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TANGENTIAL REPRESENTATIONS OF CYCLIC GROUP ACTIONS ON HOMOTOPY COMPLEX PROJECTIVE SPACES

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

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0. Introduction

Let G be a cyclic group of an odd prime order m and let t be a generator of the complex representation ring $R(G)$ of G; i.e. $R(G)=Z[t]/(1-t^m)$. Let X be a closed G manifold homotopy equivalent to $P(C^n)$ the space consisting of complex lines in C^n. Suppose G acts smoothly on X with isolated fixed points $\{p_i\}_{i=1}^n$ (Bredon's theorem asserts the number of fixed points equals n [2]). Then the tangential representation $T_{p_i}X$ of G at p_i defines a function $\psi_i(t)$ on $G-1$ (up to multiplication by t^k) for each i; see p. 137 in [12]. In particular, if X is G homotopy equivalent to $P(A)$ for some complex representation A of G (we call such X a G homotopy $P(A)$), then it has an expression

$$\psi_i(t)=\lambda_{-1}(T_{p_i}P(A))/\lambda_{-1}(T_{p_i}X)$$

where $\lambda_{-1}(V)$ is the Euler class of a G representation V. Therefore one can regard $\psi_i(t)$ as quantities which describe to what extent the action resembles a linear action on $P(C^n)$.

Petrie's conjecture in [12] suggests that $\psi_i(t)$ is independent of i for an S^1 manifold homotopy equivalent to $P(C^n)$ and the Pontrjagin classes are preserved under the homotopy equivalence. Particularly, in [13] Petrie showed that if X is S^1 homotopy equivalent to $P(A)$ for some complex representation A of S^1, then $\psi_i(t)=\pm 1$ for each i. In contrast to S^1 actions we construct infinitely many families of G homotopy $P(C^{2d})$ such that $\psi_i(t)\neq\pm\psi_j(t)$ for $i\neq j$. Here is a brief statement of our main theorem (Theorem 4.1).

Main Theorem. *Let m be an odd prime number and $2d\,|\,m-1$ for some integer $d\geq 2$. Then there are infinitely many homotopy complex projective spaces $\boldsymbol{P}^{2d-1}$ of dimension $4d-2$ such that Z_m acts on $\boldsymbol{P}^{2d-1}$ with $2d$ isolated fixed points.*

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As a corollary of Theorem 4.1, there exist counterexamples to the "mod m" version of the Petrie conjecture for all primes $m \geq 5$. These are the first examples of exotic homotopy complex projective spaces which support nontrivial Z_m actions.

We use the first Pontrjagin classes to distinguish G homotopy $P(C^n)$'s and study the relations between tangential representations at fixed points and the first Pontrjagin classes. Specializing 4.1 to $2d=4$ and applying 6.1, we are able to show that every homotopy $P(C^4)$ supports Z_m action for infinitely many prime numbers m (Theorem 6.2). This concerns a conjecture of Löffler and Raussen (strong form) which asserts that a closed simply connected manifold admits non-trivial Z_m actions for almost all primes m ([21]).

As to the method, we apply the equivariant surgery theory which is developed by T. Petrie, K.H. Dovermann and others. It consists of three steps in our construction. First, we construct a nice G homotopy equivalence $\hat{\omega}$: $\hat{V} \to \hat{U}$ between G vector bundles over $P(A)$. Next convert this map $\hat{\omega}$ to a map h which is transverse to the zero section $P(A) \subset \hat{U}$ via a proper G homotopy. The resulting transverse map h produces a G normal map (see [14]). A more general theory of these techniques is discussed in [16]. Then we check that the surgery obstruction of this G normal map vanishes. After these three steps, we obtain a homotopy complex projective space X which supports a G action with certain properties.

This paper is organized as follows. The above three steps are discussed in sections 1–3 respectively. As a consequence of these sections we obtain our first main theorem which is stated in section 4. In sections 5 and 6 we treat a related topic concerned with the relations between the first Pontrjagin classes and tangential representations and study the actions on homotopy $P(C^4)$ at the end of section 6.

Here are some conventions. Throughout this paper m stands for an odd prime number, G a cyclic group of order m and t the complex one dimensional representation of G on which a (preassigned) generator of G acts via the multiplication of $\exp(2\pi i/m)$.

