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PSEUDO-RANK FUNCTIONS ON CROSSED PRODUCTS OF FINITE GROUPS OVER REGULAR RINGS

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Let R be a regular ring with a pseudo-rank function. The collection of all pseudo-rank functions of R (See [2, Ch. 17]) is denoted by $P(R)$ which is a compact convex set, and the extreme boundary of $P(R)$ is denoted by $\partial_e P(R)$. Our main objective is to study a crossed product R^*G of a finite multiplicative group G over a regular ring R. A crossed product R^*G of G over R is an associative ring which is a free left R-module containing an element $\bar{x}\in R^*G$ for each $x\in G$ and the set generated by the symbols $\{\bar{x}\colon x\in G\}$ is a basis of R^*G as a left R-module. Hence every element $\alpha\in R^*G$ can be uniquely written as a sum $\alpha=\sum_{x\in G} r_x\bar{x}$ with $r_x\in R$. The addition in R^*G is the obvious one and the multiplication is given by the formulas

$$\bar{x}\bar{y}=t(x,y)\overline{xy} \quad r\bar{x}=\bar{x}r^{\tilde{x}}$$

for all $x, y\in G$ and $r\in R$. Here the twisting $t\colon G\times G\to U(R)$ is a map from $G\times G$ to the group of units of R and for fixed $x\in G$, the map $\tilde{x}\colon r\to r^{\tilde{x}}$ is an automorphism of R. We assume throughout this note that the order $|G|$ of G is invertible in R. The Lemma 1.1 of [9] implies that R^*G is also a regular ring. First we will study the question whether a pseudo-rank function P of R can be extended to one of R^*G. We shall show that P is extensible to R^*G if and only if P is G-invariant, i.e., $P(r)=P(r^{\tilde{x}})$ for all $r\in R$ and $x\in G$. More precisely for a G-invariant pseudo-rank function P, put $P^G(\alpha)=|G|^{-1}\sum_1^n P(r_i)$ for $\alpha\in R^*G$ if ${}_R(R^*G\alpha)\cong\bigoplus_1^n Rr_i$, where $r_i\in R$. Then P^G is a desired one of P.

R admits a pseudo-metric topology induced by each $P\in P(R)$. In [2, Ch. 19], K.R. Goodearl has studied the structure of the completion of R with respect to P-metric. Let $\bar{R}$ be the P-completion of R, let $\bar{P}$ be the extension of P on $\bar{R}$ and let $\phi\colon R\to\bar{R}$ be the natural ring map, Our theorems are following:

(1) There exists a crossed product $\bar{R}^*G$ and a ring map $\bar{\phi}\colon R^*G\to\bar{R}^*G$ such that the following diagram commute

$$\begin{array}{ccc} R & \xrightarrow{\phi} & \bar{R} \\ \downarrow & & \downarrow \\ R^*G & \xrightarrow{\bar{\phi}} & \bar{R}^*G \end{array}$$

and $\bar{P}$ is also G-invariant and we have $P^G=(\bar{P})^G\bar{\phi}$

(2) If P is in $\partial_e P(R)$, then $\bar{R}^*G$ is a P^G-completion of R^*G and $(\bar{P})^G$ is an extension of P^G. We have that $P^G=\sum_1^n \alpha_i N_i$, where $N_i \in \partial_e P(R^*G)$ and $0<\alpha_i<1$ and $\sum_1^n \alpha_i=1$.

Let θ: $P(R^*G)\rightarrow P(R)$ be the natural restriction-map and we use $N|_R$ to denote the image of $N\in P(R^*G)$ by θ. We shall show that for any $N\in \partial_e P(R^*G)$, there exists some positive real number $\alpha\leq 1$ and some $N'\in P(R^*G)$ such that $(N|_R)^G=\alpha N+(1-\alpha)N'$.

In the second section we study types of crossed products of finite groups G over directly finite, left self-injective, regular rings R. We shall show that R^*G is of Type II_f if and only if R is of Type II_f.

In the final section we study the fixed ring of a finite group of automorphisms of a regular ring. We shall show that for any $P\in\partial_e P(R)$, $P|_{R^G}$ is a finite convex combination of distinct extremal elements in $\partial_e P(R^G)$. Under the assumption that R is a finitely generated projective right R^G-module, we shall show that for any extremal element $Q\in\partial_e P(R^G)$, there exist some $P\in P(R)$ some $Q'\in P(R^G)$ and some real number $0<\alpha\leqq 1$ such that $P|_{R^G}=\alpha Q+(1-\alpha)Q'$.

1. Extensions of pseudo-rank functions

Let R be a regular ring and we use $FP(R)$ to denote the set of all finitely generated projective left R-modules. For modules A, B, $A\lesssim B$ implies that A is isomorphic to a submodule of B.

DEFINITION [2, p. 226]. A *pseudo-rank function* on R is a map N: $R\rightarrow[0, 1]$ such that

(1) $N(1)=1$.

(2) $N(rs)\leqq N(r)$ and $N(rs)\leqq N(s)$ for all r, $s\in R$.

(3) $N(e+f)=N(e)+N(f)$ for all orthogonal idempotents $e, f\in R$.

If, in addition

(4) $N(r)>0$ for all non-zero $r\in R$,

then N is called a *rank function*. We use $B(R)$ to denote the set of all pseudo-rank functions on R.

DEFINITION [2, p. 232]. A *dimension function* on $FP(R)$ is a map D: $FP(R)\rightarrow \boldsymbol{R}^+$ such that

(1) $D({}_RR)=1$
(2) If A, $B\in FP(R)$ and $A\lesssim B$, then $D(A)\leqq D(B)$.
(3) $D(A+B)=D(A)+D(B)$ for all A, $B\in FP(R)$.

