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ASYMPTOTIC PROPERTY OF AN EIGENFUNCTION OF THE LAPLACIAN UNDER SINGULAR VARIATION OF DOMAINS —THE NEUMANN CONDITION—

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1. Introduction

We consider a bounded domain Ω in $\boldsymbol{R}^2$ with smooth boundary γ. Let B_ε be the ε-disk whose center is $\tilde{w}\in\Omega$. We put $\Omega_\varepsilon=\Omega\backslash\bar{B}_\varepsilon$. We consider the following eigenvalue problems (1.1) and (1.2):

$$(1.1)\qquad \begin{aligned} -\Delta_x u(x) &= \lambda(\varepsilon)u(x)\,, && x\in\Omega_\varepsilon\,,\\ u(x) &= 0\,, && x\in\gamma\,,\\ \frac{\partial u}{\partial\nu}(x) &= 0\,, && x\in\partial B_\varepsilon\,, \end{aligned}$$

where $\partial/\partial\nu$ denotes the derivative along the inner normal vector at x with respect to the domain Ω_ε.

$$(1.2)\qquad \begin{aligned} -\Delta_x u(x) &= \lambda u(x)\,, && x\in\Omega\,,\\ u(x) &= 0\,, && x\in\gamma\,. \end{aligned}$$

Let $0<\mu_1(\varepsilon)\leq\mu_2(\varepsilon)\leq\cdots$ be the eigenvalues of (1.1). Let $0<\mu_1\leq\mu_2\leq\cdots$ be the eigenvalues of (1.2). We arrange them repeatedly according to their multiplicities. Denote by $\{\varphi_j(\varepsilon)\}_{j=1}^\infty$ ($\{\varphi_j\}_{j=1}^\infty$, respectively) a complete orthonomal basis of $L^2(\Omega_\varepsilon)$ ($L^2(\Omega)$, respectively) consisting of eigenfunction of $-\Delta$ associated with $\{\mu_j(\varepsilon)\}_{j=1}^\infty$ ($\{\varphi_j\}_{j=1}^\infty$, respectively).

In this note we consider the following problem:
Problem. What can one say about asymptotic behaviour of $\varphi_j(\varepsilon)$ as ε tends to zero?

It is well known that $\mu_j(\varepsilon)$ tends to μ_j as ε tends to zero. See Rauch-Taylor [8], Ozawa [5]. As a consequence, $\mu_j(\varepsilon)$ is simple for small $\varepsilon>0$, if we assume that μ_j is simple. Thus $\varphi_j(\varepsilon)$ is uniquely determined up to the arbitratiness of multiplication by $+1$ or -1.

We have the following Theorem 1. Theorem 2 is our main result.

Theorem 1. *Fix j. Assume that μ_j is simple. Then, the following statements* (i) *and* (ii) *hold.*
(i) *We can choose $\varphi_j(\varepsilon)$ for $\varepsilon>0$ so that*

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}(\varphi_j(\varepsilon))(x)\varphi_j(x)dx=1\,.$$

(ii) *If we choose $\varphi_j(\varepsilon)$ as in* (i), *then*

$$\|\varphi_j(\varepsilon)-\varphi_j\|_{L^\infty(\Omega_\varepsilon)}=0(\varepsilon)\,. \tag{1.3}$$

We introduce the polar coordinate $z-\tilde{w}=(r\cos\theta,\ r\sin\theta)$ to state the following

Theorem 2. *Fix j. Assume that μ_j is a simple eigenvalue. If $\varphi_j(\varepsilon)$ is chosen as in Theorem* 1, *then*

$$\begin{aligned}&\left(\frac{\partial}{\partial\theta}(\varphi_j(\varepsilon))\right)(\varepsilon\cos\theta,\ \varepsilon\sin\theta)\\ &\quad=2\frac{\partial}{\partial r}\left(\varphi_j\left(r\cos\left(\theta+\frac{\pi}{2}\right),\ r\sin\left(\theta+\frac{\pi}{2}\right)\right)\right|_{r=0}+0(\varepsilon^{(1/2)-s})\end{aligned} \tag{1.4}$$

for an arbitrary $s>0$.

REMARK. 1) Proofs of Theorems 1 and 2 are given in the section 2.
2) The remainder estimates in (1.3) and (1.4) are not uniform with respect to j.
3) Theorems 1 and 2 prove the conjecture stated in the previous work [5] of the author.
4) The celebrated Hadamard variational formula (See Garabedian-Schiffer [4]) says that

$$\frac{\partial}{\partial\varepsilon}\mu_j(\varepsilon)=-\int_{\partial B_\varepsilon}(|\mathrm{grad}_z\,\varphi_j(\varepsilon)(z)|^2-\mu_j(\varepsilon)(\varphi_j(\varepsilon))(z)^2)d\sigma_z^\varepsilon\,, \tag{1.5}$$

holds when μ_j is simple, where $d\sigma_z^\varepsilon$ denotes the line element on ∂B_ε. If we apply Theorems 1 and 2 to (1.5), then

$$\frac{\partial}{\partial\varepsilon}\mu_j(\varepsilon)=0(\varepsilon)\,.$$

Hence $\mu_j(\varepsilon)-\mu_j=0(\varepsilon^2)$. Using (1.5) once more, we can prove that

$$\mu_j(\varepsilon)-\mu_j=-(2\pi|\mathrm{grad}\,\varphi_j(\tilde{w})^2-\pi\mu_j\varphi_j(\tilde{w})^2)\varepsilon^2+0(\varepsilon^{(5/2)-s})\,, \tag{1.6}$$

while we have already obtained in [5] much stronger result

$$\mu_j(\varepsilon)-\mu_j=-(2\pi|\mathrm{grad}\,\varphi_j(\tilde{w})|^2-\pi\mu_j\varphi_j(\tilde{w})^2)\varepsilon^2+0(\varepsilon^3|\log\varepsilon|^2)\,.$$

However, discussion in [5] was very complicated. Present proof via Hadamard's variational formula (1.5) is much simpler.

See Ozawa [6], [7], Figari-Orlandi-Teta [2] for other recent developments on the asymptotic behaviour of the eigenvalues of the Laplacian under singular variation of domains.

A part of this work was done while I stayed at Courant Institute of Mathematical Sciences. I here express my sincere thanks to C.I.M.S., Professor G. Papanicolaou and Ms. Vogelsang for their hospitality.

