose $\lambda \in \mathcal{E}$ with arbitrary support, but with $G\lambda$ bounded. Then $G\lambda^{CU} \in \mathscr{F}(U)$. Since in this case, $\delta_\bullet^{CU}$ represent a projection (see (3.3)), and projections are idempotent, we get

$$(5.3) \qquad \int G\lambda \, d\delta_x^{CU} = \int G\lambda^{CU} d\delta_x^{CU}.$$

By Fubini's theorem the left-hand side of (5.3) equals $\int G\delta_x^{CU} d\lambda$, so (5.2) gives

$$\int G\delta_x^{CU} d\lambda = (g \mid G\lambda^{CU}) = (g \mid G\lambda^{CU} - G\lambda) + (g \mid G\lambda) = (g \mid G\lambda),$$

because $G\lambda^{CU} - G\lambda \in W_0(U)$ and $g \in H(U)$. Since $(g \mid G\lambda) = \int g \, d\lambda$, it follows that $g = G\delta_x^{CU}$ q.e. by variation of λ. Thus (b) follows from (a).

That (b) implies (c) is clear. The implication (c)$\Rightarrow$(d) follows from Theorem 2.1 in [19], whereas (d) implies that $G\delta_x^{CU} \in W$, according to the first part of Theorem 4.1 in [19]. In this case $\delta_x^{CU} \in \mathcal{E}$ which clearly implies (a). Furthermore, $\operatorname{cap} [\{G(x, \cdot) > t\} \setminus U] \leq \operatorname{cap} \{G\delta_x^{CU} > t\}$, since $G\delta_x^{CU} = \hat{R}_{G(x,\cdot)}^{CU}$ so that $G\delta_x^{CU}$ agrees with the Green function $G(x, \cdot)$ q.e. off U. Hence (d) implies (e). If (e) holds, then the second part of Theorem 4.1 in [19] implies that the balayage of $G(x, \cdot)$ onto U, that is $G\delta_x^{CU}$, belongs to W, so (c) holds, and the proof is complete. $\square$

5.3. REMARKS 1. The conditions in the theorem are also equivalent to the following: There is a function g in W such that $g = G(x, \cdot)$ q.e. on $M \setminus U$.
2. In the classical situation one has to require that CU be thin at x in order that x be a *BPE*, see [14]. As the following example shows, this connection between thinness and *BPE*s does not hold in general. Let M be the interval $(0, 1) \subset \mathbf{R}$, and define $(u \mid v) = \int_M u'v' \, dx$, with u and v in $C_0^\infty(M)$, say. The resulting Dirichlet space is continuously embedded in the continuous functions (by Sobolev's theorem). Hence every point of M is a *BPE* for $H(U)$, for *any* $U \subset M$, in this case.

3. There are non-trivial cases when the set of *BPE*s for $H(U)$ is empty. One can use a construction from Fernström and Polking [6, Theorem 2], to produce a compact set E with $E' \neq \emptyset$, and no *BPE*s, provided the dimension $d \geq 4$. (We are referring to the classical case in $\boldsymbol{R}^d$.) In dimension less than four, this is not possible. See the comments in [10].

5.4. Assume that M is a Euclidean space and that $G(x, y) = g(|x-y|)$. (This could probably also be carried out on a locally compact abelian group.) We also assume that $g(0) = \infty$, $g(\infty) = 0$, and that g is strictly increasing and subject to the condition $g(t) \leq \text{const. } g(2t)$, for $t > 0$. Then it is easily seen, using capacitary integrals or sums (*cf.* [14, 19]), that x is a *BPE* for $H(U)$ if and only if

$$(5.4) \qquad\qquad \sum_n g(2^{-n})^2 \, \text{cap}(A_n(x) \backslash U) < \infty,$$

where $A_n(x)$ denotes the annulus $\{y: |x-y| \in [2^{-n}, 2^{-n+1})\}$, $n \in \boldsymbol{Z}$. Consider the case of M. Riesz potentials, i.e. $G(x, y) = |x-y|^{\alpha-d}$, $0 < \alpha < d$, $\alpha \leq 2$, or Bessel potentials, i.e. $G = (1-\Delta)^{-\alpha/2}$, $0 \leq \alpha \leq 2$, in $\boldsymbol{R}^d$. Then the capacity of the annulus is comparable to $1/g(2^{-n})$; hence

Theorem. *If* $W = W^{s,2}(\boldsymbol{R}^d)$, *with* $0 \leq s \leq 1$, *or if* W *is the corresponding space of M. Riesz potentials, then* x *is a BPE for* $H(U)$ *if and only if*

$$\sum_n \frac{\text{cap}(A_n(x) \backslash U)}{\text{cap}(A_n(x))^2} < \infty.$$

See also [19, Chapter 4]. (The relation between the two parameters is $s = \alpha/2$. For the definition of Sobolev spaces, see the introduction.)