Let $D(R)$ denote the set of all dimension functions on $FP(R)$.

Pseudo-rank functions on R and dimension functions on $FP(R)$ are equivalent functions as follows.

Lemma 1 [2, Prop. 16.8]. *There is a bijection Γ_R: $P(R)\rightarrow D(R)$ such that $\Gamma_R(P)(Rr)=P(r)$ for all $P\in P(R)$ and $r\in R$.*

We always view R as a subring R^*G via the embedding $r\rightarrow r1$. Then there exists a restriction-map θ: $P(R^*G)\rightarrow P(R)$. We consider the same connections between $D(R^*G)$ and $D(R)$. For all $D\in D(R^*G)$ and $A\in FP(R)$, define $(D|_R)(A)=D(R^*G\otimes_R A)$. We can easily see that $D|_R$ is a dimension function on $FP(R)$ and $\Gamma_{R^*G}(N)|_R=\Gamma_R(N|_R)$.

Lemma 2. *Let N be in $P(R^*G)$ and D be in $D(R^*G)$. Then we have that $(N|_R)(r)=(N|_R)(r^{\bar{x}})$ and that $(D|_R)(Rr)=(D|_R)(Rr^{\bar{x}})$ for all $r\in R$ and all $x\in G$.*

Proof. Since $R^*G\otimes_R Rr\simeq R^*Gr\simeq R^*Gx^{-1}rx=R^*Grx\cong R^*G\otimes_R Rr^{\bar{x}}$, we have $(D|_R)(Rr)=(D|_R)(Rr^{\bar{x}})$ and $(N|_R)(r)=(N|_R)(r^{\bar{x}})$.

Now we shall define an extended dimension function on R^*G for a G-invariant $D\in D(R)$. Note that for $A\in FP(R^*G)$, ${}_RA\in FP(R)$.

Proposition 3. *Let D be a G-invariant dimension function on $FP(R)$. Put $D^G(A)=|G|^{-1}D({}_RA)$ for all $A\in FP(R^*G)$. Then D^G is a dimension function on $FP(R^*G)$ and $D^G|_R=D$.*

Proof. Since ${}_R(R^*G)$ isomorphic to $|G|$ copies of R, $D^G(R^*G)=1$. We can easily check that D^G satisfies the properties (2) and (3). Since ${}_R(R^*Gr)\cong\bigoplus_{x\in G}Rr^{\bar{x}}$ and D is G-invariant, then we have $D^G(R^*Gr)=|G|^{-1}\sum_{x\in G}D(Rr^{\bar{x}})=D(Rr)$ for all $r\in R$. Every $A\in FP(R)$ is isomorphic to a finite direct sum of cyclic left ideals of R. Therefore we have $(D^G|_R)(A)=D(A)$ for all $A\in FP(R)$.

Corollary 4. *Let P be a G-invariant pseudo-rank function on R. Define $P^G(\alpha)=(\Gamma_R(P))^G(R^*G\alpha)$ for all $\alpha\in R^*G$, then*
(1) *P^G is a pseudo-rank function on R^*G and $P^G|_R=P$*
(2) *We have $P^G(\alpha)=|G|^{-1}\sum_1^n P(r_i)$, if ${}_R(R^*G\alpha)\cong\bigoplus_1^n Rr_i$, where $r_i\in R$.*

Proof. (1) is clear by lemma 1 and Proposition 3. Recall that $\Gamma_R(P)$ is G-invariant dimension function on $FP(R)$ by Lemma 1. Since $P^G(\alpha)=|G|^{-1}\Gamma_R(P)({}_R(R^*G\alpha))=|G|^{-1}\sum_1^n\Gamma_R(P)(Rr_i)=|G|^{-1}\sum_1^n P(r_i)$, we have completed the proof.

Lemma 5. *Let N be a pseudo-rank function on $R*G$. Then we have that $N(\alpha)\leqq |G|(N|_R)^G(\alpha)$ for all $\alpha\in R*G$.*

Proof. Put $N|_R=P$. Since $\Gamma_{R*G}(N)|_R=\Gamma_R(P)$, then we have $\Gamma_R(P)(_R(R*G\alpha))=(\Gamma_{R*G}(N)|_R)((_R(R*G\alpha))=\Gamma_{R*G}(N)(R*G\otimes_R R*G\alpha)$. On the other hand, there exists a natural epimorphism $(R*G\otimes_R R*G\alpha)\to R*G\alpha$. Since this map splits, we have $N(\alpha)=\Gamma_{R*G}(N)(R*G\alpha)\leqq\Gamma_{R*G}(N)(R*G\otimes_R R*G\alpha)$. We have obtained that $N(\alpha)\leqq |G|P^G(\alpha)$ by Corollary 4.

DEFINITION [2, Ch. 19]. Let P be in $P(R)$. R admits a pseudo-metric δ by the rule: $\delta(r, s)=P(r-s)$. Note that δ is a metric if and only if P is a rank function. We call δ the *P-metric*. Let $\bar{R}$ be the completion of R with respect to δ and we call it the *P-completion* of R. $\bar{R}$ is a unit-regular, left and right self-injective ring by [2, Th. 19.7]. There exists a natural ring map ϕ: $R\to\bar{R}$ and a continuous map $\bar{P}$: $\bar{R}\to[0, 1]$ such that $\bar{P}\phi=P$. By [23, Th. 19.6], $\bar{P}$ is a rank function on $\bar{R}$. Put $\ker P=\{r\in R\colon P(r)=0\}$, which is a two-sided ideal. P induces the rank function $\tilde{P}$ on $R/\ker P$. Then R is equal to the $\tilde{P}$-completion of $R/\ker P$ and $\ker\phi=\ker P$.