2. Sketch of the proof

Let $G(x, y)$ be the Green function of the Laplacian in Ω under the Dirichlet condition on γ. Let $G_\varepsilon(x, y)$ be the Green function of the Laplacian in Ω_ε satisfying

$$\begin{aligned} -\Delta_x G_\varepsilon(x, y) &= \delta(x-y)\,, && x, y\in\Omega_\varepsilon \\ G_\varepsilon(x, y)_{|x\in\gamma} &= 0\,, && y\in\Omega_\varepsilon \\ \frac{\partial}{\partial\nu_x} G_\varepsilon(x, y)_{|x\in\partial B_\varepsilon} &= 0\,, && y\in\Omega_\varepsilon\,. \end{aligned}$$

Let $\boldsymbol{G}$ ($\boldsymbol{G}_\varepsilon$, respectively) be the bounded linear operator on $L^2(\Omega)$ ($L^2(\Omega_\varepsilon)$, respectively) defined by

$$(\boldsymbol{G}f)(x) = \int_\Omega G(x, y)f(y)dy\,,$$

$$(\boldsymbol{G}_\varepsilon g)(x) = \int_{\Omega_\varepsilon} G_\varepsilon(x, y)g(y)dy\,,$$

respectively. Then, (1.1) and (1.2) are transformed into the problems

$$(\boldsymbol{G}_\varepsilon u)(x) = \lambda(\varepsilon)^{-1}u(x)$$

$$(\boldsymbol{G}v)(x) = \lambda^{-1}v(x)\,.$$

We want to compare $\boldsymbol{G}_\varepsilon$ and $\boldsymbol{G}$. It should be remarked that the Green operators $\boldsymbol{G}_\varepsilon$ and $\mathbf{G}$ act on different spaces $L^2(\Omega_\varepsilon)$ and $L^2(\Omega)$. One of technical difficulties arises from here.

In order to relate $\boldsymbol{G}_\varepsilon$ with $\boldsymbol{G}$, we introduce the operators $\boldsymbol{R}_\varepsilon$ and $\tilde{\boldsymbol{R}}_\varepsilon$. To describe integral kernel of $\boldsymbol{R}_\varepsilon$ and $\tilde{\boldsymbol{R}}_\varepsilon$, we put

$$\langle\nabla_w a(x, w), \nabla_w b(w, y)\rangle = \sum_{i=1}^{2} \frac{\partial}{\partial w_i} a(x, w) \frac{\partial}{\partial w_i} b(w, y)$$

for any a, $b\in C^1(\Omega\times\Omega\backslash(\Omega\times\Omega)_d)$, where $(\Omega\times\Omega)_d$ denotes the diagonal set of $\Omega\times\Omega$. Then, $\langle\nabla_w, \nabla_w\rangle$ is invariant under any orthogonal transformation of an orthonomal coordinates (w_1, w_2). We define

$$r_\varepsilon(x, y; w) = G(x, y) + 2\pi\varepsilon^2 \langle \nabla_w G(x, w), \nabla_w G(w, y)\rangle$$

and

$$r_\varepsilon(x, y) = r_\varepsilon(x, y; \tilde{w}) .$$

Also we set

$$\tilde{r}_\varepsilon(x, y) = G(x, y) + 2\pi\varepsilon^2 \langle \nabla_w G(x, w), \nabla_w G(w, y)\rangle_{|w=\tilde{w}} \xi_\varepsilon(x)\xi_\varepsilon(y) ,$$

where $\xi_\varepsilon \in C^\infty(\boldsymbol{R}^2)$ satisfies $0 \leq \xi_\varepsilon(x) \leq 1$, $\xi_\varepsilon(x) = 1$ for $x \in \boldsymbol{R}^2 \setminus \bar{B}_\varepsilon$ and $\xi_\varepsilon(x) = 0$ for $x \in B_{\varepsilon/2}$.

The operators $\boldsymbol{R}_\varepsilon$ and $\tilde{\boldsymbol{R}}_\varepsilon$ are defined by

$$(\boldsymbol{R}_\varepsilon g)(x) = \int_{\Omega_\varepsilon} r_\varepsilon(x, y) g(y) dy , \qquad x \in \Omega_\varepsilon ,$$

$$(\tilde{\boldsymbol{R}}_\varepsilon f)(x) = \int_\Omega \tilde{r}_\varepsilon(x, y) f(y) dy , \qquad x \in \Omega ,$$

respectively. Roughly speaking, $\boldsymbol{R}_\varepsilon$ is a very good approximation of $\boldsymbol{G}_\varepsilon$. By definition it is not difficult to compare $\boldsymbol{R}_\varepsilon$ with $\tilde{\boldsymbol{R}}_\varepsilon$. Since $\tilde{\boldsymbol{R}}_\varepsilon$ acts on $L^2(\Omega)$ and not on $L^2(\Omega_\varepsilon)$, we can easily compare $\tilde{\boldsymbol{R}}_\varepsilon$ with $\boldsymbol{G}$. As a consequence we can compare $\boldsymbol{G}_\varepsilon$ with $\boldsymbol{G}$.

Proof of Theorems 1, 2 are divided into several steps.

First we show

$$|||\boldsymbol{G}_\varepsilon - \boldsymbol{R}_\varepsilon|||_{L^2(\Omega_\varepsilon)} = 0(\varepsilon^{2-s})$$

for any fixed $s > 0$ as ε tends to zero. Here $|||\ \ |||_{L^p(\Omega_\varepsilon)}$ denotes the operator norm on $L^p(\Omega_\varepsilon)$. This will be done in the section 4.

Second we consider $\tilde{\boldsymbol{R}}_\varepsilon$ as a perturbation of $\boldsymbol{G}$. We construct an approximate eigenfunction $\psi^*(\varepsilon)$ and an approximate eigenvalue $\lambda^*(\varepsilon)$ of $\tilde{\boldsymbol{R}}_\varepsilon$. Here $\lambda^*(\varepsilon)$, $\psi^*(\varepsilon)$ are explicitly constructed by usual perturbation method so that they satisfy

$$||(\tilde{\boldsymbol{R}}_\varepsilon - \lambda^*(\varepsilon))\psi^*(\varepsilon)||_{L^2\Omega)} = 0(\varepsilon^4 |\log \varepsilon|^2)$$

and

$$||\psi^*(\varepsilon)||_{L^2(\Omega)} = 1 + 0(\varepsilon^2 |\log \varepsilon|) .$$

Since $\lambda^*(\varepsilon)$ and $\psi^*(\varepsilon)$ are constructed by perturbation theory, $\lambda^*(\varepsilon)$ is close to μ_j and $\psi^*(\varepsilon)$ is close to φ_j.

A key step is to examine the following decomposition of $\varphi_j(\varepsilon)$.

$$\varphi_j(\varepsilon) = \sum_{k=1}^{3} J_k(\varepsilon) ,$$

where

$$J_1(\varepsilon) = \mu_j(\varepsilon)(\boldsymbol{G}_\varepsilon - \boldsymbol{R}_\varepsilon)(\varphi_j(\varepsilon))$$
$$J_2(\varepsilon) = \mu_j(\varepsilon)\boldsymbol{R}_\varepsilon(\varphi_j(\varepsilon) - t_\varepsilon \chi_\varepsilon \psi^*(\varepsilon))$$
$$J_3(\varepsilon) = \mu_j(\varepsilon) t_\varepsilon \boldsymbol{R}_\varepsilon(\chi_\varepsilon \psi^*(\varepsilon)) .$$

Here χ_ε is the characteristic function of Ω_ε and

$$t_\varepsilon = \operatorname{sgn} \int_{\Omega_\varepsilon} (\varphi_j(\varepsilon))(x)\varphi_j(x)dx .$$

We can prove the following facts. Here s is an arbitrary fixed positive constant:

(2.1) $$\|J_1(\varepsilon)\|_{L^\infty(\Omega_\varepsilon)} + \|J_2(\varepsilon)\|_{L^\infty(\Omega_\varepsilon)} = 0(\varepsilon^{2-s}) .$$