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1. G fiber homotopy equivalences

Let A be a complex n-dimensional representation of G with a G invariant inner product. We denote the unit sphere of A by $S(A)$. It supports an action of $G \times S^1$ via $(\rho, \zeta) \cdot v = \rho \cdot v \cdot \zeta$ for $v \in S(A)$ and $(\rho, \zeta) \in G \times S^1$ where (1) G acts on the left by $\rho \cdot (z_1, \cdots, z_n) = (\rho^{a_1} z_1, \cdots, \rho^{a_n} z_n)$ for the generator $\rho = \exp(2\pi i/m) \in G$ and some integers $a_1, \cdots, a_n$ (integers a_i's are defined as A is a representation of G), (2) S^1 acts on the right by $(z_1, z_2, \cdots, z_n) \cdot \zeta = (z_1 \zeta,$

$z_2\zeta, \cdots, z_n\zeta)$ for $\zeta\in S^1$ and $(z_1, z_2, \cdots, z_n)\in S(A)$; i.e., this is the usual action inherited from complex multiplication. Sometimes, we denote A by $t^{a_1}+t^{a_2}+\cdots+t^{a_n}$.

The orbit space $S(A)/S^1$ is a standard complex projective space, denote it by $\boldsymbol{P}(A)$. On which there is a G action via $\rho\cdot(z_1: z_2: \cdots: z_n)=(\rho^{a_1}z_1: \rho^{a_2}z_2: \cdots: \rho^{a_n}z_n)$. Suppose $a_i\not\equiv a_j \pmod m$ for $i\neq j$, then the G action on $P(A)$ have only isolated fixed points. The fixed point set $P(A)^G$ consists of n points which are $p_i=(0: 0: \cdots: 1: \cdots: 0)$, for $1\leq i\leq n$, with all coordinates zero except the i-th coordinate. Then it is easy to see $T_{p_i}P(A)=\sum_{j\neq i} t^{a_i-a_j}$ as a representation of G. See [12].

Given a proper S^1 map ω: $V\to U$ of degree one between two complex representations of S^1. Then ω yields an $G\times S^1$ map

$$1\times\omega\colon S(A)\times V\to S(A)\times U$$

where S^1 acts on $S(A)\times V$ (or $S(A)\times U$) by $(x, v)\cdot\zeta=(x\cdot\zeta, \zeta^{-1}\cdot v)$ and G acts on $S(A)$ as above and trivially on V and U. Taking their S^1 orbit spaces, it induces a fiber preserving G map

$$\hat{\omega}\colon \hat{V}=S(A)\times_{S^1}V\to\hat{U}=S(A)\times_{S^1}U$$

between complex G vector bundles $\hat{V}$ and $\hat{U}$ over $P(A)$. Since the degree of ω is one, the fiber degree of $\hat{\omega}$ is also one; so $\hat{\omega}$ turns out to be a fiber homotopy equivalence. However, it is not necessary a G fiber homotopy equivalence. The sufficient and necessary conditions for $\hat{\omega}$ to be a G fiber homotopy equivalence is that the fiber degree of the restricted map $\hat{\omega}^G$: $\hat{V}^G\to\hat{U}^G$ to G fixed point sets is also one. The details about these techniques are discussed in [16].

Let t_s be the standard complex one dimensional S^1 module. Let p and q be relatively prime integers, γ an integer prime to m and set

$$V=t_s^{\gamma p}+t_s^{\gamma q},\ U=t_s^{\gamma}+t_s^{\gamma pq}$$

Choose integers a and b such that $-ap+bq=1$ and define a proper S^1 map ω: $V\to U$ by

$$\omega(z_1, z_2)=(\bar{z}_1^a z_2^b,\ z_1^q+z_2^p)$$

It is well known that the degree of ω is one [13, 14]. In this case, $\hat{\omega}$ is a G fiber homotopy equivalence if and only if p and q are both prime to m.

2. G transversality conditions

We proceed to making a G fiber homotopy equivalence $\hat{\omega}$: $\hat{V}\to\hat{U}$ over $P(A)$ transverse to the zero section $P(A)\subset\hat{U}$ via a proper G homotopy in this section. This is not always possible; i.e., there are obstructions to achieving transversality. The complete obstruction theory is developed in [14]. In our case, G acts semifreely on $P(A)$ and fiber dimension of $\hat{V}-\hat{U}$ is zero. The following lemma is a direct consequence of Corollary 4.13 in [14].

Lemma 2.1. *Suppose $P(A)^G$ consists of isolated points $\{p_i\}$. Then one can convert $\hat{\omega}$ to a map transverse to the zero section $P(A)\subset\hat{U}$ via a proper G homotopy if $(TP(A)+\hat{V}-\hat{U})|_{p_i}$ is a true real representation of G at each fixed point p_i.*

From now on we shall identify G with the additive group Z_m. Let G^* denote the multiplicative group of units of G, which is cyclic of order $m-1$. Suppose that $2d\,|\,m-1$, then there is $a\in G^*$ with order $2d$ in G^*.