5.5. In [10], Fuglede proved that the every *BPE* was simultaneously a removable singularity[5] for all finely harmonic functions on the set U, which are also in the Dirichlet space of BLD-functions, and conversely. In our situation we have not been able to prove the corresponding full result. Let us prove a partial result.

We suppose that x is not a *BPE* (for $H(U)$). Then we can find functions $u_n = G\mu_n$, $|\mu_n| \in \mathcal{E}(\mathcal{C}U)$, with $0 \leq u_n \leq 1$, $u_n(x) = 1$, and $\|u_n\| \leq 2^{-n}$, for $n = 1, 2, \cdots$ Define $v_n = \hat{R}_{u_n}$. Then $v_n \in W$ by [19, Lemma 5.2], and $\|v_n\| \leq \|u_n\|$. To show that $v_n(x) \geq 1$ we note that $v_n \geq u_n$ q.e. (by Doob' theorem), and use fine continuity, since we may always arrange that each u_n be finely continuous. Since $u_n \in H(U)$, we also have $\delta_\cdot^{\mathcal{C}U} v_n(x) \geq 1$. It follows that the function $w = \sum \delta_\cdot^{\mathcal{C}U}(v_n)$ is in $H(U)$ (because it is the increasing limit of a sequence in $\mathcal{S} \cap H(U)$, each of norm no greater than 1) and $w(x) = +\infty$. Therefore x cannot

---

[5] In [16], we used capacitary integrals to solve a problem on removable singularities for functions in Sobolev spaces $W^{1,p}$ satisfying the Euler equation $\text{div}(\text{grad } u |\text{grad } u|^{p-2}) = 0$.

be a removable singularity for $H(U)$. (We have not given any definition; see however [9, §§ 9.14–15].) It is beyond the author's knowledge whether a converse to this result is true in the general situation considered here.

Appendix

Proof of Theorem 1.2 (ii). Let W_1 be the Dirichlet space built up from the semi-group $(e^{-t}p_t)_{t>0}$, and let $\mathcal{S}_1$ denote the corresponding excessive functions. To prove that the Choquet property follows from the quasi Lindelöf property, it suffices to show that the fine topology has a base of quasi-open sets. *Cf.* [19, Th. 3.10].

By [2, Prop. II.4.4], the fine topology has a base of finely open sets of the form $V=\{v<1\}$, where $v\in\mathcal{S}_1$ and $V\subset K$ for some compact set K. Moreover $v\equiv 1$ on K. Let U be an open and relatively compact set such that $K\subset U$, and let u denote the 1-capacitary potential of U: $u=\wedge\{w\in\mathcal{S}_1: w\geq 1 \text{ on } U\}$. Then $u\in W_1$, and $u\equiv 1$ on U.

Define $w=\min(u, v)\in\mathcal{S}_1$. Then, by Theorem 1.3 for instance, $w\in W_1$. By the definition it is clear that

$$V = \{w<1\} \cap U \, .$$

$\{w<1\}$ is quasi-open since $w\in W_1$, and the open set U is of course quasi-open. Clearly the intersection of two quasi-open sets is quasi-open, so the assertion follows.

To prove the converse we appeal to Fuglede [10a]. Since cap is a Choquet capacity, it is sequentially order continuous from below ([11, Th. 3.1.1. (ii)]). Moreover, by the Choquet property, condition (T_1) on p. 143 in [10a] is fulfilled. Since cap $E=$cap $b(E)$, also (T_2) holds. Altogether, this means that the fine topology is "compatible" with the "quasi topology" determined by cap, in the sense of [10a, Def. 4.3]. Therefore the hypotheses of Corollary 3, p. 149, in [10a] are satisfied, and the quasi Lindelöf property follows.