(2.2) $$\|\mu_j(\varepsilon)^{-1} J_3(\varepsilon) - t_\varepsilon \mu_j \varphi_j\|_{L^\infty(\Omega_\varepsilon)} = 0(\varepsilon) .$$

(2.3) $$\max_{z \in \partial B_\varepsilon} |\operatorname{grad}_z (J_1(\varepsilon))(z)| = 0(\varepsilon^{(1/2)-s}) .$$

(2.4) $$\max_{z \in \partial B_\varepsilon} |\operatorname{grad}_z (J_2(\varepsilon))(z)| = 0(\varepsilon^{2-s}) .$$

(2.5) $$\left(\frac{\partial}{\partial \theta}(J_3(\varepsilon))(z)\right)_{|z=(\varepsilon\cos\theta,\ \varepsilon\sin\theta)}$$
$$= 2t_\varepsilon \mu_j(\varepsilon)\mu_j^{-1}\left(\frac{\partial}{\partial r}(\varphi_j(r\cos(\theta+(\pi/2)),\ r\sin(\theta+(\pi/2)))\right)_{|r=0} + 0(\varepsilon^{1-s}) .$$

These will be proved in the section 6.

Here we assume (2.1)~(2.5) and we would like to prove Theorems 1 and 2. From (2.1) and (2.2) we obtain

(2.6) $$\|\varphi_j(\varepsilon) - t_\varepsilon \mu_j(\varepsilon)\mu_j^{-1}\varphi_j\|_{L^\infty(\Omega_\varepsilon)} = 0(\varepsilon) .$$

It follows from (2.3), (2.4) and (2.5) that

(2.7) $$\mu_j(\varepsilon)^{-1}\left(\frac{\partial}{\partial \theta}(\varphi_j(\varepsilon))\right)(\varepsilon\cos\theta,\ \varepsilon\sin\theta)$$
$$= 2t_\varepsilon \mu_j^{-1} \frac{\partial}{\partial r}(\varphi_j(r\cos(\theta+(\pi/2)),\ r\sin(\theta+(\pi/2)))_{|r=0} + 0(\varepsilon^{(1/2)-s}) .$$

We put (2.6) and (2.7) into (1.6) and we obtain

(2.8) $$\mu_j(\varepsilon) - \mu_j = 0(\varepsilon^2) .$$

This together with (2.6) proves Theorem 1. Theorem 2 follows from (2.7) and (2.8).

Thus, our effort to get Theorems 1, 2 will be concentrated on showing (2.1)~(2.5). This will be completed in the section 6.

Before going further, we explain the reason why $r_\varepsilon(x, y)$ approximates $G_\varepsilon(x, y)$ well. Put

$$q_\varepsilon(x, y) = r_\varepsilon(x, y) - G_\varepsilon(x, y) .$$

Then,

$$\Delta_x q_\varepsilon(x, y) = 0 , \qquad x, y \in \Omega_\varepsilon$$
$$q_\varepsilon(x, y) = 0 , \qquad x \in \gamma, y \in \Omega_\varepsilon$$

and

$$\begin{aligned}(2.9)\qquad & \frac{\partial}{\partial \nu_x} q_\varepsilon(x, y)_{|x=(\varepsilon,0)} - \frac{\partial}{\partial x_1} G(x, y)_{|x=(\varepsilon,0)} \\ & -2\pi\varepsilon^2 \frac{\partial}{\partial x_1} \langle \nabla_w S(x, w), \nabla_w G(w, y) \rangle_{|w=\tilde{w}=0, x=(\varepsilon,0)} \\ &= 2\pi\varepsilon^2 \frac{\partial}{\partial x_1} \Big(\frac{1}{2\pi} \frac{\partial}{\partial w_1} \log|x-w| \cdot \frac{\partial}{\partial w_1} G(w, y) \\ & \quad + \frac{1}{2\pi} \frac{\partial}{\partial w_2} \log|x-w| \cdot \frac{\partial}{\partial w_2} G(w, y) \Big)_{|w=\tilde{w}=0, x=(\varepsilon,0)} \\ &= -\frac{\partial}{\partial w_1} G(w, y)_{|w=\tilde{w}=0} ,\end{aligned}$$

where $S(x, y) = G(x, y) + (1/2\pi) \log|x-y|$. And using (2.9) the $L^p(\Omega_\varepsilon)$-norm of the operator $\boldsymbol{G}_\varepsilon - \boldsymbol{R}_\varepsilon$ will be estimated in the section 4.

3. Preliminary lemmas

We recall the following:

Lemma 1 (Ozawa [5]). *Assume that $u_\varepsilon \in C^\infty(\bar{\Omega}_\varepsilon)$ is harmonic in Ω_ε, $u_\varepsilon(x)=0$ for $x \in \gamma$ and*

$$\max \{ |\partial u_\varepsilon(x)/\partial \nu| \, ; x \in \partial B_\varepsilon \} = M .$$

Then,

$$|u_\varepsilon(x)| \le C\varepsilon M(1 + |\log(|x-w|/\varepsilon)|) , \qquad x \in \Omega_\varepsilon$$

holds for a constant C independent of ε.

For any periodic function $\alpha(\theta)$ of $\theta \in [0, 2\pi]$ with the Fourier expansion

$$\alpha(\theta) = u_0 + \sum_{k=1}^{\infty} (u_k \sin k\theta + t_k \cos k\theta) ,$$

we put

$$K_\vartheta(\alpha) = \sum_{k=1}^{\infty} k^\vartheta (u_k^2 + t_k^2)^{1/2} .$$

Lemma 2. *Consider the equation*

$$(3.1)\qquad \Delta v(x) = 0 , \qquad x \in \boldsymbol{R}^2 \backslash \bar{B}_1$$

$$\frac{\partial v}{\partial \nu}(x)_{|x=(\cos\theta,\,\sin\theta)}=\alpha(\theta) \tag{3.2}$$

for given $\alpha(\theta)$. *Then, there exists at least one solution* v *of* (3.1), (3.2) *satisfying*

$$|v(x)|\leq C\max_{\theta}|\alpha(\theta)|(1+|\log|x||) \tag{3.3}$$

and

$$\max_{x\in\partial B_1}|\operatorname{grad} v(x)|\leq C_{\vartheta}(\max_{\theta}|\alpha(\theta)|)K_{\vartheta}(\alpha) \tag{3.4}$$

for $\vartheta\in(1, \infty)$.

Proof. We know that

$$u_0^2+\sum_{k=1}^{\infty}(u_k^2+t_k^2)\leq 2\pi\max_{\theta}|\alpha(\theta)|^2.$$

Put

$$v(x)=u_0\log r+\sum_{k=1}^{\infty}(-k)^{-1}(u_k\sin k\theta+t_k\cos k\theta)r^{-k}.$$

Then, $v(x)$ satisfies (3.1), (3.2), (3.3) and (3.4). q.e.d.