Now we set

$$A = t+t^a+t^{a^2}+\cdots+t^{a^{2d-1}},$$
$$V = t_s^{\gamma p}+t_s^{\gamma q},\ U = t_s^{r}+t_s^{\gamma pq}.$$

Lemma 2.2. *Suppose that p, q, γ are relatively prime to m and $(p, q)=1$. If $\gamma\equiv\pm(a^r-1)$ (mod m) and $\gamma pq\equiv\pm(a^{r'}-1)$ (mod m) for $r\neq r'$ and $1\leq r$, $r'\leq 2d-1$, then $\hat{\omega}$ is properly G homotopic to a map which is transverse the to zero section $P(A)\subset\hat{U}$.*

Proof. It is easy to see that

$$TP(A)|_{p_i} = \sum_{j=1}^{2d-1} t^{a^i(a^j-1)}$$
$$\hat{V}|_{p_i} = t^{a^i\gamma p}+t^{a^i\gamma q}$$
$$\hat{U}|_{p_i} = t^{a^i\gamma}+t^{a^i\gamma pq} = t^{\pm a^i(a^r-1)}+t^{\pm a^i(a^{r'}-1)}$$

Since $t^{a^i(a^r-1)}+t^{a^i(a^{r'}-1)}$ is a factor of $\sum_{j=1}^{2d-1} t^{a^i(a^j-1)}$, $(TP(A)+\hat{V}-\hat{U})|_{p^i}$ is a true real representation of G for $i=0, 1, \cdots, 2d-1$. Hence the lemma follows from Lemma 2.1. Q.E.D.

REMARK 2.3 If $p\equiv\pm1$ or $q\equiv\pm1$ (mod m), then $\hat{V}|_{p_i}=\hat{U}|_{p_i}$ as real representations for all i. This is a special case, where we do not need to put any condition on p and q for achieving transversality. In this case, we have $\psi_i(t)=1$ for all i. If we want to avoid this, we have to put one more constraint on the choices of p, q; i.e., $p\not\equiv\pm1$ (mod m) and $q\not\equiv\pm1$ (mod m),

3. G surgery obstructions

Let h: $\hat{V}\to\hat{U}$ be a G map which is transverse to the zero section and G properly homotopic to $\hat{\omega}$. Since $\hat{\omega}$ is a G fiber homotopy equivalence, it produces a triple $\kappa=(X, f, \beta)$ where $X=h^{-1}(P(A))$, $f=h|_X$: $X\to P(A)$ is a map of degree one and β is a stable G bundle isomorphism between TX and $f^*(TP(A)+\hat{V}-\hat{U})$; i.e., κ is a G normal map.

Since m is a prime number, every G action is semifree. There are only two steps in doing equivariant surgery to convert a G normal map (X, f, β) to (X', f', β') with f' a G homotopy equivalence. The first step is on the fixed point set, second is on the free part. The reader should consult [4, 15, 16] for more general theory of G surgery.

Now we perform equivariant surgery on the G normal map κ without touching X^G to produce a new G normal map $\kappa'=(X', f', \beta')$ with f': $X'\to P(A)$ a G homotopy equivalence. In constructing the transverse G map h, we already have the bijection over the fixed point set since the map $\hat{\omega}^G$: $\hat{V}^G\to\hat{U}^G$ is bijective and h is obtained via a proper G homotopy. Therefore, the surgery obstruction we encounter is only one, written as $\sigma(f)$, which belongs to the Wall group $L^h_{4d-2}(G, 1)$ (see [5] or [9]). We shall verify that $\sigma(f)$ vanishes if we carefully choose two relatively prime integers p and q, or simply choose γ even.

Theorem 3.1. *Under conditions of Lemma* 2.2, $\sigma(f)$ *vanishes if* γ *is even.*

Proof. *Step 1.* By Rothenberg exact sequence

$$\cdots\to H^{4d-1}(Z_2, Wh(G))\to L^s_{4d-2}(G, 1)\to L^h_{4d-2}(G, 1)\xrightarrow{\alpha_G} H^{4d-2}(Z_2, Wh(G))\to\cdots.$$

Here $Wh(G)$ is the Whitehead group of G and $L^s_*(G, 1)$ is the Wall group concerning the simple homotopy equivalence. It is known that the $Wh(G)$ has no torsion (see p. 374 of [10]); so $H^{4d-1}(Z_2, Wh(G))=0$. On the other hand the recent result of Dovermann-Rothenberg (see §2 and §10 of [5]) tells us that $\alpha_G(\sigma(f))=0$ because the real representations $T_{p_i}X$ and $T_{p_{i+d}}X$ (they are conjugate as complex representations and equal as real representations) appear in pairs. These mean that $\sigma(f)$ comes from an unique element of $L^s_{4d-2}(G, 1)$, which we shall denote by $\sigma(f)$ again.

