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CHARACTERIZATIONS OF CONDITIONAL EXPECTATIONS FOR $L_1(X)$-VALUED FUNCTIONS

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Introduction. The conditional expectation of a Banach-valued function is defined by means of Bochner integral, see L. Schwartz [11]. The purpose of this paper is to study sufficient conditions for a linear operator on the space of L_1-valued integrable functions on a probability space $(\Omega, \mathcal{A}, \mu)$ to be a conditional expectation operator (in the sense of Schwartz [11]), where L_1 means the space of integrable real-valued functions over a measure space (X, S, λ). For the case of real-valued functions, such a problem has been studied by several authors, such as T. Ando [1], R.R. Bahadur [2], R.G. Douglas [4], S.C. Moy [8], M.P. Olson [9], J. Pfanzagl [10], M.M. Rao [11], and Z. Šidák [14].

For the case of strictly convex space-valued functions D. Landers and L. Rogge [7] proved that every constant-preserving contractive projection becomes conditional expectation operators. They also show that these conditions do not characterize the conditional expectation operator for the case of L_1-valued functions.

In Section 2 we shall reduce the problem of characterization of conditional expectations of L_1-valued functions to the problem of operators of scalar valued integrable functions on a product space. In Section 3 we deal with the case of a measure space with ergodic transformations. Then every constant-preserving contractive projection becomes a conditional expectation operator under the additional condition that it commutes with these transformations. Then we deal with the case that X is a locally compact Hausdorff topological group and λ is the left Haar measure on the σ-ring S generated by the class of compact sets. In Section 4 we suppose that $X=R/Z$, where Z is the class of integers, and S is the class of Borel sets and λ is the Haar measure. Then properties of translation-invariant σ-subalgebra S' of S is considered, and we will use this result to consider the case in Section 2.

1. Definitions and useful lemmas. Let E be a Banach space over the reals with the norm $||\cdot||_E$ and $(\Omega, \mathcal{A}, \mu)$ a probability space. Let $L_1(\Omega, \mathcal{A}, \mu, E)$ denote the space of all E-valued Bochner integrable functions on $(\Omega,$

$\mathcal{A}$, μ) associated with the norm defined by

$$\|f\|_L = \int \|f(\omega)\|_E \, d\mu(\omega) .$$

For the definitions and properties of Bochner integral, see Hille and Phillips [6].

DEFINITION 1. For a σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$, a function g is called the conditional expectation of f given $\mathcal{B}$ if g is weakly measurable with respect to $\mathcal{B}$, and $\int_B g \, d\mu = \int_B f \, d\mu$ for each $B \in \mathcal{B}$, where the integral is Bochner integral. We denote by $f^{\mathcal{B}}$ the conditional expectation of f given $\mathcal{B}$.

We shall denote by R the space of real numbers. For each $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$ and $a \in E$ we define $(\varphi \cdot a)(\omega) = \varphi(\omega) \cdot a$ for each $\omega \in \Omega$. Then $\|\varphi \cdot a\|_L = \|a\|_E \int |\varphi| \, d\mu$.

Lemma 1.1. *For each $f \in L_1(\Omega, \mathcal{A}, \mu, E)$ the conditional expectation $f^{\mathcal{B}}$ of f given $\mathcal{B}$ exists uniquely up to almost everywhere and satisfies $\int \|f(\omega)\|_E d\mu(\omega) = \int \|f(\omega)\|_E \, d\mu(\omega)$.*

For proof see Schwartz [12].

By the definition of conditional expectation, $(\varphi \cdot a)^{\mathcal{B}} = \varphi^{\mathcal{B}} \cdot a$ for each $\varphi \in L_1 (\Omega, \mathcal{A}, \mu, R)$ and $a \in E$.

DEFINITION 2. Let P be a linear operator of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself. P is said to be *contractive* if $\|P\| = \sup \{\|P(f)\|_L : f \in L_1(\Omega, \mathcal{A}, \mu, E)$ and $\|f\|_L = 1\} \leqq 1$, P is *constant-preserving* if $P(1_\Omega \cdot a) = 1_\Omega \cdot a$ for each $a \in E$ and P is called a *projection* if $P \circ P = P$.

In particular a contractive operator is bounded, and hence continuous.

Lemma 1.2. *The conditional expectation operator $(\cdot)^{\mathcal{B}}$ is a constant-preserving contractive projection for each σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$.*

This is a direct consequence of Definition 1 and Lemma 1.1.

Lemma 1.3 (Douglas). *If P is a constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, R)$ into itself, then there exists a σ-subalgebra C of $\mathcal{A}$ such that $P(f)$ is the conditional expectation of f given C for each $f \in L_1(\Omega, \mathcal{A}, \mu, R)$; i.e., $P(f) = f^C$ for each $f \in L_1(\Omega, \mathcal{A}, \mu, R)$.*

For proof see Douglas [4].