Lemma 3. *Fix* $q\in(1/2, \infty)$. *Then, under the same assumption as in Lemma* 1,

$$\max_{x\in\partial B_\varepsilon}|\operatorname{grad} u_\varepsilon(x)|\leq C\left(M+K_{2q}\left(\left(\frac{\partial u_\varepsilon}{\partial \nu}(z)\right)_{|z=(\varepsilon\cos\cdot,\,\varepsilon\sin\cdot)}\right)\right).$$

Proof. In the following we write $(\varepsilon\cos\theta, \varepsilon\sin\theta)=\varepsilon e(\theta)$.

Applying the similarity transformation of coordinates to Lemma 1, we have the following:

There exists at least one solution of

$$\Delta v_\varepsilon(x)=0, \qquad x\in \boldsymbol{R}^2\backslash\bar{B}_\varepsilon$$

$$\left(\frac{\partial v_\varepsilon}{\partial \nu_z}\right)(\varepsilon e(\theta))=\left(\frac{\partial u_\varepsilon}{\partial \nu_z}\right)(\varepsilon e(\theta)), \qquad \theta\in S^1\ (=\partial B_1)$$

satisfying

$$|v_\varepsilon(x)|_{x\in\partial B_\varepsilon}\leq C\varepsilon\max_{\theta}\left|\left(\frac{\partial u_\varepsilon}{\partial \nu}\right)(\varepsilon e(\theta))\right|(1+|\log(|x-\tilde{w}|/\varepsilon|))$$

and

$$\max_{\theta}|\operatorname{grad} v_\varepsilon(z)|_{z=\varepsilon e(\theta)}\leq C\left(\max_{\theta}\left|\left(\frac{\partial u_\varepsilon}{\partial \nu}\right)(\varepsilon e(\theta))\right|+K_{2q}\left(\left(\frac{\partial u_\varepsilon}{\partial \nu}\right)(\varepsilon e(\cdot))\right)\right)$$

for $q\in(1/2, \infty)$.

Then, the function v_ε may not satisfy $v_\varepsilon(x)=0$ for $x\in\gamma$. Overcome this difficulty, we apply the same argument as in Ozawa [5; Proposition 1], and we obtain the desired result. q.e.d.

We wish to replace the semi-norm $K_9(\alpha)$ by a Hölder norm. To do this we let $H^{q,2}(S^1)$ denote the L^2-Sobolev space of order q. Here q may not be an integer. It is well known that

$$C_1||\alpha||_{H^{q,2}(S^1)}\leq||\alpha||_{L^2(S^1)}+K_{2q}(\alpha)$$
$$\leq C_2||\alpha||_{H^{q,2}(S^1)}$$

holds for a constant C_1, C_2 independent of α if $q\geq 0$. We know that $H^{q,2}(S^1)$-norm of u is equivalent to the following norm:

$$||u||_{L^2(S^1)}+\left(\iint_{S^1\times S^1}|u(x)-u(y)|^2|x-y|^{-2q-1}dx\,dy\right)^{1/2}$$

when $0<q<1$. See, for example Adams [1]. Thus, we have

$$||u||_{H^{q,2}(S^1)}\leq C(||u||_{L^2(S^1)}+||u||_{C^{q+\sigma}(S^1)})$$

for any $\sigma>0$. Here $||\ \ ||_{C^\mu(S^1)}$ denotes the usual Hölder norm on S^1.

We know the interpolation inequality

$$||u||_{C^\mu(S^1)}\leq C||u||_{C^0(S^1)}^{1-(\mu/\tilde{\mu})}\,||u||_{C^{\tilde{\mu}}(S^1)}^{(\mu/\tilde{\mu})}$$

for any $0<\mu\leq\tilde{\mu}<1$.

Summing up these facts, we get

$$K_{2q}(\alpha)\leq C(||\alpha||_{L^2(S^1)}+||\alpha||_{C^0(S^1)}^{1-(\xi'/\xi)}||\alpha||_{C^\xi(S^1)}^{(\xi'/\xi)})$$

for $q\in(1/2, 1)$, $1/2<\xi'<\xi<1$.

Applying this to Lemma 3 we get the following

Corollary 1. Fix $1/2<\xi'<\xi<1$. Under the assumption of Lemma 1,

$$\max_{x\in\partial B_\varepsilon}|\text{grad}\,u_\varepsilon(x)|\leq C(M+M^{1-(\xi'/\xi)}L_\xi(\varepsilon)^{(\xi'/\xi)})\,. \tag{3.5}$$

Here

$$L_\xi(\varepsilon)=\left\|\left(\frac{\partial u_\varepsilon}{\partial \nu}\right)(z)_{|z=\varepsilon e(\cdot)}\right\|_{C^\xi(S^1)}\,.$$

4. Approximate Green's function $r_\xi(x, y)$

We use the following properties of the Green function frequently, so we here write them:

$$|G(x, y)|\leq C\,|\log|x-y||\tag{4.1}$$

$$|\nabla_x G(x, y)|\leq C\,|x-y|^{-1}\,.\tag{4.2}$$

Thus,

$$|(\boldsymbol{G}f)(x)| \leq C\|f\|_{L^p(\Omega)} \qquad (p>1) \tag{4.3}$$

$$|\mathrm{grad}_x(\boldsymbol{G}f)(x)| \leq C\|f\|_{L^p(\Omega)} \qquad (p>2)\,. \tag{4.4}$$

First we obtain the following

Lemma 5. *Let $p\in(2, \infty)$. Then, there exists a constant $C>0$ independent of ε such that*

$$|||\boldsymbol{R}_\varepsilon-\boldsymbol{G}_\varepsilon|||_{L^p(\Omega_\varepsilon)} \leq C\varepsilon^{2-(2/p)}|\log \varepsilon|\,.$$

Proof. Fix $f\in C_0^\infty(\Omega_\varepsilon)$. Then $g_\varepsilon=(\boldsymbol{R}_\varepsilon-\boldsymbol{G}_\varepsilon)f$ satisfies $\Delta g_\varepsilon(x)=0$ for $x\in\Omega_\varepsilon$ and $g_\varepsilon(x)=0$ for $x\in\gamma$.

By (2.9) we have

$$\left.\frac{\partial}{\partial\nu}g_\varepsilon(x)\right|_{x=(\varepsilon,0)} \tag{4.5}$$
$$=\frac{\partial}{\partial x_1}(\boldsymbol{G}f)(x)-\frac{\partial}{\partial w_1}(\boldsymbol{G}f)(w)+2\pi\varepsilon^2\frac{\partial}{\partial x_1}\langle\nabla_w S(x, w), \nabla_w(\boldsymbol{G}f)(w)\rangle$$

for $w=\tilde{w}$ $(=0)$.

By the Sobolev embedding theorem we have

$$\|\boldsymbol{G}f\|_{C^{1+\alpha}(\Omega)} \leq C\|f\|_{L^p(\Omega_\varepsilon)} \tag{4.6}$$

if $\alpha=1-(2/p)$, $2<p<\infty$. Here $\|\ \|_{L^p(\Omega_\varepsilon)}$ denotes the $L^p(\Omega_\varepsilon)$-norm. Therefore, (4.5) and (4.6) imply

$$\max_{x\in\partial B_\varepsilon}\left|\frac{\partial}{\partial\nu}g_\varepsilon(x)\right| \leq C\varepsilon^{1-(2/p)}\|f\|_{L^p(\Omega_\varepsilon)}\,.$$

By Lemma 1 we get the desired result. q.e.d.