Now let $R*G$ be a given crossed product of a finite group G over a regular ring R and let P be a G-invariant pseudo-rank function. Since P is G-invariant, $\ker P$ is G-invariant ideal and therefore each automorphism $\tilde{x}$ induces an automorphism $\tilde{\tilde{x}}$ of $R/\ker P$ and $\tilde{\tilde{x}}$ is uniformly continuous with respect to the induced metric. Consequently we have an automorphism of $\bar{R}$, which is again denoted by $\tilde{x}$, such that $\phi(r)^{\bar{\tilde{x}}}=\phi(r^{\tilde{x}})$ for all $r\in R$. Let a map t': $G\times G\to U(\bar{R})$ be $t'(x, y)=\phi(t(x, y))$ for all $x, y\in G$. Here of course t: $G\times G\to U(R)$ is the given map for $R*G$. We define a crossed product $\bar{R}*G$ of G over $\bar{R}$ using multiplication formula $(a\tilde{x})(b\tilde{y})=(ab^{\tilde{x}^{-1}}t'(x, y))\widetilde{xy}$ for $a, b\in R$ and $x, y\in G$, and define a map $\bar{\phi}$: $R*G\to\bar{R}*G$ by the rule: $\bar{\phi}(\sum_{x\in G} r_x\tilde{x})=\sum_{x\in G}\phi(r_x)\tilde{x}$. Then $\bar{\phi}$ is a ring homomorphism and the following diagram is commutative

$$\begin{array}{ccc} R & \xrightarrow{\ \phi\ } & \bar{R} \\ \downarrow & & \downarrow \\ R*G & \xrightarrow{\ \bar{\phi}\ } & \bar{R}*G \end{array}$$

Proposition 6. *Let P be a G-invariant pseudo-rank function on R, let $\bar{R}$ be a P-completion, let $\bar{P}$ be a continuous extension of P and let ϕ: $R\to\bar{R}$ the natural map. Then we have the relationship between P^G and $(\bar{P})^G$ such that the following diagram is commutative*

$$\begin{array}{ccc} R*G & \xrightarrow{\ P^G\ } & [0, 1] \\ {\scriptstyle\bar{\phi}}\downarrow & & \\ \bar{R}*G & \xrightarrow{\ (\bar{P})^G\ } & [0, 1] \end{array}$$

Proof. For $\alpha\in R^*G$, we assume that ${}_R(R^*G\alpha)\cong\oplus_1^n Rr_i$, where $r_i\in R$. We have

$$\begin{aligned}\Gamma_{\bar R}(\bar P)({}_{\bar R}(\bar R\otimes_R R^*G\alpha) &= \Gamma_{\bar R}(\bar P)(\oplus_1^n\bar R\phi(r_i))\\ &= \textstyle\sum_1^n \Gamma_{\bar R}(\bar P)(\bar R\phi(r_i))\\ &= \textstyle\sum_1^n \Gamma_R(P)(Rr_i)\\ &= \Gamma_R(P)({}_R(R^*G\alpha))\cdots(*)\end{aligned}$$

Consider the natural map v: $\bar R\otimes_R(R^*G\alpha)\to\bar R\bar\phi(R^*G\alpha)=(\bar R^*G)\bar\phi(\alpha)$. Since v is an epimorphism as a $\bar R$-module, we have

$${}_{\bar R}((\bar R^*G)\bar\phi(\alpha))\lesssim\bar R\otimes_R(R^*G\alpha)\,.$$

Therefore we have

$$\begin{aligned}(\bar P)^G(\bar\phi(\alpha)) &= (\Gamma_{\bar R}(\bar P))^G((\bar R^*G)\bar\phi(\alpha))\\ &= |G|^{-1}\Gamma_{\bar R}(\bar P)({}_{\bar R}(\bar R^*G)\bar\phi(\alpha))\\ &\leqq |G|^{-1}\Gamma_{\bar R}(\bar P)({}_{\bar R}(\bar R\otimes_R(R^*G\alpha))\\ &= |G|^{-1}\Gamma_R(P)({}_R(R^*G\alpha))\cdots(\text{by }(*))\\ &= P^G(\alpha)\,.\end{aligned}$$

Since $(\bar P)^G(\bar\phi(\alpha))\leqq P^G(\alpha)$ for all $\alpha\in R^*G$, we have $(\bar P)^G\bar\phi=P^G$ by [2, Lemma 16.13].

DEFINITION [2, Ch. 16 and Appendix]. For a regular ring R, we view $P(R)$ as a subset of the real vector space $\boldsymbol{R}^R$, which we equip with the product topology. Then $P(R)$ is a compact convex subset of $\boldsymbol{R}^R$ by [2, Prop. 16.17]. A *extreme point* of $P(R)$ is a point $P\in P(R)$ which cannot be expressed as a positive convex combination of distinct two points of $P(R)$. We use $\partial_e P(R)$ to denote the set of all extreme points of $P(R)$. The important result is that $P(R)$ is equal to the closure of the convex hull of $\partial_e P(R)$ by Krein-Milman Theorem.