Obviously, the above lemma holds for every finite measure space $(\Omega, \mathcal{A}, \mu)$.

Lemma 1.4. *If Q is a constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself, then for each $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$ with $0 \leqq \varphi \leqq 1$ and $a \in E$ there exists a μ-null set N such that*

$$||a||_E - ||Q(\varphi \cdot a)(\omega)||_E = ||a - Q(\varphi \cdot a)(\omega)||_E \quad \textit{for each} \quad \omega \in \Omega - N.$$

Proof. Since Q is constant-preserving and contractive and $0 \leqq \varphi \leqq 1$,

$$\begin{aligned} ||1_\Omega \cdot a||_L - ||\varphi \cdot a||_L &= ||a||_E - ||a||_E \int |\varphi| d\mu \\ &= ||a||_E \int |1_\Omega - \varphi| d\mu = \int ||a||_E |1_\Omega - \varphi| d\mu = ||1_\Omega \cdot a - \varphi \cdot a||_L \\ &\geqq ||Q(1_\Omega \cdot a - \varphi \cdot a)||_L = ||1_\Omega \cdot a - Q(\varphi \cdot a)||_L \\ &= ||1_\Omega \cdot a||_L - ||Q(\varphi \cdot a)||_L \geqq ||1_\Omega \cdot a||_L - ||\varphi \cdot a||_L . \end{aligned}$$

Therefore it holds that

$$||1_\Omega \cdot a||_L - ||Q(\varphi \cdot a)||_L = ||1_\Omega \cdot a - Q(\varphi \cdot a)||_L .$$

Hence we have

$$\int \{||a||_E - ||Q(\varphi \cdot a)(\omega)||_E\} \, d\mu(\omega) = \int ||1_\Omega(\omega) \cdot a - Q(\varphi \cdot a)(\omega)||_E \, d\mu(\omega) .$$

From the evident inequality $||a||_E - ||Q(\varphi \cdot a)(\omega)||_E \leqq ||a - Q(\varphi \cdot a)(\omega)||_E$ for each $\omega \in \Omega$, we have $||a||_E - ||Q(\varphi \cdot a)(\omega)||_E = ||a - Q(\varphi \cdot a)(\omega)||_E$ for each $\omega \in \Omega - N$, where N is a μ-null set.

Proposition 1.1. *Let Q be a constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself. If, for each $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$ and for each nonzero element a of E, there exists $\varphi' \in L_1(\Omega, \mathcal{A}, \mu, R)$ such that $Q(\varphi \cdot a) = \varphi' \cdot a$, then there exists a σ-subalgebra C of $\mathcal{A}$ such that $Q(f)$ is the conditional expectation of f given C for each $f \in L_1(\Omega, \mathcal{A}, \mu, E)$.*

Proof. φ' does not depend on the choice of the element a of E. (See the Proof of the theorem of Landers [7].) Therefore we can define an operator Q' of $L_1(\Omega, \mathcal{A}, \mu, R)$ into itself by $Q'(\varphi) \cdot a = Q(\varphi \cdot a)$ for each $a \in E$ and $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$. Clearly Q' is a constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, R)$ into itself. Therefore by Lemma 1.3 there exists a σ-subalgebra C such that $\varphi^C = Q'(\varphi)$. Therefore we have $Q(\varphi \cdot a) = \varphi^C \cdot a = (\varphi \cdot a)^C$. And hence Q is the conditional expectation operator given C by the proof of [12, Theorem 1.6.4]

In the rest of this paper we restrict ourselves to the case that $E = L_1(X, S, \lambda, R)$, where X is a measure space and S is a σ-ring and λ is a measure on S.

Lemma 1.5. *Suppose that Q is a constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself. If $K \in S$ with $\lambda(K) < \infty$, then, for every $\varphi \in L_1(\Omega, \mathcal{A}, \mu, R)$ with $0 \leqq \varphi \leqq 1$ and $A \in \mathcal{A}$, we have*

$$1 \geqq \int_A Q(\varphi \cdot 1_K)(\omega, x)\, d\mu(\omega)\, d\lambda(x) \geqq 0 \quad \textit{and}$$

$$\int_A Q(\varphi \cdot 1_K)(\omega, x)\, d\mu(\omega) = 0\,, \ \lambda\text{-}a.e.x \ on \ K^C\,.$$

Proof. By Lemma 1.4 there exists a μ-null set N such that $\|1_K\|_E - \|Q(\varphi \cdot 1_K)(\omega)\|_E = \|1_K - Q(\varphi \cdot 1_K)(\omega)\|_E$ for each $\omega \in \Omega - N$, since $1_k \in E$, where 1_k is the indicator function of K. Hence

$$\int 1_K\, d\lambda(x) - \int |Q(\varphi \cdot 1_K)(\omega, x)|\, d\lambda(x) = \int |1_K - Q(\varphi \cdot 1_K)(\omega, x)|\, d\lambda(x)\,.$$