The next lemma is stated in the introduction.

Lemma 6. *Fix $p\in(1, \infty]$. Then,*

$$|||\boldsymbol{R}_\varepsilon-\boldsymbol{G}_\varepsilon|||_{L^p(\Omega_\varepsilon)} = 0(\varepsilon^{2-s})$$

holds for any fixed $s>0$ as ε tends to zero.

Proof. Assume that $p\in(1, \infty)$. Put $\boldsymbol{Q}_\varepsilon=\boldsymbol{R}_\varepsilon-\boldsymbol{G}_\varepsilon$. The operator $\boldsymbol{Q}_\varepsilon$ is self-adjoint on $L^2(\Omega_\varepsilon)$. Thus, we get

$$|||\boldsymbol{Q}_\varepsilon|||_{L^q(\Omega_\varepsilon)} = |||\boldsymbol{Q}_\varepsilon|||_{L^{q'}(\Omega_\varepsilon)} \qquad (q^{-1}+(q')^{-1}=1)\,.$$

By the Riesz-Thorin interpolation theorem we know that

$$|||\boldsymbol{Q}_\varepsilon|||_{L^p(\Omega_\varepsilon)} \leq |||\boldsymbol{Q}_\varepsilon|||_{L^q(\Omega_\varepsilon)}$$

for any $p\in(q', q)$, $q>2$. We take sufficiently large $q>2$ and apply Lemma 5. Then we have Lemma 6 for $p\neq 1, \infty$.

Assume that $p=\infty$. Then, we get Lemma 6 with $p=\infty$ by the same argument as in the proof of Lemma 5. q.e.d.

Now we wish to compare $\boldsymbol{R}_\varepsilon$ with $\tilde{\boldsymbol{R}}_\varepsilon$. We denote by $\hat{\chi}_\varepsilon$ the characteristic function of the set B_ε. Then, $\hat{\chi}_\varepsilon=1-\chi_\varepsilon$.

We have the following

Lemma 7. *Let $p\in(1, \infty)$, $q\in(2, \infty)$ and $r\in(2, \infty)$. Then, there exists a constant C such that for any $v\in L^q(\Omega)$*

$$\begin{aligned}&||\tilde{\boldsymbol{R}}_\varepsilon v-\boldsymbol{R}_\varepsilon(\chi_\varepsilon v)||_{L^p(\Omega_\varepsilon)}\\ &\leq C(\varepsilon^{2-(2/q)}|\log\varepsilon|\,||v||_{L^q(\Omega)}+\varepsilon^{(2/r')}|\log\varepsilon|\,||v||_{L^r(B_\varepsilon)})\,.\end{aligned}$$

Proof. Put $k_\varepsilon=\chi_\varepsilon\tilde{\boldsymbol{R}}_\varepsilon v-\boldsymbol{R}_\varepsilon(\chi_\varepsilon v)$. Then, $\Delta_x k_\varepsilon(x)=0$ for $x\in\Omega_\varepsilon$ and $k_\varepsilon(x)=0$ for $x\in\gamma$.

We have

$$\begin{aligned}(4.7)\qquad &\frac{\partial}{\partial\nu}k_\varepsilon(x)_{|x=(\varepsilon,0)}\\ &=\frac{\partial}{\partial x_1}(\boldsymbol{G}(\hat{\chi}_\varepsilon v))(x)_{|x=(\varepsilon,0)}-\frac{\partial}{\partial w_1}(\boldsymbol{G}(\hat{\chi}_\varepsilon\xi_\varepsilon v))(\tilde{w})\\ &\quad+2\pi\varepsilon^2\frac{\partial}{\partial x_1}\langle\nabla_w S(x, w), \nabla_w\boldsymbol{G}(\hat{\chi}_\varepsilon\xi_\varepsilon v)(w)\rangle_{|x=(\varepsilon,0),w=\tilde{w}}\,.\end{aligned}$$

The first term minus the second term in the right hand side of (4.7) does not exceed

$$\varepsilon^\theta||\boldsymbol{G}(\hat{\chi}_\varepsilon v)||_{C^{1+\theta}(\Omega)}+\left|\frac{\partial}{\partial w_1}(\boldsymbol{G}(\hat{\chi}_\varepsilon(1-\xi_\varepsilon))v))(\tilde{w})\right|$$

for $\theta\in(0, 1)$. By (4.2) we see that

$$\begin{aligned}&|\nabla_w\boldsymbol{G}(\hat{\chi}_\varepsilon\xi_\varepsilon v)(\tilde{w})|+|\nabla_w\boldsymbol{G}(\hat{\chi}_\varepsilon(1-\xi_\varepsilon)v)(\tilde{w})|\\ &\leq C\varepsilon^{(2/r')-1}||v||_{L^r(B_\varepsilon)}\,,\end{aligned}$$

where $(r')^{-1}=1-r^{-1}$. Thus, Lemma 7 follows from these estimates and Lemma 1. q.e.d.

The following Lemma 8 asserts that $\varphi_j(\varepsilon)$ behaves well even in L^p space as ε goes to zero.

Lemma 8. *Fix j and $p\in(1, \infty]$. Then,*

$$\|\varphi_j(\varepsilon)\|_{L^p(\Omega_\varepsilon)} \leq C_p < \infty$$

holds for a constant C_p independent of ε.

Proof. We devide $\varphi_j(\varepsilon)$ as follows:

$$\varphi_j(\varepsilon) = \mu_j(\varepsilon)^{-1}(\boldsymbol{R}_\varepsilon \varphi_j(\varepsilon)) + \mu_j(\varepsilon)^{-1}((\boldsymbol{G}_\varepsilon - \boldsymbol{R}_\varepsilon)\varphi_j(\varepsilon))\,. \tag{4.8}$$

Rauch-Taylor [8] proved that

$$\lim_{\varepsilon\to 0} \mu_j(\varepsilon) = \mu_j\,. \tag{4.9}$$

By Lemma 6 we have

$$\|\mu_j(\varepsilon)^{-1}(\boldsymbol{G}_\varepsilon - \boldsymbol{R}_\varepsilon)\varphi_j(\varepsilon)\|_{L^p(\Omega_\varepsilon)} \leq 0(\varepsilon^{2-s})\|\varphi_j(\varepsilon)\|_{L^p(\Omega_\varepsilon)}\,.$$

This together with (4.8) proves that

$$\|\varphi_j(\varepsilon)\|_{L^p(\Omega_\varepsilon)} \leq C\|\boldsymbol{R}_\varepsilon \varphi_j(\varepsilon)\|_{L^p(\Omega_\varepsilon)}\,.$$

By the definition of $\boldsymbol{R}_\varepsilon$ we have

$$\|\boldsymbol{R}_\varepsilon \varphi_j(\varepsilon)\|_{L^p(\Omega_\varepsilon)} \leq C_p^*(1+\varepsilon|\log \varepsilon|^{1/2})\|\varphi_j(\varepsilon)\|_{L^2(\Omega_\varepsilon)}$$

for $p \in (1, \infty]$. Since $\varphi_j(\varepsilon)$ is a normalized eigenfunction we get the desired result. q.e.d.