Theorem 7. *Let R^*G be a crossed product of a finite group G over a regular ring R with $|G|^{-1}\in R$. Let P be a G-invariant extreme point of $P(R)$, let $\bar R$ be the P-completion of R, let ϕ: $R\to\bar R$ be the natural ring map and let $\bar P$ be the continuous extension of P over $\bar R$.*

(1) *The crossed product $\bar R^*G$ of G over $\bar R$ defined above, is the completion of R^*G with respect to P^G-metric.*

(2) *The extension P^G can be expressed as a positive convex combination of finite distinct elements in $\partial_e(R^*G)$, i.e., $P^G=\sum_1^n\alpha_iN_i$, where $N_i\in\partial_e P(R^*G)$, $0<\alpha_i<1$ and $\sum_1^n\alpha_i=1$.*

Proof. Since $P\in\partial_e P(R)$, $\bar R$ is a simple, left and right self-injective, regular ring by [2, Th. 19.2 and Th. 19.14]. Since $|G|$ is invertible in $\bar{R}$, we can easily check that $\bar{R}*G$ is self-injective on both sides by the routin way. Since $\bar{R}$ is a simple ring, $\bar{R}*G$ is a finite direct product of simple rings by [8, Cor. 3.10]. Therefore, by [2, Cor. 21.12 and Th. 21.13], $R*G$ is complete with respect to the metric induced by any rank function and so is especially with respect to the $(\bar{P})^G$-metric. We have already shown that $(\bar{P})^G\bar{\phi}=P^G$ by Proposition 6. Finally we shall show that Im $\bar{\phi}$ is dense in $\bar{R}*G$ with respect to $(\bar{P})^G$-metric. For any $\alpha=\sum_{x\in G} a_x\bar{x}\in\bar{R}*G$ and any $\varepsilon>0$, there exist $r_x\in R$ for each a_x such that $\bar{P}(a_x-\phi(r_x))<\varepsilon|G|^{-1}$. Put $\beta=\sum_{x\in G} r_x\bar{x}$. Then we have that

$$\begin{aligned}(\bar{P})^G(\alpha-\bar{\phi}(\beta)) &= (\bar{P})^G(\textstyle\sum_{x\in G}(a_x-\phi(r_x))\bar{x})\\ &\leqq \textstyle\sum_{x\in G}(\bar{P})^G((a_x-\phi(r_x))\bar{x})\\ &\leqq \textstyle\sum_{x\in G}(\bar{P})^G((a_x-\phi(r_x))\\ &< \varepsilon\,.\end{aligned}$$

Thus we have completed the proof of (1). Since the P^G-completion $\bar{R}*G$ of $R*G$ is a finite direct product of simple rings, P^G is a positive convex combination of finite distinct extreme points in $P(R*G)$ by [2, Th. 19.19].

A simple, left and right self-injective, regular ring R has a unique rank function N and it is complete with respect to N-metric and these rings are classified into two types according to the range of N, namely

(1) R is artinian if and only if the range of N is a finite set.

(2) R is non-artinian if and only if the range of N equal to [0, 1] ([4]).

For a given $Q\in\partial_e P(R)$, the Q-completion $\bar{R}$ of a regular ring R is a simple, left and right self-injective, regular ring by [2, Th. 19.14]. Hence we call Q to be *discrete* if $\bar{R}$ is artinian and to be *continuous* if $\bar{R}$ is non-artinian.

DEFINITION. Let P be a G-invariant pseudo-rank function on R. If $P^G=\sum_1^n \alpha_i N_i$, where $N_i\in\partial_e P(R*G)$, $0<\alpha_i<1$ and $\sum_1^n \alpha=1$, then we call $N_1, \cdots, N_t$ to be *associated with P*.

Proposition 8. *For a given crossed product $R*G$, let P be a G-invariant extremal pseudo-rank function on R and let $N_1, \cdots, N_t$ be extremal pseudo-rank functions associated with P. Then the following conditions are equivalent*:

(1) *P is discrete.*

(2) *N_i is discrete for some i.*

(3) *N_j is discrete for all $j=1, \cdots, t$.*

Consequently the following conditions are also equivalent:

(1) *P is continuous.*

(2) *N_i is continuous for some i.*

(3) *N_j is continuous for all $j=1, \cdots, t$.*

Proof. Let $\bar{R}$ be the P-completion of R and let $\bar{P}$ be the extension of P on R. By Theorem 7, the crosseds product $\bar{R}*G$ is the P^G-completion of $R*G$ and $(\bar{P})^G$ is the extension of P^G. Let $\bar{N}_i$ be the continuous extension of N_i on $\bar{R}*G$. The each $\ker \bar{N}_i$ is a maximal two-sided ideal and each $\bar{R}*G/\ker \bar{N}_i$ is a regular, left and right self-injective ring by [2, Th. 9.13]. Since $0=\ker(\bar{P})^G = \bigcap_{i=1}^t \ker \bar{N}_i$, then we have $\bar{R}*G\cong\prod_{i=1}^t \bar{R}*G/\ker \bar{N}_i$.

And $\bar{R}*G/\ker \bar{N}_i$ is isomorphic to the N_i-completion of $R*G$. We assume that P is discrete. So $\bar{R}=R/\ker P$ is a simple artinian ring. Then the crossed product $\bar{R}*G$ is semi-simple by [9, Lemma 1.1]. In particular each $\bar{R}*G/\ker \bar{N}_j$ is an artinian ring, and thus N_j is discrete for all j. Next we assume that some N_i (say $i=1$) is discrete. Let $\bar{\bar{N}}_1$ be the induced rank function on $\bar{R}*G/\ker \bar{N}_1$ by $\bar{N}_1$ and let π: $\bar{R}\to\bar{R}*G/\ker \bar{N}_1$ be the map obtained by compositing $\bar{R}\to\bar{R}*G\to\bar{R}*G/\ker \bar{N}_1$. Then π is monomorphism and we have $\bar{\bar{N}}_1\pi=\bar{P}$. By the assumption, the range of $\bar{\bar{N}}_1$ is a finite set and so is the range of $\bar{P}$. Then P is discrete. Since each extremal pseudo-rank function is either discrete or continuous, latter assertion is clear.