From the evident inequality

$$1_K(x) - |Q(\varphi \cdot 1_K)(\omega, x)| \leqq |1_K(x) - Q(\varphi \cdot 1_K)(\omega, x)|\,,$$

we have for each $\omega \in \Omega - N$

$$1_K(x) - |Q(\varphi \cdot 1_K)(\omega, x)| = |1_k(x) - Q(\varphi \cdot 1_K)(\omega, x)|\,,\ \lambda\text{-a.e.x.}$$

Therefore, for each $\omega \in \Omega - N$, $0 \leqq Q(\varphi \cdot 1_K)(\omega, x) \leqq 1$, λ-a.e.x, and $Q(\varphi \cdot 1_K)(\omega, x) = 0$, λ-a.e.x. on K^C. Hence

$$1 \geqq \int_A Q(\varphi \cdot 1_K)(\varphi, x)\, d\mu(\omega)\, d\lambda(x) \geqq 0 \quad \text{and}$$

$$\int_A Q(\varphi \cdot 1_K)(\omega, x)\, d\mu(\omega) = 0,\ \lambda\text{-a.e.x on } K^C\,.$$

2. The case of a general measure space. Let (X, S, λ) be a measure space. For convenience we denote $L_1(\Omega \times X, \mathcal{A} \times S, \mu \times \lambda, R)$ by $L_1(\Omega \times X)$ and $L_1(\Omega, \mathcal{A}, \mu, L_1(X, S, \lambda, R))$ by $L_1(\Omega, L_1(X))$.

Lemma 2.1. *There exists a norm isomorphism of $L_1(\Omega, L_1(X))$ onto $L_1(\Omega \times X)$.*

For proof see Treves [15, p. 464, Exercise 46.5]

Let Q be a mapping of $L_1(\Omega, L_1(X))$ into itself. Let i be the isomorphism of $L_1(\Omega, L_1(X))$ onto $L_1(\Omega \times X)$. Then $Q' = i \circ Q \circ i^{-1}$ is a mapping of $L_1(\Omega \times X)$ into itself.

Lemma 2.2. *Q is a contractive projection iff Q' is a contractive projection.*

This lemma is a direct consequence of the definition of i.

For $K \in S$ such that $0 < \lambda(K) < \infty$ we denote $L_1(K, S \cap K, \lambda / S \cap K)$ by

$L_1(K)$ and $L_1(\Omega\times K,\ \mathcal{A}\times(S\cap K),\ \mu\times(\lambda/S\cap K))$ by $L_1(\Omega\times K)$. We may regard $L_1(\Omega\times K)$ as a subspace of $L_1(\Omega\times X)$ by a canonical way.

Lemma 2.3. *If Q is a constant-preserving contractive projection, then $Q'(L_1(\Omega\times K))\subset L_1(\Omega\times K)$.*

Proof. If $f\in L_1(\Omega,\ L_1(X))$ and f is an $L_1(K)$-valued function, then by Lemma 1.5, $Q(f)$ is an $L_1(K)$-valued function. By Lemma 2.1 there exists a norm isomorphism of $L_1(\Omega,\ L_1(K))$ onto $L_1(\Omega\times K)$, therefore $Q'(L_1(\Omega\times K))\subset L_1(\Omega\times K)$.

Lemma 2.4. *Let Q be a bounded transformation of $L_1(\Omega,\ L_1(X))$ into itself. Then Q is the conditional expectation operator given $\mathcal{B}$ iff $Q'/L_1(\Omega\times K)$ is the conditional expectation operator of $L_1(\Omega\times K)$ into itself given $\mathcal{B}\times(S\cap K)$.*

Proof. Suppose that Q is a conditional expectation operator given $\mathcal{B}$. Then for every $M\in\mathcal{B}$ and $N\in S\cap K$, we have $Q(1_M\cdot 1_N)=(1_M)^{\mathcal{B}}\cdot 1_N$. It follows that $Q'(L_1(\Omega\times K))\subset L_1(\Omega\times K)$. For any $M\in\mathcal{B}$, $N\in S\cap K$ and $f\in L_1(\Omega\times K)$, we have

$$\int_{M\times N} Q'(f)\,d\mu\times d\lambda=\int_N\{\int_M Q(f)\,d\mu\}\,d\lambda=\int_N\}\int_M f\,d\mu\}\,d\lambda$$
$$=\int_{M\times N} f\,d\mu\times d\lambda\,.$$