5. An approximate eigenfunction of $\tilde{\boldsymbol{R}}_\varepsilon$

Let $\boldsymbol{G}_w$ denote the functional $v(x) \mapsto (\boldsymbol{G}v)(w)$. Put

$$A(\varepsilon)\colon v \mapsto 2\pi\langle \nabla_w G(\cdot, w), \nabla_w \boldsymbol{G}_w(\xi_\varepsilon v)\rangle|_{w=\tilde{w}}\,.$$

Then, $\tilde{\boldsymbol{R}}_\varepsilon = \boldsymbol{G} + \varepsilon^2 A(\varepsilon)$. We wish to construct an approximate eigenvalue $\lambda^*(\varepsilon)$ and an approximate eigenfunction $\psi^*(\varepsilon)$ of $\tilde{\boldsymbol{R}}_\varepsilon$ in such a way that

$$\|(\tilde{\boldsymbol{R}}_\varepsilon - \lambda^*(\varepsilon))\psi^*(\varepsilon)\|_{L^2(\Omega)} = o(\varepsilon^2) \tag{5.1}$$

and

$$\|\psi^*(\varepsilon)\|_{L^2(\Omega)} = 1 + O(\varepsilon^2|\log \varepsilon|) \tag{5.2}$$

By virtue of perturbation theory, we may take

$$\lambda^*(\varepsilon) = \mu_j^{-1} + \varepsilon^2 \lambda(\varepsilon)\,,$$

where $\lambda(\varepsilon) = (A(\varepsilon)\varphi_j, \varphi_j)_{L^2}$. Here $(\ ,\ )_{L^2}$ denotes the inner product on $L^2(\Omega)$. And we may assume that $\psi^*(\varepsilon)$ is of the form

$$\psi^*(\varepsilon) = \varphi_j + \varepsilon^2 \psi(\varepsilon)\,,$$

where $\psi(\varepsilon)$ should satisfy (5.3) and (5.4):

$$(5.3) \qquad (\boldsymbol{G}-\mu_j^{-1})\psi(\varepsilon) = (\lambda(\varepsilon)-A(\varepsilon))\varphi_j$$

$$(5.4) \qquad \int_\Omega (\psi(\varepsilon))(x)\varphi_j(x)dx = 0\,.$$

Note that $\boldsymbol{G}$ is a compact operator and that the right hand side of (5.3) is orthogonal to φ_j. Thus, the unique solution $\psi(\varepsilon)$ of (5.3), (5.4) exists.
We see that

$$(5.5) \qquad (\tilde{\boldsymbol{R}}_\varepsilon-\lambda^*(\varepsilon))\psi^*(\varepsilon) = \varepsilon^4(A(\varepsilon)-\lambda(\varepsilon))\psi(\varepsilon)\,.$$

To estimate the left hand sides of (5.1) and (5.2), we need the following

Lemma 9. *For a constant C independent of ε, we have*

$$(5.6) \qquad |||A(\varepsilon)|||_{L^p(\Omega)} \leq C\varepsilon^{(2-p)/p}|\log\varepsilon|^{1/2}\,, \qquad (p>2)$$

$$(5.7) \qquad |||A(\varepsilon)|||_{L^2(\Omega)} \leq C|\log\varepsilon|$$

and

$$||\psi(\varepsilon)||_{L^p(\Omega)} \leq C\varepsilon^{(2-p)/p}|\log\varepsilon|^{1/2}\,, \qquad (p>2)$$

$$||\psi(\varepsilon)||_{L^2(\Omega)} \leq C|\log\varepsilon|\,.$$

Proof. By a Hölder inequality and (4.1) we obtain (5.6) and (5.7). Using (5.7) we have

$$\begin{aligned}||(\lambda(\varepsilon)-A(\varepsilon))\varphi_j||_{L^2(\Omega)} &\leq C'|||A(\varepsilon)|||_{L^2(\Omega)} \\ &\leq C|\log\varepsilon|\,.\end{aligned}$$

Thus, by virtue of the Fredholm theory we obtain a bound for $L^2(\Omega)$-norm of $\psi(\varepsilon)$. Similarly we get L^p estimates. q.e.d.

By (5.5) and Lemma 9 we have the following fact, which is stronger than (5.1).

Lemma 10. *For a constant C independent of ε*

$$(5.8) \qquad ||(\tilde{\boldsymbol{R}}_\varepsilon-\lambda^*(\varepsilon))\psi^*(\varepsilon)||_{L^2(\Omega)} \leq C\varepsilon^4|\log\varepsilon|^2\,.$$

Since $\boldsymbol{G}_\varepsilon$ is approximated by $\boldsymbol{R}_\varepsilon$ (Lemma 6) and $\boldsymbol{R}_\varepsilon$ is approximated by $\tilde{\boldsymbol{R}}_\varepsilon$ (Lemma 7), we may consider $\psi^*(\varepsilon)$ as an approximate eigenfunction of $\boldsymbol{G}_\varepsilon$. More precisely we have

Lemma 11. *For a constant C independent of ε*

$$(5.9) \qquad ||(\boldsymbol{G}_\varepsilon-\lambda^*(\varepsilon))(\chi_\varepsilon\psi^*(\varepsilon))||_{L^2(\Omega_\varepsilon)} = 0(\varepsilon^{2-s})$$

holds, where s being an arbitrary fixed positive constant.

Proof. We see that the left hand side of (5.6) does not exceed

$$(5.10) \qquad \|(G_\varepsilon - R_\varepsilon)(\chi_\varepsilon \psi^*(\varepsilon))\|_{L^2(\Omega_\varepsilon)} + \|\tilde{R}_\varepsilon \psi^*(\varepsilon) - R_\varepsilon(\chi_\varepsilon \psi^*(\varepsilon))\|_{L^2(\Omega_\varepsilon)} + \|(\tilde{R}_\varepsilon - \lambda^*(\varepsilon))\psi^*(\varepsilon)\|_{L^2(\Omega_\varepsilon)} .$$

The last term is estimated by Lemma 10. By Lemma 7, the second term of (5.10) does not exceed

$$C\varepsilon^{2-(2/q)} |\log \varepsilon| \|\psi^*(\varepsilon)\|_{L^q(\Omega)} + C\varepsilon^{(2/r')} |\log \varepsilon| \|\psi^*(\varepsilon)\|_{L^r(B_\varepsilon)} .$$

We see from the definition of $\psi^*(\varepsilon)$ that

$$\|\psi^*(\varepsilon)\|_{L^r(B_\varepsilon)} \leq \|\varphi_j\|_{L^r(B_\varepsilon)} + \varepsilon^2 \|\psi(\varepsilon)\|_{L^r(B_\varepsilon)} .$$

We apply Lemma 9 to this and we have

$$\|\psi^*(\varepsilon)\|_{L^r(B_\varepsilon)} \leq C(\varepsilon^{3/r} + \varepsilon^{2+(2-r)/r} |\log \varepsilon|^{1/2})$$

for $r>2$. Thus, the second term of (5.10) is $0(\varepsilon^{2-s})$. The first term of (5.10) is also $0(\varepsilon^{2-s})$, since we have Lemma 6 and $\|\psi^*(\varepsilon)\|_{L^2(\Omega)} = 0(1)$. Summing up these facts we obtain (5.9). q.e.d.

