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# A CLASS OF TRANSCENDENTAL FUNCTIONS CONTAINING ELEMENTARY AND ELLIPTIC ONES

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## 0. Introduction

In this paper we shall present a differential extension field which is wider than Liouville's one and contains elliptic functions. The irreducibility of ordinary differential operators over our field will be investigated.

Liouville proved in [6] that if a linear homogeneous differential equation of the second order over the rational function field $\boldsymbol{C}(x)$ admits a non-trivial solution which is liouvillian over $\boldsymbol{C}(x)$ then it admits a non-trivial solution whose logarithmic derivative is algebraic over $\boldsymbol{C}(x)$ (cf. Ritt [11, chapter 4]). In his [13], Rosenlicht extended this result to the case of general order. As was mentioned there, the theorem of his can be obtained through Picard-Vessiot theory (confer with Kolchin [3]). We shall further extend this.

In [14], Siegel proved a similar theorem. That is to say, if a linear homogeneous differential equation of the second order over $\boldsymbol{C}(x)$ admits a non-trivial solution which satisfies an algebraic differential equation over $\boldsymbol{C}(x)$ then it admits a non-trivial solution whose logarithmic derivative is algebraic over $\boldsymbol{C}(x)$. This result was generalized by Goldman [1] in the case of general order, and further by Singer [15] in the non homogeneous case. Their methods depend upon respectively the Low Power Theorem of Ritt and the valuation theory. The latter was utilized effectually first by Rosenlicht [Publ. Math. Inst. HES., 36 (1969), 15–22]. Another generalization was established by Oleinikov [9]: Let $F$ be a differential field consisting of meromorphic functions in some domain. If a linear homogeneous differential equation of order $n$ over $F$ admits a non-trivial solution which satisfies an algebraic differential equation over $F$ of order less than $n$, then it admits a non-trivial solution which satisfies a homogeneous differential equation over $F$ of order less than $n$. His method is aalytical. We shall give a differential-algebraic proof of this theorem through considering formal infinite series in an arbitrary constant (cf. Ritt [12, chapter 3]).

Let $K$ be an ordinary differential field of characteristic 0 with a differentiation $D$. Throughout this paper we fix a universal differential field extension

$U$ of $K$ and assume that every differential subfield of $U$ discussed below has $U$ as a universal differential field extension. For a differential subfield $F$ of $U$ we denote by $C_F$ the field of constants of $F$ and by $\bar{F}$ the algebraic closure of $F$ in $U$. Let $F$ be a differential field extension of $K$. As usual $F\{Y\}$ indicates the differential polynomial algebra in a differential indeterminate $Y$, and $F[D]$ indicates the algebra of differential operators with coefficients in $F$. In $F[D]$ each element $L \neq 0$ will be written in the form

$$L = \sum f_i D^{n-i};\, f_i \in F,\, f_0 \neq 0\,,$$

and we denote $n = \deg_D L$. The multiplication is determined by

$$D \cdot f = fD + (Df)$$

for any $f$ in $F$.

Let $F$ be an intermediate differential field between $K$ and $U$, and $L \neq 0$ be an element of $F[D]$. The *minimal admissible order* $\mu_F(L)$ over $F$ for the equation $LY=0$ is defined to be the minimum among trans.deg $F\langle x\rangle/F$, where $x$ runs through all elements of $U$ which are transcendental over $F$ and satisfy $Lx=0$. Immediately we see that $\mu_F(L)$ does not exceed $\deg_D L$. We call the equation $LY=0$ *differentially irreducible* over $F$ if $\mu_F(L)=\deg_D L$ and $Ly \neq 0$ for any non-zero $y$ in $\bar{F}$.

As usual an element $L \neq 0$ of $F[D]$ is called reducible over $F$ if it is the product of two elements of $F[D]$ with positive degrees in $D$, or else irreducible over $F$.

The following notion was suggested by Hardy [2, p. 62] (cf. Kolchin [4, p. 809]).

A differential subfield $F$ of $U$ will be called an *H-extension* of $K$ if there exists a finite chain of differential subfields of $U$: $K=F_0 \subseteq F_1 \subseteq \cdots \subseteq F_m = F$ such that for each $j$ $(1 \leqq j \leqq m)$ $F_j$ is finitely algebraic over $F_{j-1}\langle t_j\rangle$, where $t_j$ is primitive, exponential or weierstrassian over $F_{j-1}$.