Thus $Q'/L_1(\Omega\times K)$ is the conditional expectation operator given $\mathcal{B}\times(S\cap K)$. Conversely, suppose that $Q'/L_1(\Omega\times K)$ is the conditional expectation operator given $\mathcal{B}\times(S\cap K)$ for each $K\in S$ with $0<\lambda(K)<\infty$. Let $\varphi\in L_1(\Omega,\ \mathcal{A},\ \mu,\ R)$ and $K\in S$ with $0<\lambda(K)<\infty$. Then, for any $M\in\mathcal{A}$ and $N\in S$, we have

$$\int_N\{\int_M Q(\varphi\cdot 1_K)\,d\mu\}\,d\lambda=\int_{M\times N} Q'(\varphi\cdot 1_K)\,d\mu\times d\lambda$$
$$=\int_{M\times N}\varphi^{\mathcal{B}}\cdot 1_K\,d\mu\times d\lambda=\int_N\{\int_M \varphi^{\mathcal{B}}\cdot 1_K\,d\mu\}\,d\lambda\,.$$

It follows that $Q(\varphi\cdot 1_K)=\varphi^{\mathcal{B}}\cdot 1_K$. By linearity and continuity $Q(\varphi\cdot a)=\varphi^{\mathcal{B}}\cdot a$ for all $a\in L_1(X)$. By the proof [12, Theorem 1.6.4], Q is the conditional expectation operator given $\mathcal{B}$.

Let Q be a constant-preserving contractive projection on $L_1(\Omega,\ L_1(X))$. Then by Lemmas 1.3 and 2.3, for any $K\in S$ with finite measure, there is a σ-subalgebra F_K of $\mathcal{A}\times(S\cap K)$ such that $Q'/L_1(\Omega\times K)$ is the conditional expectation operator given F_K. Moreover, by Lemma 2.4, Q is a conditional expectation operator on $L_1(\Omega,\ L_1(X))$ if and only if there is a σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that $F_K=\mathcal{B}\times(S\cap K)$ for all K.

3. The case of a measure space with ergodic transformations. Let

(X, S, λ) be a measure space, S a σ-algebra, $S(\lambda)=\{K; K\in S$ and $\lambda(K)<\infty\}$ and $S_l(\lambda)=\{K\subset X; K\cap E\in S$ for each $E\in S(\lambda)\}$. For each $K\in S_l(\lambda)$ let $\bar{\lambda}(K)=\sup\{\lambda(K\cap E); E\in S(\lambda)\}$.

DEFINITION 3. A measure space (X, S, λ) is localizable if each nonempty collection $\mathcal{V}\subset S(\lambda)$ has $\sup\mathcal{V}\in S$, in the sense that for each $K\in\mathcal{V}$, $\lambda(K-\sup\mathcal{V})=0$ and if $H_1\in S$ and $\lambda(K-H_1)=0$ for each $K\in\mathcal{V}$, then $\lambda(\sup\mathcal{V}-H_1)=0$.

DEFINITION 4. A measure space (X, S, λ) is locally localizable if each nonempty collection $\mathcal{V}\subset S(\lambda)$ has $\sup\mathcal{V}\in S_l(\lambda)$, in the sense that for each $K\in\mathcal{V}$, $\lambda(K-\sup\mathcal{V})=0$ and if $H_1\in S_l(\lambda)$ and $\lambda(K-H_1)=0$ for each $K\in\mathcal{V}$, then $\bar{\lambda}(\sup\mathcal{V}-H_1)=0$.

DEFINITION 5. A measure space (X, S, λ) has the finite subset property if for each $K\in S$, $\lambda(K)>0$, there is $K'\in S$ with $K'\subset K$ and $0<\lambda(K')<\infty$.

Lemma 3.1. *If (X, S, λ) is a locally localizable measure space with the finite subset property, then $(X, S_l(\lambda), \bar{\lambda})$ is a localizable space which satisfies the finite subset property and $\bar{\lambda}/S=\lambda$.*

For proof see Ghosh, Morimoto and Yamada [5].

DEFINITION 6. A class $\{f(x, K): K\in S(\lambda)\}$ of S-measurable functions on (X, S, λ) is called a cross-section

$$\text{if } f(x, K)=0 \text{ on } K^C \text{ and } 1_{K_1\cap K_2}(x)\cdot f(x, K_1) = 1_{K_1\cap K_2}(x)\cdot f(x, K_2) \quad \text{(a.e.x)} \quad \text{for each } K_1, K_2\in S(\lambda).$$

Lemma 3.2. *Suppose that a measure space (X, S, λ) is localizable. Then for each cross-section $\{f(x, K): K\in S(\lambda)\}$ there exists a S-measurable function f such that $f(x)\cdot 1_K(x)=f(x, K)$ (λ-a.e.x) for each $K\in S(\lambda)$.*

For proof see Zaanen [16]

In the rest of this section we assume that $(X. S. \lambda)$ is a localizable space with the finite subset property.