For $N\in\partial_e P(R*G)$, we have the following relationship between N and $(N\,|_R)^G$.

Theorem 9. *Let $R*G$ be a crossed product of a finite group G over a regular ring R with $|G|^{-1}\in R$ and let N be extremal pseudo-rank function on $R*G$. Then we have $(N\,|_R)^G=\alpha N+(1-\alpha)N'$ for some $N'\in P(R*G)$ and some positive real number $\alpha\leq 1$.*

Proof. Put $N\,|_R=P$, then P is G-invariant by Lemma 2. Let T be the P^G-completion on $R*G$ and let $\overline{P^G}$ be the extension of P^G on T. Since N is uniformly continuous with respect to P^G-metric by Lemma 5, we have the continuous extension $\bar{N}$ of N on T. By [2, Th. 19.22], there exists a non-zero central idempotent $e\in T$ such that $\ker \bar{N}=(1-e)T$ and Te is a simple ring. Since Te has the unique rank function, $\overline{P^G}$ and $\bar{N}$ induce the same rank function Q on Te, i.e., $Q(te)=\overline{P^G}(e)^{-1}\overline{P^G}(te)=\bar{N}(te)=\bar{N}(t)$ for all $te\in Te$. Put $L(t)=\overline{P^G}(1-e)^{-1}\overline{P^G}(t(1-e))$ for any $t\in T$, then L is a pseudo-rank function on T. We have

$$\overline{P^G}=\overline{P^G}(e)N+\overline{P^G}(1-e)L$$

by investigating the decomposition $T=Te\oplus T(1-e)$. Let ϕ: $R*G\to T$ be the natural map and let $N'=L\phi$ and let $\alpha=\overline{P^G}(e)$. Then we have that $(N\,|_R)^G=\alpha N+(1-\alpha)N'$.

REMARK. For a G-invariant element $P\in\partial_e P(R)$, let $N_1,\cdots, N_t$ be elements in $\partial_e P(R*G)$ associative with P. We can easily prove that $\{N_1, \cdots, N_t\}$ is equal to the set $\{N\in\partial_e P(R^*G)\colon \theta(N)=N|_R=P\}$, where $\theta\colon P(R^*G)\to P(R)$, by theorem 7 and Theorem 9. Unfortunately we don't know whether $N|_R$ is always extremal for any extremal pseudo-rank function N on R^*G or not.

Now we consider a pseudo-rank function P which is not necessarily G-invariant. For each $x\in G$, put $P^x(r)=P(r^{\tilde{x}^{-1}})$ for all $r\in R$. Then P^x is also a pseudo-rank function and $\ker P^x=(\ker P)^{\tilde{x}}$. Put $t(P)=\sum_{x\in G}|G|^{-1}P^x$, then $t(P)$ is G-invariant pseudo-rank function with $P\leqq|G|t(P)$. We call $t(P)$ to the *trace* of P.

Proposition 10. *Let R^*G be a crossed product of a finite group G over a regular ring R whit $|G|^{-1}\in R$. Let P be in $\partial_e P(R)$ which is not necessarily G-invariant and let $t(P)$ be the trace of P. Then the extension $t(P)^G$ can be expressed as a positive convex combination of finite distinct elements in $\partial_e P(R^*G)$.*

Proof. Let $\bar{R}$ be the $t(P)$-completion of R. Since $t(P)$ is a finite convex combination of extreme points in $P(R)$, $\bar{R}$ is a finite direct product of simple regular self-injective rings by [2, Th. 19.19], R^*G is also a finite direct product of simple regular self-injective rings. In the same way as in the proof of Theorem 7, we can prove that $\bar{R}^*G$ is the $t(P)^G$-completion of R^*G and that $t(P)^G=\sum_1^n\alpha_iN_i$, where $N_i\in\partial_e P(R^*G)$, $0<\alpha_i<1$ and $\sum_1^n\alpha_i=1$.

Corollary 11. *Let R^*G be a crossed product of a finite group G over a regular ring R with $|G|^{-1}\in R$. If $\partial_e P(R)$ is a finite set, then $\partial_e P(R^*G)$ is also a finite set.*

Proof. Let $\partial_e P(R)=\{P_1,\cdots,P_t\}$ and let $\{N_{ij}\colon j=1,\cdots,s(i)\}$ be extremal pseudo-rank functions associated with $t(P_i)$ for each $i=1,\cdots,t$ by Proposition 10. We shall show that $\partial_e P(R^*G)=\{N_{ij}\colon i=1,\cdots,t\ j=1,\cdots,s(i)\}$. We choose $N\in\partial_e P(R^*G)$ and put $P=N|_R$. Since $P(R)$ is equal to the convex-hull of $\{P_1,\cdots,P_t\}$ by [2, A.6], $P=\sum_1^n\alpha_iP_i$, for some $0<\alpha_i<1$ and $\sum_1^n\alpha_i=1$. Put $Q=\sum_1^t\alpha_it(P_i)$, then Q is G-invariant and $Q^G=\sum_1^t\alpha_it(P_i)^G$. Since $P_i\leqq|G|t(P_i)$ for each $i=1,\cdots,t$, $P\leqq|G|Q$ and so $P^G\leqq|G|Q^G$. Let T be the Q^G-completion of R^*G and $\overline{Q^G}$ (resp. $\overline{P^G}$) be the extension of Q^G (resp. P^G) on T. Since $N\leqq|G|P^G$ on R^*G by Lemma 5, $\bar{N}\leqq|G|\overline{P^G}$ on T, where $\bar{N}$ is the extension of N on T. Since $\bar{N}\leqq|G|^2\overline{Q^G}$ on T, $\ker\overline{Q^G}\subset\ker\bar{N}$. Let $\bar{N}_{ij}$ be the extension of N_{ij} on T for each i,j. Since Q^G is a convex combination of $\{N_{ij}\colon i=1,\cdots,t\ j=1,\cdots,s(i)\}$ in $\partial_e P(R)$, $\overline{Q^G}$ is a convex combination of $\{\bar{N}_{ij}\colon i=1,\cdots,t\ j=1,\cdots,s(i)\}$ in $P(T)$. Then we have $\cap_{i,j}\ker\bar{N}_{ij}=\ker\overline{Q^G}$ and therefore $\ker\bar{N}_{ij}\subset\ker\bar{N}$ for some i,j by primeness of $\ker\bar{N}$. Since $\ker\bar{N}_{ij}$ is also a maximal ideal by [2, Th. 19.22], $\ker\bar{N}_{ij}=\ker\bar{N}$. Consequently we have $\bar{N}_{ij}=\bar{N}$ by [5, Prop. II. 14.5] and hence $N_{ij}=N$.

