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ON MIXED BOUNDARY VALUE PROBLEMS FOR PARABOLIC EQUATIONS IN SINGULAR DOMAINS

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1. Introduction. In this paper we continue our investigation on boundary value problems for elliptic and parabolic equations in singular domains. The problem is thoroughly investigated if the boundary is smooth. See [1] for general boundary value problems for elliptic equations and [8] and [11] for the parabolic case.

Elliptic boundary value problems in singular domains have been studied by many authors using different approaches. See [10], [12]–[15]. Comparatively little is known in the case of parabolic equations. One of the reasons for that is the fact that the methods used in the elliptic case do not extend completely to the parabolic case. In [3] we have introduced a method for investigating the Dirichlet problem for elliptic equations in plane domains with corners. This method was then modified to study different boundary value problems for elliptic equations in n-dimensional domains with edges (cf. [4], [5]) and initial-Dirichlet problem for parabolic equations (cf. [6]). The method is based on obtaining a bound for the solution near the singular part of the boundary. This is done by constructing a suitable barrier function. Then using a Schauder-type estimate we obtain bounds for the derivatives of the solution and then its smoothness properties. In [7], we applied this method to investigate the smoothness properties of solutions of initial-mixed boundary value problems for parabolic equations and obtained C^{ν} statements for these solutions, $1<\nu\leq 2$. In this paper, we study the same problem, and give conditions sufficient for the solution to belong to $C^{m+2+\alpha}$, $m\geq 0$, $0<\alpha<1$.

2. The problem. Consider a simply connected bounded domain $G\subset R^2$ with boundary consisting of finite number of $C^{m+2+\alpha}$ curves $\Gamma_1, \cdots, \Gamma_q$. Here $m\geq 0$ is an integer and $\alpha\in(0, 1)$. Γ_k, Γ_{k+1} meet at the point $x^{(k)}=(x_1^{(k)}, x_2^{(k)})$ forming there an interior angle γ_k; $0<\gamma_k<2\pi$, $k=1, \cdots, q$, $\Gamma_{q+1}=\Gamma_1$. In $\Omega=G\times J$ where $J=\{t: 0\leq t<T\}$ consider the parabolic operator $Lu\equiv a_{ij}(x, t)u_{ij}+a_i(x, t)u_i+a(x, t)u-u_t$ Here $x=(x_1, x_2)$, $u_i=\dfrac{\partial u}{\partial x_i}$, $u_{ij}=\dfrac{\partial^2 u}{\partial x_i \partial x_j}$, $i, j=1, 2$ and we use the summation convention. Consider in Ω the initial mixed boundary value

problem

$$Lu = f(x, t), \quad (x, t) \in \Omega \tag{1}$$

$$u(x, 0) = 0 \quad \text{on } \bar{G} \tag{2}$$

$$\eta_k u + (1-\eta_k) \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_k \times J \tag{3}$$

The coefficients on L and f belong to $C^{m+\alpha}(\bar{\Omega})$ and η_k in (3) is either 0 or 1 and $\eta_k + \eta_{k+1} \neq 0$, $k=1, \cdots, q$; $\eta_{q+1} = \eta_1$. Under these assumptions, it is known [9] that

$$u \in C^{m+2+\alpha}(\Omega_1) \cap C^0(\bar{\Omega}) \tag{4}$$

where $\Omega_1 = G_1 \times J$ and G_1 is any compact subregion of $\bar{G}$ with positive distance from the corner points. To investigate the smoothness of the solutions near the edges, consider a fixed point $(x^{(k)}, t_0)$, on the edge; $x^{(k)} = \Gamma_k \cap \Gamma_{k+1}$, $t_0 \in J$. Transforming the equation $a_{ij}(x^{(k)}, t_0)\, u_{ij} = 0$ to canonical form, the angle γ_k at $(x^{(k)}, t_0)$ will be transformed to the angle ω_k, where $\omega_k = \omega(x^{(k)}, t_0, \gamma_k)$ is defined by

$$\omega(x, t, \gamma) = \arctan \frac{[a_{11}(x, t)\, a_{22}(x, t) - a_{12}^2(x, t)]^{1/2}}{a_{22}(x, t) \cot \gamma - a_{12}(x, t)} .$$