Then we shall prove the following:

**Theorem 1.** *Suppose that an element $L$ of $K[D]$ satisfies $Ly \neq 0$ for any non-zero $y$ in $\bar{K}$. Then we have*

$$\mu_K(L) = \mu_W(L)$$

*for any H-extension $W$ of $K$.*

**Corollary.** *If $L$ in $K[D]$ is differentially irreducible over $K$ then so is it over any H-extension of $K$.*

The proof of Theorem 1 will be divided into two parts. The first part relates to primitive and exponential elements, the second relates to weierstrassian elements and needs the following theorem which is due to Rosenlicht

[13].

**Lemma 1.** *Let $F$ be a differential subfield of $U$ and $E$ be a differential field extension of $F$ which is finitely algebraic over a Picard-Vessiot extension of $F$. Then any weierstrassian element of $E$ over $F$ is algebraic over $F$.*

For a nonhomogeneous differential equation $LY=f$ over $F$, we define the *minimal admissible order* $\mu_F(L; f)$ over $F$ as the minimal among trans.deg $F\langle x\rangle/F$, where $x$ runs through all elements of $U$ that are transcendental over $F$ and satisfy $Lx=f$. Then Oleinikov's theorem can be stated in the following form:

**Lemma 2.** *Let $F$ be a differential subfield of $U$, $L\notin F$ be an element of $F[D]$ and $f$ be an element of $F$. Then there exist a non-zero element $x$ of $U$ and a homogeneous differential polynomial $H$ in $F\{Y\}$ such that $Lx=H(x)=0$ and $0<\mathrm{ord}_Y H \leqq \mu_F(L; f)$.*

From this it is derived that $\mu_F(L)\leqq\mu_F(L; f)$ for any $f$ in $F$ and $L$ in $F[D]$ such that $Ly\neq 0$ for any non-zero $y$ in $\bar{F}$. Lemma 2 contains Singer's theorem. Combining Theorem 1 and Lemma 2, we have the following:

**Corollary.** *Let $L$ be in $K[D]$ and $k$ be in $K$. Suppose that there is a solution of $LY=k$ which belongs to some $H$-extension of $K$ but not to $K$. Then there is a non-zero solution of $LY=0$ whose logarithmic derivative is algebraic over $K$.*

This contains Rosenlicht's theorem mentioned above.

For the irreducibility of differential operators, a similar fact to Corollary to Theorem 1 holds.

**Theorem 2.** *Suppose that an element $L$ of $K[D]$ is irreducible in $\bar{K}[D]$. Then it is irreducible over any $H$-extension of $K$.*

REMARK. For details about elliptic functions or rather weierstrassian elements, refer to Kolchin [4], Rosenlicht [13], Nishioka [8], Otsubo [10] and §§7–13 of Matsuda [7]. In particular it is a well-known fact that elliptic functions are not liouvillian over the complex number field. This is found from Lemma 1 as well.

We shall prove Theorem 1 together with Lemma 1 in §2, Lemma 2 in §1, Theorem 2 in §3 and the second corollary in §4.

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## 1. Proof of Lemma 2

Before proceeding we note two facts.

Let $E$, $F$ and $G$ be differential subfields of $U$ such that $E$ and $G$ are differential field extensions of $F$ with transcendence degrees over $F$ being finite. Then there exists a differential subfield $E^*$ of $U$ such that $E^*$ is a differential field extension of $F$ differentially isomorphic to $E$ over $F$ and $E^*$ and $G$ are linearly disjoint over $F$. For by defining $D(a\otimes b)=(Da)\otimes b+a\otimes(Db)$, we can make $G\underset{F}{\otimes}E$ into a differential integral domain. Hence the quotient field $I$ of $G\underset{F}{\otimes}E$ is considered as a differential field extension of $G$ with the transcendence degree over $G$ being finite. Since $U$ is also universal over $G$, there is a differential subfield $I^*$ of $U$ which contains $G$ and is differentially isomorphic to $I$ over $G$. As $E^*$ we may take the image of $E$.

Let $C$ be an algebraically closed field of characteristic 0 and $B$ be a finitely generated field extension of $C$. Suppose that we have an element $b$ of $B$ which is transcendental over $C$. Then $B$ can be embedded into the field $C((t))$ of formal power series in $t$ subject to $\nu(b)<0$, where $\nu$ indicates the order function with respect to $t$. This is implied by the fact that the field of Puiseux series has the infinite transcendence degree over the coefficient field.