Step 2. The multisignature map Sign: $L^s_{4d-2}(G, 1)\to R(G)$ sends $\sigma(f)$ into $R(G)$ (see [11] or [18]). The evaluated value (Sign $\sigma(f)$) (g) at $g\in G-1$ has a description

$$(\mathrm{Sign}\,\sigma(f))\,(g)=\mathrm{Sign}(g, X)-\mathrm{Sign}(g, P(A))$$

where Sign(g, M) denotes the G signature of M evaluated at g ([1]). We apply the Atiyah-Singer G signature theorem to compute Sign(g, X) and Sign$(g, P(A))$. Since the representations $T_{p_i}X$ and $T_{p_{i+d}}X$ are of an odd complex dimension and conjugate to each other, the contributions to Sign(g, X) are cancelled. Therefore, Sign(g, X) vanishes. Similarly, Sign$(g, P(A))$ vanishes.

Step 3. It is known that the kernel of Sign is isomorphic to Z_2 and is detected by the Kervaire invariant of the normal map κ forgetting the group action (see p. 168 in [18]). The proof of Theorem 3.1 is completed by assuming the following lemma. Q.E.D.

Lemma 3.2. *The Kervaire invariant of $\sigma(f)$ is equal to 0 if γ is even.*

Proof. We lift the fiber homotopy equivalence $\hat{\omega}$: $\hat{V}\to\hat{U}$ over $P(C^{2d})$ to $P(R^{4d})$ through the Hopf map: $P(C^{2d})\to P(R^{4d})$ and then restrict it to $P(R^{4d-1})$. Obviously the resulting fiber homotopy equivalence over $P(R^{4d-1})$ is given by

$$\tilde{\omega}\colon \tilde{V}=(S(R^{4d-1})\times V)/Z_2\to\tilde{U}=(S(R^{4d-1})\times U)/Z_2$$

where $\tilde{\omega}$ is induced from ω as usual. Here we note that the Kervaire invariant of κ coincides with that of a normal map obtained from $\tilde{\omega}$ (see pp. 196–197 of [18]). Since exponents of irreducible factors in U and V are all even, the actions of Z_2 on them are trivial and so $\tilde{\omega}$ splits into the direct product $1\times\omega$: $P(R^{4d-1})\times V\to P(R^{4d-1})\times U$. Namely, $\tilde{\omega}$ is a pullback of the fiber homotopy equivalence ω over a point. The Sullivan formula (see §13B in [18]) then implies that the Kervaire invariant of $\tilde{\omega}$ vanishes. Q.E.D.

REMARK 3.3. A more general version of Lemma 3.2 was proved by the first author ([8]) and R. Schultz ([17]) independently. It is stated as follows: the Kervaire invariant $c(\hat{\omega})$ of

$$\hat{\omega}\colon S(A)\times_{S^1}(t_s^{\gamma p}+t_s^{\gamma q})\to S(A)\times_{S^1}(t_s^{\gamma}+t_s^{\gamma pq})$$

is given by

$$c(\hat{\omega})=\begin{cases}1 & \text{if } \gamma^2(p^2-1)(q^2-1)\equiv 24 \pmod{48}\\ 0 & \text{if } \gamma^2(p^2-1)(q^2-1)\equiv 0 \pmod{48}\,.\end{cases}$$

Hence, we can prove that the surgery obstruction $\sigma(f)$ vanishes if $\gamma^2(p^2-1)(q^2-1)\equiv 0 \pmod{48}$.

4. Main theorem

Now as a consequence of previous sections, we have established

Theorem 4.1. *Suppose G is a cyclic group of odd prime order m. If $2d|m-1$ for some integer $d\geq 2$, then there exists a closed G manifold $X(\gamma, p, q)$ together with a G homotopy equivalence f: $X(\gamma, p, q)\to P(A)$ such that*

(1) *$TX(\gamma, p, q)$ is stably isomorphic to $f^*(TP(A)+\hat{V}-\hat{U})$ as G vector bundles,*

(2) *$X(\gamma, p, q)^G$ consists of $2d$ points,*

(3) $\psi_i(t)=(1-t^{a^i\gamma})(1-t^{a^i\gamma pq})/(1-t^{a^i\gamma p})(1-t^{a^i\gamma q})$,

where A, $\hat{V}$, $\hat{U}$ are as in Lemma 2.2 (i.e. a, p, q and γ satisfy the assumptions of Lemma 2.2) and moreover γ is even.