2. Directly finite left self-injective regular rings

In this section, we consider a crossed product of a finite group G over a directly finite, left self-injective, regular ring R with $|G|^{-1} \in R$. K.R. Goodearl has constructed a structure theory on self-injective regular rings. Now we refer to [3, Ch. 10] for definitions and notations. We study types of crossed products R^*G. We begin with the following lemma.

Lemma 12 [4, II. 14.5]. *Let R be a directly finite, left self-injective, regular ring. We define a map v: $\partial_e P(R) \to \mathrm{Max}(R)$ by the rule: $v(P) = \ker P$. Then v is a bijection.*

Theorem 13. *Let R be a directly finite, left self-injective, regular ring and let G be a finite group such that $|G|^{-1} \in R$. Then the following conditions are equivalent.*

(1) *A crossed product R^*G of G over R is of Type II_f.*

(2) *R is of Type II_f.*

Proof. We know that R^*G is a directly finite, left self-injective, regular ring.

(1)$\Rightarrow$(2). It suffices to prove that R has no simple artinian homomorphic images, by [3, Th. 7.10 and Th. 10.24]. Assume that there exists $M \in \mathrm{Max}(R)$ such that R/M is artinian. By Lemma 12, we have $P \in \partial_e P(R)$ such that $\ker P = M$. Let H be the stabilizer of M in G and let Λ be a transversal for H in G with $1 \in \Lambda$. Let $J = \bigcap_{y \in \Lambda} M^{\tilde{y}}$, then J is G-invariant and $R/J \cong \prod_{y \in \Lambda} R/M^{\tilde{y}}$. Since each $\tilde{x}$ induces an automorphism on R/J, there gives rise to a crossed product $(R/J)^*G$ of G over R/J with the natural map ψ: $R^*G \to (R/J)^*G$. Since R/J is a semi-simple ring, so is the crossed product $(R/J)^*G$ by [9, Lemma 1]. Since ψ is epimorphism, R^*G has a simple artinian homomorphic image. This contradicts that R^*G is of Type II_f by [2, Th. 10.29].

(2)$\Rightarrow$(1). Assume that there exists $N \in \partial_e P(R^*G)$ such that $R^*G/\ker N$ is artinian. Put $P = N|_R$. Since $\ker P = \ker N \cap R$ and $\ker N$ is a maximal ideal of R^*G, $\ker P = \bigcap_{x \in G} I^{\tilde{x}}$, where I is a maximal ideal of R by [7, p. 295]. Let K be the stabilizer of I in G and let Λ be a transversal for K in G with $1 \in \Lambda$. Then $R/\ker P \cong \prod_{y \in \Lambda} R/I^{\tilde{y}}$, where $R/I^{\tilde{y}}$ is a simple, left and right self-injective, regular ring. We claim that all $R/I^{\tilde{y}}$ is artinian. Since $\ker P$ is a G-invariant ideal, there exists a crossed product $(R/\ker P)^*G$ of G over $R/\ker P$ such that $R^*G/(\ker P)^*G \cong (R/\ker P)^*G$. By [8, Cor. 3.10], $R^*G/(\ker P)^*G$ is a finite direct of simple, left and right self-injective, regular rings. Since $(\ker P)^*G \subset \ker P^G \subset \ker N$, $R^*G/\ker N$ is isomorphic to a simple component of $R^*G/(\ker P)^*G \cong (R/\ker P)^*G$. By considering $\prod_{y \in \Lambda} R/I^{\tilde{y}} \cong R/\ker P \subset (R/\ker P)^*G$, we find a ring homomorphism f: $\prod_{y \in \Lambda} R/I^{\tilde{y}} \to R^*G/\ker N$. Then we have a ring-monomorphism f': $T = \prod_{y \in \Lambda'} R/I^{\tilde{y}} \to R^*G/\ker N$ for some

$\Lambda'\subset\Lambda$. Let N be the unique rank function of $R^*G/\ker N$ and let P_y be the unique rank function of $R/I^{\tilde{y}}$ and put $Q=Nf'$. This is a rank function on T. Let e_y be a central idempotent of T which is identity element for $R/I^{\tilde{y}}$. By the uniqueness of rank function on $R/I^{\tilde{y}}$, we have $P_y(\alpha)=Q(e_y)^{-1}Q(\alpha)$ for all $\alpha\in R/I^{\tilde{y}}$. By our assumption, the range of N is a finite set and so is the range of Q. Consequently the range of P_y is a finite set. Therefore $R/I^{\tilde{y}}$ is a simple artinian ring by [4]. This is a contradiction by [2, Th. 10.29].