Lemma 2 is proved in the following form.

**Proposition 1.** *Let $F$ be a differential subfield of $U$, $L$ be an element of $F[D]$ and $f$ be an element of $F$. Let $E$ be a differential subfield of $U$ which contains $F$, a fundamental system of zeroes of $LY$ and a zero of $LY-f$. Assume we have an $A$ in $F\{Y\}$ and a $y$ in $U$ which is transcendental over $F$ and satisfies $A(y)=Ly-f=0$. Then there exists a non-zero element $x$ of $\bar{E}$ with $Lx=H(x)=0$, where $H$ indicates the portion of highest degree included in $A$.*

Proof. To prove this it is sufficient to consider the case where $E$ and $F$ are both algebraically closed in $U$. Let $x_1, x_2, \cdots, x_n$ be elements of $E$ constituting a fundamental system of zeroes of $LY$ and $g$ be a zero of $LY-f$ in $E$. From the above note we may assume that $E$ and $F\langle y\rangle$ are linearly disjoint over $F$. Since $L(y-g)=0$, we have $n$ elements $c_1, c_2, \cdots, c_n$ of $C_{E\langle y\rangle}$ with

$$y=g+\sum_{i=1}^{n} c_i x_i .$$

It follows that $E\langle y\rangle=E(c_1, c_2, \cdots, c_n)$ because the determinant of the matrix $(D^i x_j)$ is not zero. Noting $C_E$ is algebraically closed and

$$\begin{aligned} &\text{trans.deg } C_E(c_1, c_2, \cdots, c_n)/C_E \\ &= \text{trans.deg } E(c_1, c_2, \cdots, c_n)/E \\ &= \text{trans.deg } E\langle y\rangle/E \\ &= \text{trans.deg } F\langle y\rangle/F \end{aligned}$$

is positive since $E$ and $F\langle y\rangle$ are linearly disjoint over $F$, we may consider $C_E(c_1,$

$c_2, \cdots, c_n)$ as a differential subfield of the field $C_E((t))$ of formal power series in $t$ with $\nu(c_1)<0$. Since $C_{E\langle y\rangle}=C_E(c_1, c_2, \cdots, c_n)=C_E(c_1, c_2, \cdots, c_n)$, $E\langle y\rangle$ is a regular extension of $C_E(c_1, c_2, \cdots, c_n)$. We regard the quotient field of the differential integral domain $E\langle y\rangle\otimes C_E((t))$ as a differential subfield of $E((t))$, where the differentiation $D$ operates as $D\sum a_i t^i=\sum (Da_i)t^i$. Thus $E\langle y\rangle$ becomes a differential subfield of $E((t))$. Since $y=g+\sum c_i x_i$ and we may express as $c_i=\sum a_{ij}t^j$, it follows that

$$\begin{aligned} y &= g+\sum_j (\sum_i a_{ij}x_i)t^j \\ &= g+\sum_{j=p}^{\infty} b_j t^j , \end{aligned}$$

where the $b$'s are in $E$ with $b_p\neq 0$. Denoting by $\nu$ the order function of $E((t))$ with respect to $t$, we find $p$ to be negative. For by our assumption $a_{1\nu(c_1)}$ is non-zero and $x_1, x_2, \cdots, x_n$ are linearly independent over constants, hence $b_{\nu(c_1)}=\sum a_{i\nu(c_1)}x_i$ is non zero and $p$ does not exceed $\nu(c_1)$. Let $H$ be the portion of highest degree in $A$. From

$$D^h y = D^h g+\sum_{j=p}^{\infty}(D^h b_j)t^j ,$$

it is derived that $H(b_p)=Lb_p=0$, because $p$ is negative. This completes the proof.