Proof of (2.1). Let $\mu\in\mathcal{E}$, and define $B=CE$, where E is finely open. We may assume that $B=b(B)$, and that μ is finite. Then the approximation theorem [2, I.1.13] shows that there is a decreasing sequence of open sets $\omega_n\supset B$, such that

$$T_{\omega_n} \uparrow T_B \quad \text{a.s.} \quad P^x, \, \forall x \, .$$

We may assume that E is relatively compact, hence that this holds for each $K_n=\mathcal{C}\omega_n$ too. Now

$$\pi_{W_0(K_n)}G\mu = E^{\cdot}[A(T_{\omega_n})] \, ,$$

where $A(\cdot)$ denotes the positive continuous additive functional associated with $G\mu$. By monotone convergence

$$E^{\cdot}[A(T_{\omega_n})] \uparrow E^{\cdot}[A(T_B)] = \pi_{W_0(E)} G\mu .$$

This proves (2.1).

References

[1] J. Bliedtner and W. Hansen: *Markov processes and harmonic spaces*, Z. Wahrsch. Verv. Gebiete **42** (1978), 309–325.

[2] R.M. Blumenthal and R.K. Getoor: Markov processes and potential theory, Academic, New York-London, 1968.

[3] M. Brelot: Sur l'allure des fonctions harmoniques et sous-harmoniques à la frontière, Math. Nachr. **4** (1950–51), 169–197.

[4] K.L. Chung: Lectures from Markov processes to Brownian motion, Springer, Berlin-Heidelberg-New York, 1982.

[5] C. Constantinescu and A. Cornea: Potential theory on harmonic spaces, Springer, Berlin-Heidelberg-New York, 1972.

[6] E.B. Dynkin: Markov processes I–II, Springer, Berlin-Heidelberg-New York, 1965. (Translation of the Russian edition from 1962)

[7] C. Fernström and J.C. Polking: *Bounded point evaluations and approximation in L^p by solutions of elliptic partial differential equations*, J. Funct. Anal. **28** (1978), 1–20.

[8] D. Feyel and A. de la Pradelle: *Le rôle des espaces de Sobolev en topologie fine*, Séminaire de Théorie du Potentiel, Paris, Lecture Notes in Math. 563, Springer, Berlin, 1976.

[9] B. Fuglede: Finely harmonic functions, Lecture Notes in Math. 289, Springer, Berlin, 1972.

[10] ————: *Fonctions BLD et fonctions finement surharmoniques*, Séminaire de Théorie du Potentiel, Paris, Lecture Notes in Math. 906, Springer, Berlin, 1982, 126–157.

[10a] ————: *The quasi topology associated with a countably subadditive set function*, Ann. Inst. Fourier **21**, 1 (1971), 123–169.

[11] M. Fukushima: Dirichlet forms and Markov processes, North-Holland/Kodansha, Amsterdam/Tokyo, 1980.

[12] W. Hansen: Potentialtheorie harmonischer Kerne, Seminar über Potentialtheorie, Lecture Notes in Math. 69, Springer, Berlin, 1968.

[13] M. Itô: *On α-harmonic functions*, Nagoya Math. J. **26** (1966), 205–221.

[14] T. Kolsrud: *Bounded point evaluations and balayage*, Ark. Mat. **20** (1982), 137–146.

[15] ————: *A uniqueness theorem for higher order elliptic partial differential equations*, Math. Scand. **51** (1982), 323–332.

[16] ————: *Condenser capacities and removable sets in $W^{1,p}$*, Ann. Acad. Sci. Fenn. Ser. AI Math. **8** (1983), 343–348.

[17] ————: Capacitary integrals and fine potential theory in certain function spaces, Univ. of Stockholm, Report, 4, 1983.

[18] ————: *Localisation of superharmonic functions in not necessarily local Dirichlet spaces*, Séminaire de Théorie du Potentiel, Orsay 1983, Lecture Notes in

Math. 1096, Springer, Berlin, 1984, 401–411.

[19] ————: *Capacitary integrals in Dirichlet spaces*, Math. Scand. **55** (1984), 95–120.

[20] ————: *Diffusions and maximal Dirichlet spaces*, to appear.

[21] ————: On the Markov property for certain Gaussian random fields, Techn. Report Univ. of Stockholm/ Forschungszentrum, Bielefeld-Bochum-Stochastik/ ZiF, Universität Bielefeld, 1984.

[22] P.-A. Meyer: Processus de Markov: la frontière de Martin, Lecture notes in Math. 77, Springer, Berlin, 1068.

[23] ————: Probability and potentials, Ginn and Blaisdell, Massachusetts, 1966. (Translation of the 1966 French edition).

Department of Mathematics
University of Stockholm
Box 6701
S-113 85 Stockholm
Sweden