Even when R is a self-injective regular ring, $N|_R$ is not necessarily extremal for $N\in\partial_eP(R^*G)$. If each maximal ideal of R is G-invariant, then $N|_R$ is extremal. In fact, since $\ker(N|_R)=\ker N\cap R$ is a maximal ideal by [7, p. 295], $N|_R$ is extremal by Lemma 12. Hence we shall consider the map θ: $\partial_eP(R^*G)\rightarrow\partial_eP(R)$. We denote the set of all central idempotents of R by $B(R)$.

Lemma 14. *Let R be a directly finite, left self-injective, regular ring and let G be a finite group of automorphisms of R. The following conditions are equivalent;*

(1) *Every maximal ideal of R of G-invariant.*

(2) *Every extremal pseudo-rank function on R is G-invariant.*

(3) *Every central idempotent of R is G-invariant.*

Proof. (1)$\Rightarrow$(2) It is clear by Lemma 12.

(1)$\Rightarrow$(3) Take $e\in B(R)$ and $g\in G$. For $M\in\mathrm{Max}(R)$, we have $e\in M$ or $1-e\in M$ by [3, Th. 8.20]. Since $e-e^g=(1-e^g)-(1-e)$, $e-e^g\in\cap\{M\colon M\in\mathrm{Max}(R)\}$. By [3, Cor. 8.19], we conclude $e=e^g$.

(3)$\Rightarrow$(1). Let M be any maximal ideal of R and let g be any element in G. By [3, Th. 8.20 and Cor. 8.22], $(B(R)\cap M)R$ is a G-invariant, minimal prime ideal. Since any minimal prime ideal of R is contained in a unique maximal ideal by [3, Cor. 8.23], $M=M^g$.

In [4], the Grothendieck group $K_0(R)$ of a regular ring R is investigated as a partially ordered abelian group with order-unit. We refer to [4, 8] for the terminologies of partially ordered abelian groups.

We shall study conditions under which θ is a homeomorphism.

Theorem 15. *Let R be a left self-injective, regular ring of Type II_f and R^*G be a crossed product of a finite group G over R with $|G|^{-1}\in R$. We assume any $M\in\mathrm{Max}(R)$ is G-invariant. Let θ: $\partial_eP(R^*G)\rightarrow\partial_eP(R)$ be a natural restriction map. Then the following conditions are equivalent:*

(1) *θ is a homeomorphism.*

(2) *The natural map f: $K_0(R)\rightarrow K_0(R^*G)$, defined by $f([A])=[R^*G\otimes_R A]$ for $A\in FP(R)$, is an isomorphism as a partially ordered abelian group with order-unit.*

(3) $B(R)=B(R^*G)$.

Proof. We know that R^*G is a left self-injective regular ring of Type II_f by Theorem 13.

(1)⇒(2). By Lemma 12, $\partial_e P(R)$ and $\partial_e P(R^*G)$ are compact. Combining [8, Th. 3.6] with [9, Prop. II. 3.13], we see that $(K_0(R), [R])\cong(C(\partial_e P(R), \boldsymbol{R}), 1)$ and $(K_0(R^*G), [R^*G])\cong(C(\partial_e P(R^*G), \boldsymbol{R}), 1)$, where 1 is the constant function with value 1. Therefore we have that f: $(K_0(R), [R])\cong(K_0(R^*G), [R^*G])$ is an isomorphism.

(2)⇒(3). Let e be any element in $B(R^*G)$. For the element $[R^*Ge]\in K_0(R^*G)$, we choose an element $[A]\in K_0(R)$, such that $f([A])=[R^*Ge]$, where $A\in FP(R)$. First we shall show that $A\lesssim R$. In fact, since $[R^*G\otimes_R A]=[R^*Ge]$, $R^*G\otimes_R A\cong R^*Ge$ by [3, Prop. 15.2]. Let $A\cong\oplus_1^n Rr_i$, where $r_i\in R$. For any $P\in\partial_e P(R)$

$$\begin{aligned}\textstyle\sum_1^n P(r_i) &= \textstyle\sum_1^n \Gamma_R(P)(Rr_i)\\ &= \Gamma_R(P)(A)\\ &= \Gamma_{R^*G}(P^G)(R^*G\otimes A)\\ &= \Gamma_{R^*G}(P^G)(R^*Ge)\\ &\leqq 1\end{aligned}$$

Then we have $A\lesssim R$ by [8, Cor. 2.7]. We may assume that $R^*Ge\cong R^*Gh$ for some idempotent $h\in R$. As e is central, we have $e=h$. On the other hand, since any $h'\in B(R)$ is G-invariant by Lemma 14, h' is central in R^*G.

(3)⇒(1). In general, θ is a continuous epimorphism. We shall that θ is a monomorphism. Assume that there exist $N_1\neq N_2\in\partial_e P(R^*G)$ such that $\theta(N_1)=\theta(N_2)$. By Lemma 12, $\ker N_1\neq\ker N_2$ and so $B(R^*G)\cap\ker N_1\neq B(R^*G)\cap\ker N_2$ by [3, Th. 8.25]. Then there exists $e\in B(R^*G)$ such that $N_1(e)=0$ and $N_2(e)=1$. However since $e\in B(R)$ and $\theta(N_1)=\theta(N_2)$, we have a contradiction. Hence θ is a monomorphism. Next let W be any clopon set in $\partial_e P(R^*G)$. Then $W=\{N\in\partial_e P(R^*G)\colon N(e)=0\}$ for some $e\in B(R^*G)$. Now it is easy to see that $\theta(W)=\{P\in\partial_e P(R)\colon P(e)=0\}$. Therefore $\theta(W)$ is an also clopon set in $\partial_e P(R)$ and so θ is a homeomorphism.