## 2. Proof of Theorem 1

Let $L$ be an element of $K[D]$ such that $Ly\neq 0$ for any non-zero $y$ in $\bar{K}$. Then we show that $\mu_K(L)\geqq\mu_W(L)$ for any $H$-extension $W$ of $K$. In fact let $x$ be an element of $U$ with $Lx=0$ and trans.deg $K\langle x\rangle/K=\mu_K(L)$. By the fact noted in §1 we have an element $y$ of $U$ such that $\bar{K}\langle x\rangle$ and $\bar{K}\langle y\rangle$ are differentially isomorphic over $\bar{K}$ and $\bar{K}W$ and $\bar{K}\langle y\rangle$ are linearly disjoint over $\bar{K}$. Our assertion is justified by the following:

$$\begin{aligned} &\text{trans.deg } W\langle y\rangle/W \\ &= \text{trans.deg } \bar{K}W\langle y\rangle/\bar{K}W \\ &= \text{trans.deg } \bar{K}\langle y\rangle/\bar{K} \\ &= \text{trans.deg } \bar{K}\langle x\rangle/\bar{K} \\ &= \mu_K(L) . \end{aligned}$$

Thus for the proof it remains that $\mu_K(L)\leqq\mu_W(L)$. We shall prove this by the induction on the transcendence degree of $W$ over $K$. In the following the notations of [7] are used.

**Proposition 2.** *Let $F$ be a differential subfield of $U$ being algebraically closed in $U$ and $E$ be a differential algebraic function field of one variable over $F$.*

*Suppose that we have an F-place of E with $\nu_P(Dt_P)>0$, where $t_P$ in E indicates a uniformizing variable at P and $\nu_R$ the order function with respect to $t_P$. Let L be in $F[D]$ such that $Ly\neq 0$ for any non-zero y in F. Then $\mu_F(L)\leqq\mu_E(L)$.*

Proof. Let $y$ be an element of $U$ with $Ly=0$ and trans.deg $E\langle y\rangle/E=\mu_E(L)$. Let $r$ in $F$ be a constant term of $Dt_P/t_P$ in $F((t_P))$. It may happen to be zero. Bring a differential subfield $J$ of $U$ which is algebraically closed in $U$ and contains $F$, a fundamental system of zeroes of $LY$ and an elements of $U$ with $Ds=rs$. In addition we may consider that $J$ and $E\langle y\rangle$ are linearly disjoint over $F$. According to Proposition 1 we obtain an element $x$ of $\bar{E}\bar{J}$ and a homogeneous differential polynomial $H$ in $\bar{E}\{Y\}$ such that $x$ is non-zero and satisfies $Lx=H(x)=0$ and $0<\mathrm{ord}_Y H\leqq\mu_E(L)$. Let $I$ be a finitely algebraic extension of $EJ$ containing $x$ and all the coefficients of $H$. Then $I$ is a differential algebraic function field over $J$. A $J$-place $P_1$ of $EJ$ lying above $P$ is determined uniquely through the coefficient extension. Let $P_2$ be an arbitrary $J$-place of $I$ lying above $P_1$ and $e$ be the ramification index of $P_1$ with respect to $P_2$. An element $t=t_P/s$ of $J$ can be taken as a uniformizing variable at $P_1$. Let $\tau$ be a uniformizing variable at $P_2$ satisfying $\tau^e=t$. By $\nu_i$ we denote the order functions at $P_i$ respectively. Then

$$\begin{aligned}\nu_1(Dt/t) &= \nu_1(Dt_P/t_P-Ds/s)\\ &= \nu_P(Dt_P/t_P-r)\end{aligned}$$

is positive, therefore

$$\nu_2(D\tau/\tau)=\nu_2(Dt/t)=e\nu_1(Dt/t)$$

is positive. Since $J((\tau))$ is the completion of $I$ at $P_2$, we have an expression

$$x=\sum_{i=p}^{\infty}u_i\tau^i\ u_i\in J,\ u_p\neq 0\,.$$

We may express

$$H=\sum_{i=0}^{\infty}H_it_P^i;\ H_i\in F\{Y\},\ H_0\neq 0\,,$$

and so in $J((\tau))$

$$H=\sum_{i=0}^{\infty}s^iH_i\tau^{ei}\,.$$

From $Lx=H(x)=0$ and for each $j$

$$D^jx=(D^ju_P)\tau^p+(\text{terms of order at least } p+1 \text{ in } \tau)$$

since $\nu_2(D\tau/\tau)>0$, it follows that $Lu_p=H_0(u_p)=0$. The element $u_p$ of $J$ does not belong to $F$. Thus we have $\mu_F(L)\leqq\mu_E(L)$.