Lemma 3.3. *Let Q be a constant-preserving contractive projection of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself, where $E=L_1(X, S, \lambda, R)$. Then for each $\varphi\in L_1(\Omega, \mathcal{A}, \mu, R)$, $0\leqq\varphi\leqq 1$, and $A\in\mathcal{A}$ there exists a λ-a.e. unique S-measurable function b such that*

$$0\leqq b(x)\leqq 1 \ (\lambda\text{-}a.e.x) \quad and \quad b\cdot 1_K=1_K\cdot\int_A Q(\varphi\cdot 1_K)\,d\mu \quad (\lambda\text{-}a.e.x) \quad for\ each \quad K\in S(\lambda).$$

Proof. By Lemma 1.5, $\int_A Q(\varphi\cdot 1_K)\,d\mu=0$ on K^C for each $K\in S(\lambda)$. For each K_1, $K_2\in S(\lambda)$ $1_{K_1\cap K_2}\cdot\int_A Q(\varphi\cdot 1_{K_1})\,d\mu=1_{K_1\cap K_2}(\int_A Q(\varphi\cdot 1_{K_1\cap K_2})\,d\mu$, since $\int_A Q(\varphi\cdot 1_{K_1-K_2})\,d\mu=0$ on $K_1\cap K_2$. Similarly

$$1_{K_1\cap K_2}\cdot\int_A Q(\varphi\cdot 1_{K_2})\,d\mu=1_{K_1\cap K_2}\cdot\int_A Q(\varphi\cdot 1_{K_1\cap K_2})\,d\mu\,.$$

Therefore $\{\int_A Q(\varphi\cdot 1_K)\,d\mu\colon K\in S(\lambda)\}$ is a cross section, and hence by Lemma 3.2 there exists a S-measurable function b such that $b\cdot 1_K\cdot=1_K\cdot\int_A Q(\varphi\cdot 1_K)\,d\mu$ (λ-a.e.x) for each $K\in S(\lambda)$. What remains is to prove the uniqueness of b. Suppose that there exists a S-measurable function b' such that $b\cdot 1_K=b'\cdot 1_K$ (λ-a.e.x) and $\lambda(\{x\colon b(x)\neq b'(x)\})>0$. By the finite subset property of (X, S, λ) there exists $E\in S(\lambda)$, $E\subset\{x\colon b(x)\neq b'(x)\}$, which leads to a contradiction, since $b\cdot 1_E=b'\cdot 1_E$ (a.e.x). We have proved that $b(x)=b'(x)$ (λ-a.e.x). Similarly by Lemma 1.5 and the finite subset property of (x, S, λ) we have $0\leqq b(x)\leqq 1$ (λ-a.e.x).

DEFINITION 7. Let T be a one to one transformation of (X, S, λ) onto itself, then T is called a bounded measurable transformation if T is a measurable transformation and there exists a positive number k such that $\lambda(T^{-1}(A))\leqq k\cdot\lambda(A)$ for each $A\in S$.

DEFINITION 8. Let $\{T\colon T\in\mathcal{T}\}$ be a class of bounded measurable transformations of X onto X such that $T^{-1}(S(\lambda))=S(\lambda)$ for each $T\in\mathcal{T}$. $(X, S, \lambda, T\colon T\in\mathcal{T})$ is called ergodic if $\lambda(A\Delta T^{-1}(A))=0$ for each $T\in\mathcal{T}$ implies $\lambda(A)=0$ or $\lambda(A^c)=0$.

Lemma 3.4. *If $(X, S, \lambda, T\colon T\in\mathcal{T})$ is an ergodic space, then for each bounded measurable function f on X $f(x)=f(T(x))$ a.e.x for each $T\in\mathcal{T}$ implies that $f(x)=$ const. λ-a.e.x.*

Proof. Let f be a bounded measurable function on X and $f(x)=f(T(x))$, λ-a.e.x for each $T\in\mathcal{T}$. For each real number d let $E_d=f^{-1}((d, \infty))$. Then $\lambda(E_d\Delta T^{-1}(E_d))\leqq\lambda(f(x)\neq f(T(x)))=0$. By the definition of erogdicity $\lambda(E_d)=0$ or $\lambda(E_d^c)=0$, f is bounded, and hence there exists a real number M such that $|f(x)|\leqq M$, a.e.x. If $d>M$ then, $\lambda(E_d)=0$. If $d<-M$, then $\lambda(E_d^c)=0$. Let $c=\inf\{d\colon \lambda(E_d)=0\}$. Then $f=c$, λ-a.e.x.

Let $(X, S, \lambda, T\colon T\in\mathcal{T})$ be an ergodic measure space and $E=L_1(X, S, \lambda, T\colon T\in\mathcal{T})$. For each real valued measurable function a on X and $T\in\mathcal{T}$ we write $T\cdot a(x)=a(T(x))$. Then T can be seen as a bounded linear operator of $L_1(X, S, \lambda, \mathcal{R})$ into itself.