Using Theorem 4.1 (1), the total Pontrjagin class $p(X)$ of $X=X(\gamma, p, q)$ the can be computed as follows:

$$p(X)=(1+x^2)^{2d}(1+\gamma^2 p^2 x^2)(1+\gamma^2 q^2 x^2)(1+\gamma^2 x^2)^{-1}(1+\gamma^2 p^2 q^2 x^2)^{-1}$$

where x is a generator of $H^2(X; Z)$. So, the first Pontrjagin class of X is equal to $\{2d+\gamma^2(p^2-1)(1-q^2)\}\, x^2$. It is easy to see that there exist infinitely many prime numbers m such that $\psi_{r_i}(t)\neq\psi_{r_j}(t)$, $i\neq j$ for each d. Moreover, there are infinitely many homotopy complex projective spaces which support Z_m actions for any prime $m\geq 5$. They are distinct from $P(C^{2d})$ because p, $q\neq 1$. (We distinguish them by the first Pontrajgin classes).

Corollary 4.2. *There exists $X(\gamma, p, q)$ with a nontrivail G action and $p_1(X(\gamma, p, q))\not\equiv 2dx^2$ (mod m) for every prime $m\geq 5$.*

Proof. *Case 1*: $m\geq 7$. Choose an even γ such that $\gamma\equiv a-1$ (mod m) for $a\in G^*$ with order $2d$. Then choose two relatively prime odd integers p and q such that $\gamma pq\equiv a^{r'}-1$ (mod m) for $2\leq r'\leq 2d-1$. Such γ, p and q always exist. The above two congruences mean that $pq\equiv 1+a^{+}\cdots+a^{r'-1}$ (mod m). Denote $1+a^{+}\cdots+a^{r'-1}$ by x. There exists an integer p such that $p\not\equiv 0$, ± 1, $\pm x$ (mod m), since $m\geq 7$. Suppose s be the order of p in the multiplicative group G^*, then $x\equiv xp^s\equiv p(xp^{s-1}+m^{\mu})$ (mod m). It is easy to see that p and $xp^{s-1}+m^{\mu}$ are relatively prime for $\mu\geq 1$ and $xp^{s-1}\not\equiv\pm 1$ (mod m). Therefore, $q=xp^{s-1}+m^{\mu}$ fit our requirement for $\mu\geq 1$.

Case 2: $m=5$. Choose $a=3$, $\gamma=2$ and an integer $p\equiv 2$ (mod 5), $q=p+5^{\mu}$, $\mu\geq 1$.

Hence, we can construct the required $X(\gamma, p, q)$ because $\gamma^2(p^2-1)(1-q^2)\not\equiv 0$ (mod m) for above choices of p, q. This completes the proof. Q.E.D.

These are the first examples of exotic homotopy complex projective spaces which support Z_m actions with isolated fixed points. The above corollary implies that the "mod m" version of the Petrie conjecture is false for $m\geq 5$.

5. Tangential representations and first Pontrjagin classes

It is well-known that tangential representations of a G manifold are closely related to its Pontrjagin classes. We shall restrict our concern to the first Pontrjagin class and discuss some relations between it and tangential representations.

We begin with the following lemma.

Lemma 5.1. *Let X be a homotopy $P(C^n)$. Then the first Pontrjagin class $p_1(X)$ of X is of the form*

$$p_1(X)=(n+24k(X))x^2 \qquad (k(X)\in Z)$$

where $x\in H^2(X; Z)$ is a generator.

Proof. It is well-known that X admits a $\mathrm{Spin}^c(2n-2)$ structure; i.e., there is a principal $\mathrm{Spin}^c(2n-2)$ bundle over X with total space P such that

$$TX = P \underset{\mathrm{Spin}^c(2n-2)}{\times} R^{2n-2}$$

(see [12]). The half $\mathrm{Spin}^c(2n-2)$ modules Δ_+ and Δ_- yield vector bundles E_+ and E_- over TX

$$E_\pm = P \underset{\mathrm{Spin}^c(2n-2)}{\times} (R^{2n-2} \times \Delta_\pm)$$

and there is a complex over TX; $E_+ \to E_-$ which defines an element δ_X of a K group $K(TX)$.