Proof of Lemma 1. Here we use the notations of [4]. It is sufficient to prove this in the case where $F=\bar{F}$. Let $J$ be a Picard-Vessiot extension of $F$

over which $E$ is finitely algebraic. Take a normal extension $N$ of $J$ containing $E$. Let $w$ be an element of $E$ which is weierstrassian over $F$, that is, a solution of the equation over $F$

$$(Dw)^2 = a^2(4w^3 - g_2 w - g_3),$$

where $a$ is a non-zero element of $F$ and $g_2$ and $g_3$ are in $C_F$ with $27g_3^2 - g_2^3 \neq 0$. Let $w = w_1, w_2, \cdots, w_r$ be all conjugates of $w$. Then those are contained in $N$ and satisfy

$$(Dw_j)^2 = a^2(4w_j^3 - g_2 w_j - g_3).$$

Since $w_i \neq w_j$ for $i \neq j$, we have elements $b_j, c_j$ $(j=2, \cdots, r)$ of $C_N$ such that

$$(1\colon w_j\colon a^{-1}Dw_j) = (1\colon w\colon a^{-1}Dw)\,(1\colon b_j\colon c_j).$$

Multiplying these we obtain an element $v$ of $J$ satisfying

$$\prod_{j=1}^{r} (1\colon w_j\colon a^{-1}Dw_j) = (1\colon w\colon a^{-1}Dw) \prod_{j=2}^{r} (1\colon b_j\colon c_j)$$

$$= (1\colon v\colon (na)^{-1}Dv) \prod_{j=2}^{r} (1\colon b_j\colon c_j),$$

because the left hand side is left invariant under any automorphism of $N$ over $J$ and therefore rational over $J$. The differential field $J\langle v\rangle$ is a strongly normal extension of $F$ and contained in a Picard-Vessiot extension $J$ of $F$. Then by Kolchin's theorem [5], it is a Picard-Vessiot extension of $F$. Suppose that $v$ is transcendental over $F$. Then the Galois group of differential automorphisms of $F\langle v\rangle$ over $F$ is an affine group over $C_F$ and an abelian variety over $C_F$, hence being trivial. This is a contradiction. Thus $v$ lies in $F$. The point $(1\colon w\colon a^{-1}Dw)$ is an $n$-division point of $(1\colon v\colon (na)^{-1}Dv)$, hence $w$ lies in $F$.

**Proposition 3.** *Let $F$ be a differential subfield of $U$ being algebraically closed in $U$ and $E$ be a differential algebraic function field of one variable over $F$. Suppose that $E$ contains an element $w$ which is weierstrassian over $F$ and not in $F$. Let $L$ be in $F[D]$ such that $Ly \neq 0$ for any non-zero $y$ in $F$. Then $\mu_F(L) \leqq \mu_E(L)$.*

Proof. Bring a zero $y$ of $LY$ with trans.deg $E\langle y\rangle/E = \mu_E(L)$. Let $J$ be a Picard-Vessiot extension of $F$ generated by a fundamental system of zeroes of $LY$. We may assume that $J$ and $E\langle y\rangle$ are linearly disjoint over $F$. According to Proposition 1, we have an element $x \neq 0$ of $\bar{E}\bar{J}$ and a homogeneous differential polynomial $H$ in $\bar{E}\{Y\}$ such that $Lx = H(x) = 0$ and $0 < \mathrm{ord}_Y H \leqq \mu_E(L)$. First consider the case where $x$ is algebraic over $J$. Since

$$\text{trans.deg } F\langle x\rangle/F = \text{trans.deg } E\langle x\rangle/E \leqq \mathrm{ord}_Y H$$

$$\leqq \mu_E(L),$$

it follows that $\mu_F(L) \leqq \mu_E(L)$. Next consider the case where $x$ is transcendental over $J$. Then $x$ and $w$ are algebraically dependent over $J$ since both are in $\bar{E}\bar{J}$, hence trans.deg $J\langle x\rangle/J=1$. Let $P$ be a $\bar{J}$-place of $\bar{J}\langle x\rangle$ which is a pole of $x$. Then $\nu_P(Dt_P)>0$, where $t_P$ is a uniformizing variable at $P$ and $\nu_P$ is the order function with respect to $t_P$ of $\bar{J}\langle x\rangle$. In fact assume the converse. Then $n_P=1-\nu_P(Dt_P/t_P)$ is a positive integer and $\nu_P(D^j x)=\nu_P(x)-jn_P$ holds for each non-negative $j$. Hence $\nu_P(Lx)=\nu_P(x)-n_P \deg_D L$. But this contradicts $Lx=0$. Let $Q$ be a $\bar{J}$-place of $\bar{J}\langle w\rangle$ which lies below some $\bar{J}$-place of $\bar{J}\langle x, w\rangle$ lying above $P$, $u_Q$ be a uniformizing variable at $Q$ and $\nu_Q$ indicate the order function with respect to $u_Q$. The inequality $\nu_Q(Du_Q)>0$ holds because of $\nu_P(Dt_P)>0$. We show that $\nu_Q(w)\geqq 0$. In fact assume the converse. Then