DEFINITION 9. Let Q be a transformation of $L_1(\Omega, \mathcal{A}, \mu, E)$ into itself, then Q is called covariant under $\mathcal{I}$ if $Q(\varphi\cdot(T\cdot a))=T\cdot Q(\varphi\cdot a)$ for each $\varphi\in L_1(\Omega, \mathcal{A}, \mu, R)$ and $a\in E$ and $T\in\mathcal{I}$.

Theorem 1. *Let Q be a constant-preserving contractive projection which is invariant under $\mathcal{I}$. Then $Q=(\cdot)^{\mathcal{B}}$ for some σ-subalgebra $\mathcal{B}$ of $\mathcal{A}$.*

Proof. Let $\varphi\in L_1(\Omega, \mathcal{A}, \mu, R)$, $0\leqq\varphi\leqq 1$ and $A\in\mathcal{A}$ and $T\in\mathcal{I}$. By Proposition 1.1 it is sufficient to prove that there exists $\varphi'\in L_1(\Omega, \mathcal{A}, \mu, R)$ such that $Q(\varphi\cdot 1_K)=\varphi'\cdot 1_K$ for each $K\in S(\lambda)$. By Lemma 3.3 there exists a S-measurable function b such that $0\leqq b(x)\leqq 1$ (a.e.x) and $b\cdot 1_K=\int_A Q(\varphi\cdot 1_K)d\mu$ (λ-a.e.x) for each $K\in S(\lambda)$.

$$\begin{aligned}&(T\cdot b)\quad 1_{T^{-1}(K)}=T(b\cdot 1_K)=T\int_A Q(\varphi\cdot 1_K)d\mu\\ &=\int_A T\cdot Q(\varphi\cdot 1_K)d\mu=\int_A Q(\varphi\cdot(T\cdot 1_K))d\mu\\ &=\int_A Q(\varphi\cdot 1_{T^{-1}(K)})d\mu.\quad \text{Since}\quad T^{-1}(S(\lambda))=S(\lambda)\quad \text{we have}\\ &(T\cdot b)\cdot 1_K=\int_A Q(\varphi\cdot 1_K)d\mu=b\cdot 1_K\quad \text{for each}\quad K\in S(\lambda)\,.\end{aligned}$$

By the uniqueness of b $T\cdot b=b$(λ-a.e.x). By Lemma 3.4 there exists a positive number $k(A)$ such that $b(x)=1_X\cdot k(A)$ (λ-a.e.x). Hence $b\cdot 1_K=1_K\cdot k(A)$.

Let $\{A_n, n=1, 2, \cdots\}$ be a sequence of elements of $\mathcal{A}$ and $A_n\cap A_m=\phi(n\neq m)$.

$$\begin{aligned}1_K\cdot k(\bigcup_{n=1}^{\infty} A_n)&=\int_{\bigcup_{n=1}^{\infty} A_n} Q(\varphi\cdot 1_K)d\mu=\sum_{n=1}^{\infty}\int_{A_n} Q(\varphi\cdot 1_K)d\mu\\ &=\sum_{n=1}^{\infty} 1_K\cdot K(A_n)=1_K\cdot(\sum_{n=1}^{\infty} k(A_n))\,.\end{aligned}$$

Therefore $k(\bigcup_{n=1}^{\infty} A_n)=\sum_{n=1}^{\infty} k(A_n)$, this shows that $k(\cdot)$ is a measure on $\mathcal{A}$. k is absolutely continuous with respect to μ, since $1_K\cdot k(A)=\int_A Q(\varphi\cdot 1_K)d\mu$. By the Radon-Nykodym theorem there is $\varphi'\in L_1(\Omega, \mathcal{A}, \mu, R)$ such that

$$\int_A \varphi'\cdot 1_K\,d\mu=1_K\cdot\int_A \varphi'\,d\mu=1_K\cdot k(A)=\int_A Q(\varphi\cdot 1_K)d\mu\,.$$

Therefore $Q(\varphi\cdot 1_K)=\varphi'\cdot 1_K$.

REMARK. If (X, S, λ) is σ-finite measure space, then Theorem 1 can be proved without the condition that $T^{-1}(S(\lambda))=S(\lambda)$.

Let G be a locally compact group and λ a left Haar measure on the σ-algebra S generated by open sets (cf. Berberian [3, Exercise 79.6, p. 263]). Then (G, S, λ) is a locally localizable measure space with the finite subset property (cf. Segal [13]). Let $\mathcal{T}$ be the set of all translations on G. Then it is easy to see that (G.S. .$\mathcal{T}$) is an ergodic measure space. Thus we obtain the following.