The next Lemma states that $\mu_j(\varepsilon)$ is close to $\lambda^*(\varepsilon)$ and $\varphi_j(\varepsilon)$ is close to $\chi_\varepsilon \psi^*(\varepsilon)$.

Lemma 12. *Under the same assumption as in Theorem* 1

$$(5.11) \qquad \lambda^*(\varepsilon) - \mu_j(\varepsilon) = 0(\varepsilon^{2-s})$$

and

$$(5.12) \qquad \|\varphi_j(\varepsilon) - t_\varepsilon \chi_\varepsilon \psi^*(\varepsilon)\|_{L^2(\Omega_\varepsilon)} = 0(\varepsilon^{2-s})$$

hold.

Proof. We know from (5.9) and a spectral theory of compact self-adjoint operator that there exists at least one eigenvalue $\lambda_*(\varepsilon)$ of G_ε satisfying

$$\lambda_*(\varepsilon) - \lambda^*(\varepsilon) = 0(\varepsilon^{2-s}) .$$

Rauch-Taylor [8] showed that $\mu_k(\varepsilon)$ tends to μ_k as ε tends to zero for any k. Thus, we get $\lambda_*(\varepsilon) = \mu_j(\varepsilon)^{-1}$.

By the eigenfunction expansion

$$G_\varepsilon f = \sum_{k=1}^{\infty} \mu_k(\varepsilon)^{-1} \langle \varphi_k(\varepsilon), f \rangle \varphi_k(\varepsilon) ,$$

we have

$$\|(G_\varepsilon - \lambda^*(\varepsilon))(\chi_\varepsilon \psi^*(\varepsilon))\|^2_{L^2(\Omega_\varepsilon)} = \sum_{k=1}^{\infty} |\mu_k(\varepsilon)^{-1} - \lambda^*(\varepsilon)|^2 |\langle \varphi_k(\varepsilon), \chi_\varepsilon \psi^*(\varepsilon) \rangle|^2 .$$

Since $\lambda^*(\varepsilon) \to \mu_j^{-1}$ and $\mu_k(\varepsilon)^{-1} \to \mu_k^{-1}$ as $\varepsilon \to 0$, we have

$$\sum_{k=1,k\neq j}^{\infty} |\langle \varphi_k(\varepsilon), \chi_\varepsilon \psi^*(\varepsilon)\rangle|^2 = 0(\varepsilon^{4-2s}) .$$

This implies

$$||\chi_\varepsilon \psi^*(\varepsilon) - \langle \varphi_j(\varepsilon), \chi_\varepsilon \psi^*(\varepsilon)\rangle \varphi_j(\varepsilon)||_{L(\Omega_\varepsilon)} = 0(\varepsilon^{2-s}) .$$

Thus,

$$|\langle \varphi_j(\varepsilon), \chi_\varepsilon \psi^*(\varepsilon)\rangle^2 - 1| = 0(\varepsilon^{4-2s})$$

and we obtain (5.12). q.e.d.

6. Proof of (2.1)∼(2.5)

In this section we shall complete the proof of Theorems 1, 2 by giving proofs of (2.1)∼(2.5).

Recall the definition of $J_k(\varepsilon)$.

$$J_1(\varepsilon) = \mu_j(\varepsilon)(\boldsymbol{G}_\varepsilon - \boldsymbol{R}_\varepsilon)(\varphi_j(\varepsilon))$$
$$J_2(\varepsilon) = \mu_j(\varepsilon)\boldsymbol{R}_\varepsilon(\varphi_j(\varepsilon) - \chi_\varepsilon \psi^*(\varepsilon))$$
$$J_3(\varepsilon) = \mu_j(\varepsilon)\boldsymbol{R}_\varepsilon(\chi_\varepsilon \psi^*(\varepsilon)) .$$

Here we should state that we choose $\varphi_j(\varepsilon)$ so that $t_\varepsilon = 1$, because we see in the final part of the section 5 that $t_\varepsilon^2 = 1$ for small $\varepsilon > 0$.

Lemma 13. *Fix an arbitrary* $s>0$. *Then,*

$$||J_1(\varepsilon)||_{L^\infty(\Omega_\varepsilon)} = 0(\varepsilon^{2-s})$$

and (2.3) *hold.*

Proof. Let $\tilde{\varphi}_j(\varepsilon)$ be the extension of $\varphi_j(\varepsilon)$ to Ω putting its value zero on B_ε. We know that $J_1(\varepsilon)$ is harmonic in Ω_ε and zero on γ. We have

$$\begin{aligned}(6.1)\qquad & \mu_1(\varepsilon)\frac{\partial}{\partial \nu_z}(J_1(\varepsilon))(z)_{|z=\varepsilon e(\theta)} \\ &= \frac{\partial}{\partial r}((\boldsymbol{G}\tilde{\varphi}_j(\varepsilon)))(r\cos\theta, r\sin\theta)_{|r=\varepsilon} \\ &\quad - \frac{\partial}{\partial r}((\boldsymbol{G}\tilde{\varphi}_j(\varepsilon))(r\cos\theta, r\sin\theta))_{|r=0} \\ &\quad + 2\pi\varepsilon^2\left(\frac{\partial}{\partial r}\langle \nabla_w S(x, w), \nabla_w(\boldsymbol{G}\tilde{\varphi}_j(\varepsilon))(w)\rangle_{|x=\varepsilon e(\theta), w=\tilde{w}} \right.\end{aligned}$$

Thus, by the same argument as in the proof of Lemma 5 we have

$$(6.2)\qquad \max_{x\in\partial B_\varepsilon}\left|\frac{\partial}{\partial \nu}(J_1(\varepsilon))(x)\right| \le C\varepsilon^{1-(2/p)}||\varphi_j(\varepsilon)||_{L^p(\Omega_\varepsilon)}$$

for $p>2$. By Lemma 8 we see that (6.2) does not exceed $C'\varepsilon^{1-(2/p)}$. This fact together with Lemma 1 show that

$$||J_1(\varepsilon)||_{L^\infty(\Omega_\varepsilon)} = 0(\varepsilon^{2-s}) .$$

We now wish to apply Corollary 1 to $J_1(\varepsilon)$ to prove (2.3). We know that $S(x, w) \in C^\infty(\Omega)$. Then, $C^\xi(S^1)$ norm of the third term in the right hand side of (6.1) (considering it as a function of θ) does not exceed C. Here we used (4.4) and Lemma 8. By the fact

$$||\boldsymbol{G}f||_{C^{1+\xi}(\Omega)} \leq C||f||_{L^\infty(\Omega)} \qquad (\xi<1)$$

we see that the $C^\xi(S^1)$ norm of the first and the second term in the right hand side of (6.1) do not exceed C''_ξ for $\xi<1$. From Corollary 1 we obtain

$$\max_{z\in\partial B_\varepsilon} |\operatorname{grad}_z (J_1(\varepsilon))(z)| \leq C(\varepsilon^{1-s} + C_\xi(\varepsilon^{1-s})^{(1-(\xi/\xi'))}) . \tag{6.3}$$

We take $\xi'>1/2$, $\xi<1$ such that $|\xi'-1/2|+|\xi-1|$ is sufficiently small and we get (2.3). q.e.d.