On the other hand let η_X be a complex line bundle over X whose first chern class is x and consider an element $(\eta_X - 1)^{n-3}$ of $K(X)$. Since $K(TX)$ has a natural $K(X)$ module structure, the above two element yield an element

$$(\eta_X - 1)^{n-3}\, \delta_X \in K(TX)\,.$$

We apply the Atiyah-Singer index homomorphism $Id_X\colon K(TX) \to Z$ to this element. Then the Atiyah-Singer index theorem implies that

$$Id_X((\eta_X - 1)^{n-3}\, \delta_X) = \langle ch(\eta_X - 1)^{n-3}\, e^{nx/2}\, \hat{A}(X),\ [X]\rangle \in Z$$

where ch denotes the Chern character, $\hat{A}(X)$ the $\hat{A}$ class of X and $[X]$ a fundamental class of X. Write $p_1(X) = (n+24k(X))x^2$ with a rational number $k(X)$. Then the above integrality condition reduces to $(n-1)(n-2)/2 - k(X) \in Z$ which shows the integrality of $k(X)$. Q.E.D.

REMARK 5.2. For dimensions, in which framed closed manifolds with the Kervaire invariant one exist, the function $k(X)$ takes any integer (at present $n=2, 4, 8, 16, 32$ are known to be such dimensions). For other even n one can see that $k(X)$ can take any even integer. Conversely an improved Sullivan formula for spin manifolds in terms of Wu class (see p. 255 in [18]) implies that $k(X)$ must be even in case $n \equiv 2 \pmod 4$. For odd n the value of $k(X)$ are more restrictive and complicated.

Now we need some convention about an orientation: for W a homotopy $P(C^n)$ we choose a generator z of $H^2(W; Z)$ and define a fundamental class $[W]$ of W by $\langle z^{n-1}, [W]\rangle = 1$. Therefore, in case n is odd, an orientation on W is uniquely determined, but in case n is even, it depends on the choice of a generator z.

Recall that if X supports an action of G with isolated fixed points $\{p_i\}$, then the tangential representations $T_{p_i}X$ admit a complex structure (which is not unique). We choose a complex structure on it such that the orientations induced from the complex structure agree with the given one and specify the resulting complex G representations by $T_{p_i}X^{\sim}$.

Theorem 5.3. *Let X and Y be oriented homotopy $P(C^n)$ with actions of G whose fixed point set consists of isolated points $\{p_i\}_{i=1}^n$ and $\{q_i\}_{i=1}^n$ respectively. Suppose that for each i*

(i) *$T_{p_i}X = T_{q_i}Y$ as real representations of G.*

(ii) *The orientation on $T_{q_i}Y$ ihduced from the complex structure of $T_{p_i}X^{\sim}$ via* (i) *agrees with the given one on Y.*

Then $k(X)\equiv k(Y)$ (mod m). *In case $n\equiv 2$* (mod 4), *the congruence is improved to $k(X)\equiv k(Y)$* (mod $2m$).

Proof. The argument used in the proof of Lemma 5.1 still holds in equivariant category; we shall use the same notations in equivariant category.

We apply the equivariant Atiyah-Singer index theorem to $Id_X((\eta_X-1)^{n-3}\,\delta_X)$ and $Id_Y((\eta_X-1)^{n-3}\,\delta_Y)$. It is easy to see that they are elements of $R(G)$ and their values evaluated at each element of $G-1$ are equal by hypotheses. This means that we can write

$$Id_X((\eta_X-1)^{n-3}\,\delta_X)-Id_Y((\eta_Y-1)^{n-3}\,\delta_Y)=f(t)\,(t^{m-1}+t^{m-2}+\cdots+t+1)$$

with some element $f(t)$ of $R(G)$. If we substitute t by the identity element 1 in this equation, it turns into

$$\langle ch(\eta_X-1)^{n-3}\,e^{nx/2}\,\hat{A}(X),\,[X]\rangle-\langle ch(\eta_Y-1)^{n-3}\,e^{ny/2}\,\hat{A}(Y),\,[Y]\rangle=mf(1)\,.$$

Since $f(1)$ is an integer, this equation reduces to $-k(X)+k(Y)\equiv 0$ (mod m), which verifies the former assertion.

When $n\equiv 2$ (mod 4), both $k(X)$ and $k(Y)$ are even by Remark 5.2. This implies the latter assertion. Q.E.D.

6. The realization of Theorem 5.3

We shall consider a realization problem of Theorem 5.3 and show that every homotopy $P(C^4)$ supports Z_m-actions for infinitely many prime numbers m. It is motivated by a conjecture of Löffler and Raussen (see §0).