$$\nu_Q(Dw/w) = \nu_Q(Du_Q/u_Q)>0 ,$$

that is, $\nu_Q(Dw)>\nu_Q(w)$. Since $w$ is weierstrassian over $F$, it satisfies

$$(Dw)^2 = a^2(4w^3-g_2w-g_3) ,$$

where $a$ is a non-zero element of $F$ and $g_2$ and $g_3$ are in $C_F$ with $27g_3^2-g_2^3\neq 0$. Considering the orders of both sides,

$$3\nu_Q(w) = 2\nu_Q(Dw)\geqq 2\nu_Q(w) ,$$

we have a contradiction to $\nu_Q(w)<0$. Thus we obtain an element $z$ of $\bar{J}$ with $\nu_Q(w-z)>0$, which satisfies

$$(Dz)^2 = a^2(4z^3-g_2z-g_3) ,$$

that is to say, being weierstrassian over $F$. According to Lemma 1, it belongs to $F$ but not to $C_F$, since $\nu_Q(Du_Q)>0$. This implies that the $F$-place of $F\langle w\rangle$ lying below $Q$ fulfils the condition of Proposition 2. Consequently we have $\mu_F(L)\leqq \mu_F(L)$.

Let us turn to the proof of Theorem 1. We work it by the induction on the transcendence degree of $W$ over $K$. There is nothing to say when $W$ is algebraic over $K$. Suppose that the theorem is true for any $H$-extension with the transcendence degree less than $m$ over $K$ and let $W$ be an $H$-extension of $K$ with $m=$trans.deg $W/K>0$. There is a differential subfield $W_1$ of $W$ with trans.deg $W_1/K=m-1$ such that $W$ is finitely algebraic over $W_1(w)$, where $w$ is primitive, exponential or weierstrassian over $W_1$. Put $F=\bar{W}_1$ and $E=W\bar{W}_1$. Then they satisfy the conditions of Propositions 2 and 3 respectively in the first two cases of $w$ and in the last one (cf. Otsubo [10]). From this it follows that either there is an element $y$ of $\bar{W}_1$ with $Ly=0$ or $\mu_{W_1}(L)\leqq \mu_W(L)$. In the latter case by induction hypothesis we have the required result. In the former case by our assumption on $L$ we see that $y$ is transcendental over

$K$. Since $W_1(y)$ is an $H$-extension of $K$, we have two $H$-extensions $W_2$ and $W_3$ of $K$ such that $W_2$ contains $y$ and $W_3$, $y$ is transcendental over $W_3$ and trans.deg $W_2/W_3=1$. We see $\mu_{W_3}(L)=1$ because trans.deg $W_3/K<m$ and $\mu_{W_3}(L)=1$. Since $\mu_{W_1}(L)\leqq\mu_W(L)\leqq\mu_K(L)=1$ we have $\mu_W(L)=\mu_K(L)=1$. This completes the proof.

## 3. Proof of Theorem 2

**Proposition 4.** *Let $E$, $F$ and an $F$-place $P$ be the same as in Proposition 2. Let $L$ be an element of $F[D]$ which is reducible in $E[D]$. Then $L$ is reducible in $F[D]$.*

Proof. Let $L_1$ be an element of $E[D]$ which is a right divisor of $L$ and set $H=L_1Y\in E\{Y\}$. By the proof of Proposition 2, we obtain a non-zero element $u$ of $U$ and a homogeneous differential polynomial $H_0$ in $F\{Y\}$ such that $Lu=H_0(u)=0$ and tot.deg $H_0\leqq$tot.deg $H$. In the present case $H_0$ is linear and can be expressed as $H_0=L_0Y$ with $L_0$ in $F[D]$. We get $L_0u=0$. Bring an element $M$ of $F[D]$ with the least degree in $D, M\neq 0$ and $Mu=0$. Then $M$ is a right divisor of $L$. For we have an expression $L=L_2M+L_3$, where either $L_3=0$ or $\deg_D L_3<\deg_D M$. Then

$$0 = Lu = L_2Mu+L_3u = L_3u\,,$$

and so $L_3=0$ by the minimality of $\deg_D M$. This shows that $L$ is reducible in $F[D]$.