We also introduce the following notations

$$\beta_k = 2\omega_k/(\eta_k + \eta_{k+1}), \quad k = 1, \cdots, q$$
$$\beta = \operatorname{Sup} \beta_k ,$$

where the Sup is taken over $k=1, \cdots, q$ and $t_0 \in J$.
We now state our main result

Theorem I. *Any bounded solution of* (1)–(3) *belongs to* $C^{\nu}(\bar{\Omega})$, *where,*

$$\nu = \min\,(m+2+\alpha,\ \pi/\beta - \varepsilon),$$
$\varepsilon > 0$ *is arbitrarily small.*

From (4) it follows that it is sufficient to investigate the smoothness of u in a neighborhood of the edge point $(x^{(k)}, t_0)$. As a matter of fact, we shall prove that

$$u \in C^{\nu_k}(N), \tag{5}$$

where $\nu_k = \min\,(m+2+\alpha,\ \pi/\beta_k - \varepsilon)$, $\varepsilon > 0$ is arbitrarily small and N is the intersection of $\bar{\Omega}$ with a small ball centered at $(x^{(k)}, t_0)$. Finally, we remark that it is sufficient to prove (5) in the case of a cylindrical sector. This is true since the general case may be transformed to the cylindrical sector case using a locally injective $C^{m+2+\alpha}$ transformation, cf. [5].

3. The cylindrical sector case. Let $t_0 \in J$ be fixed, $\Omega_\sigma = G_\sigma \times I_\sigma$ where

$$G_\sigma = \{(r, \theta);\ 0<r<\sigma,\ 0<\theta<\omega\}$$
$$I_\sigma = \{t;\ t\in J,\ |t-t_0|<\sigma\}$$

and let

$$\Gamma_1 = \{(r, \theta),\ r<\sigma,\ \theta=0\}$$
$$\Gamma_2 = \{(r, \theta),\ r<\sigma,\ \theta=\omega\}$$

We now state a theorem equivalent to Theorem 1 but in the cylindrical sector case.

Theorem 2. *Let u be a bounded solution of the problem*

$$Lu = f(x, t) \quad \text{in } \Omega_\sigma \tag{6}$$
$$u(x, 0) = 0$$

$$u = 0 \quad \text{on } \Gamma_1 \times J \tag{7}$$

$$\eta u + (1-\eta)\, u_n = 0 \quad \text{on } \Gamma_2 \times J \tag{8}$$

*where a_{ij}, a_i, a and f belong to $C^{m+\alpha}(\bar{\Omega}_\sigma)$,
$a_{ij}(0, t_0) = \delta_{ij}$, $i, j = 1, 2$ and η in* (8) *is either* 0 *or* 1. *Let $\beta = 2\omega/(\eta+1)$, then*

$$u \in C^\nu(\bar{\Omega}_\delta) \tag{9}$$

where $\nu = \min(m+2+\alpha,\ \pi/\beta - \varepsilon)$, $\varepsilon > 0$ is arbitrarily small and $\delta < \sigma$.

We shall discuss only the case when $\nu = m+2+\alpha$. The other cases can be discussed in a similar way. The two cases $\eta = 1$ and $\eta = 0$ with $m = 0$, $\beta < \pi$ were given in [6]. In proving (9) we use the method introduced in [3] to investigate smoothness properties of solutions of the Dirichlet problem for elliptic equations in singular domains. The main step of this method consists of deriving bounds for solutions of (6)–(8) of the form

$$|u(x, t)| \le M r^\nu \tag{10}$$

Using this bound, we then estimate, in the second step, the partial derivatives of u to get bounds of the form

$$|D^k u(x, t)| \le M_k r^{\nu-k}, \quad k = 1, \cdots, [\nu]\,. \tag{11}$$

Then, finally, we can get the required smoothness results. The last two steps follow from (10) almost in the same way as it was done in [5] and [6]. To obtain estimates of the form (10) we need the right hand side of (6) to have enough zeros at the edge point, namely

$$D^p f(x, t)|_{x=0} = 0, \quad |p| \leq m, \tag{12}$$

Where D^p is any partial derivative of order p with respect to x. This will be done by adding to u a suitable $C^{m+2+\alpha}$ function. This function will be constructed using the next two lemmas.