We have the following

Lemma 14. *Fix an arbitrary* $s>0$. *Then*

$$||J_2(\varepsilon)||_{L^\infty(\Omega_\varepsilon)} = 0(\varepsilon^{2-s}) \tag{6.4}$$

and (2.4) *hold.*

Proof. Put $\chi_\varepsilon = \varphi_j(\varepsilon) - \chi_\varepsilon \psi^*(\varepsilon)$. Then, $J_2(\varepsilon) = \mu_j(\varepsilon)\boldsymbol{R}_\varepsilon \kappa_\varepsilon$. By the definition of $\boldsymbol{R}_\varepsilon$ and (4.2), (4.3) and (4.4) we have

$$||J_2(\varepsilon)||_{L^\infty(\Omega_\varepsilon)} \leq C(||\kappa_\varepsilon||_{L^2(\Omega_\varepsilon)} + \varepsilon||\kappa_\varepsilon||_{L^p(\Omega_\varepsilon)}) \tag{6.5}$$

for $p\in(2, \infty)$. Lemma 8 asserts that

$$||\kappa_\varepsilon||_{L^p(\Omega_\varepsilon)} \leq C', \qquad p\in(2, \infty) , \tag{6.6}$$

while Lemma 12 gives us the estimate

$$||\kappa_\varepsilon||_{L^2(\Omega_\varepsilon)} = 0(\varepsilon^{2-s}) . \tag{6.7}$$

Let s' be an arbitrary fixed number. Then, by the Riesz-Thorin interpolation theorem we get

$$||\kappa_\varepsilon||_{L^p(\Omega_\varepsilon)} = 0(\varepsilon^{2-s'}) \tag{6.8}$$

for $p>2$ close to 2. Thus, (6.4) is proved by (6.5), (6.6) and (6.7).

By the definition of $J_2(\varepsilon)$,

(6.9) $$|\partial_{x_i}\partial_{x_j}G(x, y)| \leq C|x-y|^{-2}$$

and (4.4) we have

$$\max_{z\in\partial B_\varepsilon}|\operatorname{grad}_z(J_2(\varepsilon))(z)| \leq C\|\kappa_\varepsilon\|_{L^p(\Omega_\varepsilon)}$$

for $p\in(2, \infty)$. Thus, (2.4) is proved by (6.8). q.e.d.

Finally we have the following

Lemma 15. *Fix an arbitrary* $s>0$. *Then,* (2.2) *and* (2.5) *hold.*

Proof. We see that $\mu_j(\varepsilon)^{-1}J_3(\varepsilon)$ can be written as $\Pi(\varepsilon)+\Pi'(\varepsilon)$. Here

$$\Pi(\varepsilon) = \boldsymbol{G}\varphi_j + 2\pi\varepsilon^2\langle\nabla_w G(\cdot, w), \nabla_w \boldsymbol{G}(\chi_\varepsilon\varphi_j)(w)\rangle_{|w=\tilde{w}}$$

and

$$\Pi'(\varepsilon) = \boldsymbol{G}((\chi_\varepsilon-1)\varphi_j)+\varepsilon^2\boldsymbol{G}(\chi_\varepsilon\psi(\varepsilon)) + 2\pi\varepsilon^4\langle\nabla_w G(\cdot, w), \nabla_w\boldsymbol{G}(\chi_\varepsilon\psi(\varepsilon))(w)\rangle_{|w=\tilde{w}}\,.$$

We have

(6.10) $$\|\Pi'(\varepsilon)\|_{L^\infty(\Omega_\varepsilon)} \leq C(\|\varphi_j\|_{L^p(B_\varepsilon)}+\varepsilon^2\|\psi(\varepsilon)\|_{L^r(\Omega)})$$

for $p>1$, $r>2$. Thus, (6.10) is estimated by Lemma 9 and we get

(6.11) $$\|\Pi'(\varepsilon)\|_{L^\infty(\Omega_\varepsilon)} = 0(\varepsilon^{2-s})$$

for any $s>0$.

On the other hand, by (4.4) we have

(6.12) $$\|\Pi(\varepsilon)-\mu_j^{-1}\varphi_j\|_{L^\infty(\Omega_\varepsilon)} = 0(\varepsilon)\,.$$

Thus, (6.11) and (6.12) imply (2.2).

We wish to show (2.5). By (4.4) and (6.9) we see that $\max\{|\operatorname{grad}_z(\Pi'(\varepsilon))(z)|\,;\, z\in\partial B_\varepsilon\}$ does not exceed

$$C(\|\varphi_j\|_{L^r(B_\varepsilon)}+\varepsilon^2\|\psi(\varepsilon)\|_{L^r(\Omega)})$$

for $r>2$. Thus,

(6.13) $$\max_{z\in\partial B_\varepsilon}|\operatorname{grad}_z(\Pi'(\varepsilon))(z)| = 0(\varepsilon^{1-s})$$

by Lemma 9. By the similar calculation as in (2.9) we see that

(6.14) $$\left(\frac{\partial}{\partial\theta}(2\pi\varepsilon^2\langle\nabla_w G(\cdot, w), \nabla_w(\boldsymbol{G}\varphi_j)(w)_{|w=\tilde{w}})\right)(\varepsilon\cos\theta, \varepsilon\sin\theta)$$

$$= \mu_j^{-1}\left(\frac{\partial}{\partial\theta}\varphi_j\right)(\varepsilon\cos\theta, \varepsilon\sin\theta)+0(\varepsilon^2)|\nabla_w(\boldsymbol{G}\varphi_j)(\tilde{w})|\,.$$

Thus,

$$(6.15)\qquad \left(\frac{\partial}{\partial\theta}(\Pi(\varepsilon)-\mu_j^{-1}\varphi_j)\right)(\varepsilon\cos\theta,\ \varepsilon\sin\theta)$$
$$=\mu_j^{-1}\left(\frac{\partial}{\partial\theta}\varphi_j\right)(\varepsilon\cos\theta,\ \varepsilon\sin\theta)+0(\varepsilon^2)||\varphi_j||_{L^r(\Omega)}$$
$$+0(1)|\nabla_w(G(\hat{\chi}_\varepsilon\varphi_j))(\tilde{w})|$$

for $r>2$. Thus, by Lemma 9, (4.4), (6.15) and

$$(6.16)\qquad \left(\frac{\partial}{\partial\theta}\varphi_j\right)(\varepsilon\cos\theta,\ \varepsilon\sin\theta)$$
$$=\frac{\theta}{\partial r}(\varphi_j(r\cos(\theta+(\pi/2)),\ r\sin(\theta+(\pi/2)))_{|r=0}+0(\varepsilon)\,,$$

we get (2.5). q.e.d.

We have thus proved all of (2.1)~(2.5) which were stated in the section 2. Therefore our proofs of Theorem 1 and 2 are complete.

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