3. Fixed subrings of a finite group of automorphisms

In this section, let R be a regular ring and let G be a finite group of automorphisms of R with $|G|^{-1}\in R$. We shall consider a relationship between $P(R)$ and $P(R^G)$. For any $P\in P(R)$, the restriction of P on R^G, which is denoted by $P|_{R^G}$, is also a pseudo-rank function on R^G. If P is extremal, then we have the following result.

Proposition 16. *For $P\in\partial_e P(R)$, $P|_{R^G}$ can be the expressed as a positive convex combination of finite distinct elements in $\partial_e P(R^G)$.*

Proof. Since P is not necessarily G-invariant, we consider the trace $t(P)$ of P instead of P. Let $\bar{R}$ be the $t(P)$-completion of R. Since $t(P)$ is a finite convex combination of extreme points in $P(R)$, $\bar{R}$ is a finite direct product of simple regular self-injective rings by [2, Th. 19.19]. Let $\overline{t(P)}$ be the extension of $t(P)$ on $\bar{R}$. Since $P|_{R^G}=t(P)|_{R^G}$ on R^G, $(\bar{R})^G$ is the $P|_{R^G}$-completion of R^G. By [8, Cor. 3.10], $(\bar{R})^G$ is also a direct product of simple regular self-injective rings. Therefore $P|_{R^G}$ can be the expressed as a positive convex combination of finite distinct elements in $\partial_e P(R^G)$ by [2, Th. 19.19].

In this section, R^*G implies the skew group ring of G over R. Put $e=|G|^{-1}\sum_{g\in G} g$ in R^*G, then e is an idempotent. Between eR^*Ge and R^G, there exists an isomorphism by the rule: $a\to ea$. Put $X=eR^*G$, then X is a (R^G, R^*G)-bimodule. Throughout this section, we assume

(*) R is a finitely generated projective right R^G-module

Since $\mathrm{Hom}_{R^*G}(X, R^*G)\cong R^*Ge\cong R$ as a right R^G-module, $\mathrm{Hom}_{R^*G}(X, A)$ is a finitely generated projective right R^G-module for all $A\in FP(R^*G)$. Therefore, for $D\in D(R^G)$, $D(\mathrm{Hom}_{R^*G}(X, A))$ gives an unnormalized dimension function on $FP(R^*G)$. We note that $D(R_{R^G})\geqq 1$, because $R_{R^G}\supset R^G$. We define

$$D^{R^*G}(A) = D(R_{R^G})^{-1}D(\mathrm{Hom}(X, A)) \qquad \text{for} \quad A\in FP(R^*G)\,,$$

then D^{R^*G} is a dimension function on $FP(R^*G)$. For a given pseudo-rank function Q on R^G, put $D_Q=\Gamma_{R^G}(Q)$. We define

$$N_Q(x) = D_Q(R_{R^G})^{-1}D_Q(\mathrm{Hom}(X, xR^*G)) \qquad \text{for} \quad x\in R^*G\,.$$

Then by Lemma 1, N_Q is a pseudo-rank function on R^*G. Especially for an idempotent $x\in R^*G$, we have

$$N_Q(x) = D_Q(R_{R^G})^{-1}D_Q((xR^*Ge)_{R^G})\,,$$

because $\mathrm{Hom}_{R^*G}(X, xR^*G)\cong xR^*Ge$ as a right R^G-module. For the induced pseudo-rank function $N_Q\in P(R^*G)$ by $Q\in P(R^G)$, the restriction-function on R, denoted by P_Q, is also a pseudo-rank function on R. $P_Q|_{R^G}$ is not necessarily equal to Q, but we have the following relations between them.

Lemma 17. *Let R be a regular ring, let G be a finite group of automorphisms of R with $|G|^{-1}\in R$ and let R^*G be a skew group ring of G over R. We assume that R satisfies the condition (*). Then for a given $Q\in P(R^G)$, we have the following relation;*

$$Q(a)\leqq D_Q(R_{R^G})(P_Q|_{R^G})(a) \qquad \textit{for all} \quad a\in R^G\,.$$

Proof. For any idempotent $b \in R^G$,

$$Q(b) = D_Q(bR^G) = D_Q(beR^*Ge) = D_Q(ebR^*Ge).$$

Since there exists a natural epimorphism $bR^*Ge \rightarrow ebR^*Ge$ as a R^G-module, we have $ebR^*Ge \lesssim bR^*Ge$. Then we have

$$Q(b) \leqq D_Q(bR^*Ge) = D_Q(R_{R^G})(P_Q|_{R^G})(b).$$

Proposition 18. *Let R be a regular ring and let G be a finite group of automorphisms of R with* $|G|^{-1} \in R$. *We assume that R satisfies the condition* (*). *Then, for a given extremal pseudo-rank function Q on* R^G, *we have*

$$P_Q|_{R^G} = \alpha Q + (1-\alpha)Q'$$

for some $Q' \in P(R^G)$ *and some* $0 < \alpha \leqq 1$.

Proof. We consider R as a ring with $P_Q|_{R^G}$-metric. By Lemma 17, Q is continuous with respect to the metric. Therefore there exist some $Q' \in P(R^G)$ and some real number $0 < \alpha \leqq 1$ such that $P_Q|_{R^G} = \alpha Q + (1-\alpha)Q'$, using the same way as Theorem 9.

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