First we show the realization part of Theorem 5.3.

Theorem 6.1. *Let X be an oriented homotopy $P(C^{2d})$ with an action of G whose fixed point set consists of isolated points $\{p_i\}$. Suppose we are given an integer k such that*

(i) *$k\equiv k(X)$* (mod m) *if $d=2$, 4, 8, 16,*

(ii) *$k\equiv k(X)$* (mod $2m$) *otherwise.*

Then there exists an oriented G manifold Y together with an orientation preserxing G map f: $Y\to X$ such that

(1) *f is a G homotopy equivalence,*

(2) *Y satisfies the conditions* (i) *and* (ii) *in Theorem* 5.3,

(3) $k(Y)=k$.

Proof. We apply the equivariant surgery theory again. The method is almost the same as that developed in §1–§3. We start with the following G fiber homotopy equivalence.

Let S be an S^1 bundle over X whose first Chern class is a generator of $H^2(X; Z)$. Since $H^1(X; Z)=0$ and $H^2(X; Z)^G=H^2(X; Z)$, the action of G on X lifts to that on S ([6]). We consider two S^1 maps

$$\omega_1\colon V_1 = t_s^2+t_s^{2m+1} \to U_1 = t_s+t_s^{4m+2}$$
$$\omega_2\colon V_2 = t_s^2+t_s^{-2m+1} \to U_2 = t_s+t_s^{-4m+2}$$

defined in §1. By using the construction of §1 to S (instead of $S(A)$), one gets G fiber homotopy equivalences $\hat{\omega}_i$ ($i=1, 2$). Generally a G fiber homotopy equivalence $\hat{\omega}\colon \hat{V}\to\hat{U}$ has a homotopy inverse which is a map from $\hat{U}$ to $\hat{V}$. We denote it by $-\hat{\omega}\colon -\hat{V}\to-\hat{U}$. Then our desired G fiber homotopy equivalence is given by

$$([k-k(X)]/m)\,(-\hat{\omega}_1\oplus\hat{\omega}_2)\colon ([k-k(X)]/m)\,(-\hat{V}_1\oplus\hat{V}_2)$$
$$\to ([k-k(X)]/m)\,(-\hat{U}_1\oplus\hat{U}_2)\,.$$

We shall abbreviate this as $\hat{\omega}\colon \hat{V}\to\hat{U}$. One can see that

$$(\hat{V}-\hat{U})\,|_{p_i} = 0 \text{ for each } i\,.$$

In particular, the conditions of Lemma 2.1 are satisfied. Therefore, we can convert $\hat{\omega}$ transverse to the zero section $X\subset\hat{U}$. This produces a G normal map $\kappa=(W, g, \beta)$.

From §3, g^G is a bijection. Therefore, the next and final surgery obstruction $\sigma(g)$, which belongs to $L^s_{4d-2}(G, 1)$, emerges.

We shall investigate it along the same line as in §3. Since the tangential representations of W and X are equal at every corresponding fixed point, steps 1 and 2 in §3 work. It suffices to show that the Kervaire invariant of g vanishes. We need to distinguish cases (i) and (ii) in Theorem 6.1.

In the case (ii), $k-k(X)$ is even by the assumption. This means that the fiber homotopy equivalence $\hat{\omega}\colon \hat{V}\to\hat{U}$ is two times a fiber homotopy equivalence. Therefore, the vanishing of the Kervaire invariant of g follows from the Sullivan formula (see §13B in [18]).

In the case (i), if the Kervaire invariant of g vanishes, then we have nothing to do. Suppose it does not vanish. There exists a closed framed manifold M with the Kervaire invariant one. We do connected sum of W with m copies of M equivariantly. Since the Kervaire invariant is additive with respect to connected sum operation and m is odd, the Kervaire invariant of the resulting normal map vanishes. The stable isomorphism β is preserved after doing connected sums because M is framed.

Thus we can apply equivariant surgery to achieve a G normal map (Y, f, β) such that

(1) $f: Y \to X$ is a G homotopy equivalence,

(2) $\beta: TY \to f^*(TX + \hat{V} - \hat{U})$ is a stable G vector bundle isomorphism.

This is a desired one. In fact, the conditions (i) and (ii) in Theorem 5.3 are obtained by restricting the above (2) to p_i and an elementary calculation shows $k(Y)=k$. This completes the proof of Theorem 6.1. Q.E.D.