Corollary 1. *A constant-preserving contractive projection on $L_1(\Omega, L_1(G))$ which is covariant under all translations is a conditional expectation operator given some σ-subalgebra of $\mathcal{A}$.*

4. Properties of translation-invariant σ-algebras on R/Z and a characterization of conditional expectation for $L_1(R/Z)$-valued function. Let $X=R/Z$, where Z is the class of integers. Let λ be the Haar measure and S the λ-completion of the class of Borel sets on X. Let $\mathcal{N}$ be the σ-ring of λ-null sets and α an irrational number.

We define a mapping T_α of X onto X by $T_\alpha(x)=x+\alpha$ (mod 1). A σ-subalgebra S' of S is said to be T_α-invariant if $T_\alpha(K)\in S'$ for each $K\in S'$. For $n=1, 2, \cdots$. Let $S_n=\{K\in S, K=K+1/n \ (\lambda\text{-a.e.x})\}$.

Lemma 4.1. *Let U and V be σ-subalgebras of S containing $\mathcal{N}$. Then $U=V$ iff*

$$(e^{2\pi jkx})^U=(e^{2\pi ikx})^V \ \lambda\text{-}a.e.x \quad for\ any \quad k\in Z\,.$$

Proof. For each complex integrable function f and a positive bumber $\varepsilon>0$ there exist complex numbers $c_1, c_2, \cdots, c_n$ such that $\|f-\sum_{j=1}^{n} c_j\, e^{2\pi ijx}\|_{L_1(X)}$. Since conditional expectation operator is linear continuous, we have this lemma.

Lemma 4.2. *Let S' be a σ-subalgebra of S containing $\mathcal{N}$. Then*

$$S'=S_n \ iff\ (e^{2\pi ikx})^S=\begin{cases}0 \ (k\not\equiv 0 \ (\mathrm{mod}\ n)) \\ e^{2\pi ikx} \ (k\equiv 0 \ (\mathrm{mod}\ n))\end{cases} \quad a.e.x \quad for\ any \quad k\in Z\,.$$

Proof. If $k\not\equiv 0$ (mod n), then $\int_K e^{2\pi ikx}\,dx=0$ for each $K\in S'$. This lemma is a direct consequence of this fact and Lemma 4.1.

Lemma 4.3. *Let S' be a T_α-invariant σ-subalgebra of S containing $\mathcal{N}$. Then*

$$(e^{2\pi ikx})\,(T_\alpha(x))^{S'}=e^{2\pi ik\alpha}(e^{2\pi ikx})^{S'}(x) \ a.e.x \quad for\ any \quad k\in Z\,.$$

Proof. Let $f(x)=(e^{2\pi ikx})^{S'}(x)$. Since λ and S' are T_α-invariant, for any $K\in S'$

$$\int_K f(T_\alpha(x))\,d\lambda(x)=\int_{T\alpha(K)} f(x)\,d\lambda(x)=\int_{T\alpha(K)} e^{2\pi ikx}\,d\lambda(x)$$

$$= \int_K e^{2\pi i k T_\alpha(x)}\, d\lambda(x) = e^{2\pi i k\alpha}\int_K e^{2\pi i k x}\, d\lambda(x)$$
$$= e^{2\pi i k\alpha}\int_K f(x)\, d\lambda(x).$$

Therefore $f(T_\alpha(x))=e^{2\pi i k\alpha}f(x)$.

Lemma 4.4. *Let $f\in L_2(X.\ S.\ \lambda.\ R)$ such that $f(T_\alpha(x))=e^{2\pi j k\alpha}f(x)$ a.e.x. Then $f(x)=Ce^{2\pi i k x}$ a.e.x, where C is a constant.*

Proof. $\{e^{2\pi i j x}, j=1, 2, \cdots\}$ is a complete orthogonal system in $L_2(X, S, \lambda, R)$. Let $f(x)=\sum_{j=1}^{\infty} c_j e^{2\pi i j x}$. Since $f(T_\alpha(x))=e^{2\pi i k\alpha}f(x)$ a.e.x, it holds that $c_j e^{2\pi i j\alpha}= c_j e^{2\pi i k\alpha}$ for any positive integer j. Therefore $c_j=0$ except for $j=k$.