Lemma 1. *Given a function $F(t) \in C^{k+\alpha}$ defined on $x=0$, there exists an extension $F^*(x, t) \in C^{k+\alpha}(R^3)$ which coincides with $F(t)$ when $x=0$ and*

$$x_1^{k_1} x_2^{k_2} F^*(x, t) \in C^{k+k_1+k_2+\alpha}(R^3)$$

The proof of this lemma goes along the same lines of the proof of Lemma 1 in [5]. See also [2].

Lemma 2. *There exists a function $v(x, t) \in C^{m+2+\alpha}(\bar{\Omega}_\sigma)$ satisfying* (7) *and* (8) *and*

$$D^p(Lv-f)|_{x=0} = 0, \quad |p| \leq m.$$

The idea of proving this lemma is to construct first the function v as a polynomial in x with coefficients depending on t. All the terms of this polynomial are of the form $x_1^{k_1} x_2^{k_2} F(t)$, where $F(t) \in C^{k+\alpha}(J)$ and $k_1+k_2+k=m+2$. Then we replace $F(t)$ by $F^*(x, t)$ constructed in Lemma 1. See [4] and [5].

From these two lemmas, it follows that the function $w=u-v$ satisfies conditions (7) and (8) of Theorem 2, and in $\bar{\Omega}_\sigma$ it satisfies an equation of the form (6) with the right hand side satisfying (12). For simplicity, we shall still use u and f in Theorem 1 with f satisfying now (12).

Proof of Theorem 2. As mentioned before, to prove the theorem it is sufficient to show that any bounded solution of (6)–(8) with f satisfying (12), will satisfy the estimation

$$|u(x, t)| \leq Mr^\nu$$
$$\nu = m+2+\alpha < \pi/\beta.$$

We shall consider first the case when $\eta=0$. The modifications in the proof for the case $\eta=1$ will be given in the end of the proof. Consider the function

$$U(x) = -Mr^\nu \cos \lambda(\omega-\theta),$$

where $\lambda = \dfrac{\pi-2\Delta}{2\omega} > m+2+\alpha$, $\Delta>0$. In virtue of $a_{ij}(0, t_0)=\delta_{ij}$ and (12) it can be easily verified that $LU \geq f(x, t)$ in G_δ, provided that M is sufficiently large and δ is sufficiently small. i.e.

$$L(u-U)\leq 0 \quad \text{in } G_\delta .$$

We aim to apply the maximum principle in G_δ. We note first that $U_n=0$ on $\Gamma_2\times J$. (i.e.) $(u-U)_n=0$ on $\Gamma_2\times J$ and the maximum of $u-U$ cannot be attained on this part of the boundary of G_δ. On the rest of the boundary of G_δ we can make $u-U\geq 0$ by taking M sufficiently large. Finally, taking δ sufficiently small we conclude that $u-U\geq 0$ in the interior of G_δ as well. i.e.

$$u\geq -M r^\nu \cos \lambda(\omega-\theta)\geq -M r^\nu .$$

Similarly we can prove that in $\bar{G}_\delta$

$$u\leq M r^\nu ,$$

provided that M is taken sufficiently large and δ sufficiently small. This proves the theorem in the case when $\eta=0$. When $\eta=1$, we may take as a barrier function

$$\tilde{U}= -M r^\nu \cos \tilde{\lambda} \left(\frac{\omega}{2}-\theta\right),$$

where $\tilde{\lambda}=(\pi-2\tilde{\Delta})/\omega>m+2+\alpha$, $\tilde{\Delta}>0$,
and proceed as before.

This concludes the proof of the theorem.

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