Now we pose

DEFINITION. For a smooth closed manifold M we define $n(M)=\sup\{m \text{ prime} \mid M \text{ admits a nontrivial smooth actions of } Z_m\}$.

Clearly if $N(M)$ (the degree of symmetry of M) is positive (i.e. M admits a circle action), then $n(M)=\infty$. Therefore, it is reasonable to ask

QUESTION. Is there a closed (simply connected) manifold M such that $N(M)=0$ but $n(M)=\infty$?

Our theorem is the following:

Theorem 6.2.[(†)] *Each X_k (homotopy equivalent to $P(C^4)$) with $p_1(X_k)=(4+24k)x^2$ admits a smooth action of Z_m (with isolated fixed points) if m is a prime number such that $m \equiv 1$* (mod $4(6k+1)(6k-1)$).

Given an integer k, it is easy to see that there are infinitely many primes m satisfying the above congruence. On the other hand, $N(X_k)>0$ if and only if $k=0$ [3,20]. Hence X_k $(k \neq 0)$ are manifolds which answer the above question. Theorem 6.2 is also related to the conjecture of Löffler and Raussen.

Proof of Theorem 6.2. In this proof a congruence is modulo m unless otherwise stated. Let $\alpha \in (Z_m)^*$ with order 4. We take $\gamma \equiv \alpha - 1$ and $\gamma pq \equiv \alpha^3 - 1$ with γ even. Then there is a Z_m homotopy $P(C^4)$ with $p_1(X)=(4-\gamma^2(p^2-1)(q^2-1))x^2$ by Theorem 4.1. An easy computation shows

$$(*) = -\gamma^2(p^2-1)(q^2-1) \equiv -2\alpha(p^2-p^{-2}).$$

Set $p \equiv A\alpha + B$, $p^{-1} \equiv -A\alpha + B$ $(A, B \in Z)$ where $A^2+B^2 \equiv 1$ by $p \cdot p^{-1} \equiv 1$ and $\alpha^2 \equiv -1$. Then we have $(*) \equiv 8AB$.

Now we impose $AB \equiv 3k$ $(\Rightarrow (*) \equiv 24k)$. Namely our requirements are:

(1) $A^2+B^2 \equiv 1$,

(2) $AB \equiv 3k$.

If these equations have solutions, then we obtain a Z_m homotopy $P(C^4)$ X with $p_1(X)=(4+24k+hm)x^2$, $h \in Z$. However, since $24k+hm$ must be divisible by 24, h is divisible by 24. Here we use Theorem 6.1. This together with the fact that homotopy $P(C^4)$ are classified by their first Pontrjagin classes ([19]) tells us that X_k with $p_1(X_k)=(4+24k)x^2$ admits a smooth action of Z_m with isolated fixed points. Therefore we only have to verify that the equations (1) and (2) have solutions if $m\equiv 1 \pmod{4(6k+1)(6k-1)}$.

The proof is as follows. By the quadratic reciprocity law of Legendre-Jacobi symbol $(\div)$ we have

$$\left(\frac{6k+1}{m}\right)\left(\frac{m}{6k+1}\right)=(-1)^{\frac{m-1}{2}\cdot\frac{6k+1-1}{2}}=1\,.$$

This implies

$$\left(\frac{6k+1}{m}\right)=\left(\frac{m}{6k+1}\right)=\left(\frac{1}{6k+1}\right)=1,\ \text{i.e.}\quad 6k+1\equiv K^2 \ \text{for some}\ K\,.$$

Similarly we get $\left(\frac{1-6k}{m}\right)=1$. We notice $(A+B)^2\equiv 1+6k\equiv K^2$, i.e., $A+B\equiv\pm K$. Consider an equation $x^2\pm Kx+3k\equiv 0$ (the solutions are precisely A and B). This is equivalent to $(2x\pm K)^2\equiv 1-6k$. Here since $\left(\frac{1-6k}{m}\right)=1$, this equation has solutions. This completes the proof. Q.E.D.

We conclude this paper with a remark.

REMARK 6.3. Our argument developed in this paper works for groups of prime power order m^r with little modification if we consider a semifree action of Z_{m^r}. It only requires to choose $a\in(Z_{m^r})^*$ with order $2d$ where $2d\,|\,m-1$ (see §2). As a consequence, we can improve Theorem 6.2 that if m is a prime number as in Theorem 6.2, then each X_k admits a nontrivial smooth action of Z_{m^r} for any positive integer r.

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(†) Note added in proof.
K.H. Devermann and the first named author could improve this theorem in some cases. For example we could prove that X_{-4} admits a $\boldsymbol{Z}_m$ action for every prime numbr m.

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