Theorem 2. *Let S' be a σ-subalgebra of S containing $\mathcal{N}$. Then S' is T_α-invariant iff $S'=\mathcal{N}$ or $S'=S_n$ for some positive integer n.*

Proof. Suppose that S' is T_α-invariant. By Lemma 5.3 and Lemma 4.4 there exists a complex number C_k such that $(e^{2\pi i k x})^{S'}=c_k e^{2\pi i k x}$ a.e.x for each positive integer k. If $S'\neq\mathcal{N}$, then there exists a positive integer k such that $(e^{2\pi i k x})^{S'}\neq 0$ (a.e.x). Let $n=\text{Min}\{k\colon k$ is a positive integer and $(e^{2\pi i k x})^{S'}\not\equiv 0$ (a.e.x)$\}$. Then $e^{2\pi i n x}$ is S'-measurable and $c_n=1$. Since S' is T_α-invariant and $e^{2\pi i n x}$ is S_n-measurable, $S_n\subset S'$. Therefore for each k such that $k=0$ (mod n) $C_K=0$. For any positive integer k there exist positive integers h and j such that $k=h\cdot n+j$ $(0\leqq j<n)$. Since $e^{2\pi i h n x}$ is S_n-measurable, it is S'-measurable. Hence $(e^{2\pi i k x})^{S'}=e^{2\pi i h n x}(e^{2\pi i j X})^{S'}=0$ a.e.x. By Lemma 4.2 $S'=S_n$. Conversely if $S'=\mathcal{N}$ or $S'=S_n$ for some positive integer n, then S' is T_α-invariant.

DEFINITION 11. Let $\psi(x)=x-[x]$. Then ψ is a mapping of R onto R/Z. A subset K of R/Z is said to be an interval if $K=\psi([a, b])$ for some real numbers $a, b\in R$.

DEFINITION 12. For $K\in S$ define

$$k(K) = \text{Max}\{\lambda(H)\colon\ H \text{ is an interval and } H\subset K\}\ .$$

DEFINITION 13. For each $a\in L_1(X, S, \lambda, R)$ and $x_0\in X$, let $(T_{x_0}\cdot a)(x)= a(x_0\cdot x)$. Let P be a transformation of $L_1(\Omega\times X, \mathcal{A}\times S, \mu\times\lambda, R)$ into itself. P is said to be translation invariant if $T_x\cdot P(\varphi\cdot a)=P(\varphi\cdot T_x\cdot a)$ for each

$$\varphi\in L_1(\Omega, \mathcal{A}, \mu, R),\ a\in L_1(X, S, \lambda, R) \text{ and } x\in X\,.$$

Theorem 3. *Let P be a translation invariant constant-preserving contractive projection of $L_1(\Omega\times X, \mathcal{A}\times S, \mu\times\lambda, R)$ into itself. If there exists $K\in S$*

such that $k(K)>1/2$, $\lambda(K)<1$ *and* $P(1_{\Omega\times K})=1_{\Omega\times K}$, *then there exists a* σ*-subalgebra* $\mathcal{B}$ *of* $\mathcal{A}$ *such that* $P(f)=f^{\mathcal{B}\times S}$ *for each*

$$f\in L_1(\Omega\times X, \mathcal{A}\times S, \mu\times\lambda, R).$$

Proof. By Lemma 1.3 there exists a σ-subalgebra C of $\mathcal{A}\times S$ such that $P(f)=f^C$ for each $f\in L_1(\Omega\times X, \mathcal{A}\times S, \mu\times\lambda, R)$. Let i be the isomorphism of $L_1(\Omega, \mathcal{A}, \mu, L_1(X, S, \lambda, R))$ onto $L_1(\Omega\times X, \mathcal{A}\times S, \mu\times\lambda, R)$ and $Q=i^{-1}\circ P\circ i$, then Q is a translation invariant contractive projection of $L_1(\Omega, \mathcal{A}, \mu, L_1(X, S, \lambda, R))$ into itself. Write $S'=\{K: \Omega\times K\in C\}$. Since P is translation invariant, S is a T_α-invariant σ-subalgebra of S. Therefore by Theorem 2 $S'=\mathcal{N}$ or S_n for some positive integer n. Since $1>k(K)>1/2$, $S'=S$. This implies that for each $K\in S$ $P(1_{\Omega\times K})=1_{\Omega\times K}$. Therefore $Q(1_\Omega\cdot 1_K)=1_\Omega\cdot 1_K$. By the arbitrariness of K we have $Q(1_\Omega\cdot a)=1_\Omega\cdot a$ for each $a\in L_1(X, S, \lambda, R)$, and hence Q is a constant-preserving contractive projection. Therefore by Corollary 1 there exists $\mathcal{B}$ such that $Q(f)=f^{\mathcal{B}}$ for each $f\in L_1(\Omega, \mathcal{A}, \mu, L_1(X, S, \lambda, R))$. By Lemma 2.4 $P(f)=f^{\mathcal{B}\times S}$ for each $f\in L_1(\Omega\times X, \mathcal{A}\times S, \mu\times\lambda, R)$.

REMARK. In Theorem 3 for the transformation P of $L_1(\Omega\times X, \mathcal{A}\times S, \mu\times\lambda, R)$ into itself contsant-preserving means $P(1_{\Omega\times X})=1_{\Omega\times X}$.

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