**Proposition 5.** *Let $E$ and $F$ be the same as in Proposition 3. Let $L$ be an element of $F[D]$ which is reducible in $E[D]$. Then $L$ is reducible in $F[D]$.*

Proof. Let $M$ be a right divisor of $L$ in $E[D]$. Take a zero $y$ of $LY$ such that trans.deg $E\langle y\rangle/E=\deg_D M$. We have a Picard-Vessiot extension $J$ of $F$ with generators consisting of a fundamental system of zeroes of $LY$ such that $E\langle y\rangle$ and $J$ are linearly disjoint over $F$. By Proposition 1, there is a non-zero element $x$ of $\bar{E}\bar{J}$ with $Mx=0$. First suppose that $x$ is algebraic over $J$. By linearly independent elements $a_i$ of $E$ over $F$ we represent $M=\sum a_iL_i$, where $L_i$ is in $F[D]$. Then

$$0 = Mx = \sum a_iL_ix$$

and noting each $L_ix$ is algebraic over $J$, we get $L_ix=0$. Hence there is a non-zero element $N$ of $F[D]$ with $Nx=0$ and similarly to the proof of Proposition 4 we find that $L$ is reducible in $F[D]$. Next suppose that $x$ is transcendental over $J$. Then by the same argument as in the proof of Proposition 3 we obtain an $F$-place $P$ of $E$ satisfying the condition of Proposition 2. According to Proposition 4, $L$ is seen to be reducible in $F[D]$.

Let us prove Theorem 2. We shall show that if $L$ in $K[D]$ is reducible in $W[D]$ for some $H$-extension $W$ of $K$ then it is reducible in $\bar{K}[D]$. When $W$ is algebraic over $K$ there is nothing to show. Suppose that our assertion is true for $H$-extensions of $K$ with the transcendence degree less than $m$ over $K$ and let $W$ be an $H$-extension of $K$ such that trans.deg $W/K=m>0$ and $L$ is reducible in $W[D]$. By the definition, there is a differential subfield $W_1$ of $W$ such that $W$ is finitely algebraic over $W_1(w)$ with trans.deg $W/W_1=1$, where $w$ is primitive, exponential or weierstrassian over $W_1$. Setting $F=\bar{W}_1$ and $E=W\bar{W}_1$ and applying Propositions 2 and 3, we see that $L$ is reducible in $\bar{W}_1[D]$. Since we obtain an $H$-extension $W_2$ which is finitely algebraic over $W_1$ and over which $L$ is reducible, by the induction hypothesis we conclude $L$ is reducible in $\bar{K}[D]$.

## 4. Proof of Corollary to Theorem 1

If we have a non-zero solution of $LY=0$ which is algebraic over $K$, then its logarithmic derivative is of course algebraic over $K$. Hence we assume in the following that $Lh\neq 0$ for any non-zero $h$ in $\bar{K}$. By the supposition there is an $H$-extension of $K$ which contains a solution $y$ of $LY=k$ not lying in $K$. If $y$ is algebraic over $K$, let $z$ be another conjugate of $y$ in $\bar{K}$. Then $y-z$ satisfies $L(y-z)=0$ and it is a non-zero element of $\bar{K}$. This implies a contradiction to our assumption. Hence $y$ is transcendental over $K$. As the preceding there exists an $H$-extension $W$ of $K$ such that trans. deg $W\langle y\rangle/W=1$. This shows $\mu_W(L; k)=1$ and according to Lemma 2 there exist a non-zero solution $x$ of $LY=0$ and a homogeneous differential polynomial $H$ in $W\{Y\}$ such that $H(x)=0$ and $\mathrm{ord}_W H=1$. Since $x$ is not in $\bar{K}$, we have $\mu_W(L)=1$. Theorem 1 yields $\mu_K(L)=1$ and again Lemma 2 gives us a non-zero solution $w$ of $LY=0$ and a homogeneous differential polynomial $G$ in $K\{Y\}$ such that $G(w)=0$ and $\mathrm{ord}_Y G=1$. The logarithmic derivative $Dw/w$ is algebraic over $K$ and this completes the proof.

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