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# RIEMANNIAN SUBMERSIONS OF SPHERES WITH TOTALLY GEODESIC FIBRES

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## Introduction

Let $M$ and $B$ be $C^\infty$ Riemannian manifolds. By a *Riemannian submersion* we mean a $C^\infty$ mapping $\pi: M\to B$ such that $\pi$ is of maximal rank and $\pi_*$ preserves the lengths of horizontal vectors, i.e., vectors orthogonal to the fibre $\pi^{-1}(x)$ for $x\in B$.

In his paper [7] Richard H. Escobales, JR dealt with the problem of classifying (upto equivalence) the Riemannian submersions of standard spheres $S^n(1)$ of unit radius in the Euclidean space $\boldsymbol{R}^{n+1}$ onto various Riemannian manifolds $B$ under the assumption that the fibres are *connected and totally geodesic.* He proved (Theorem 3.5 [7]) that as a fibre bundle, the Riemannian submersion $\pi: S^n\to B$ is one of the following types:

a) $$\begin{array}{ccc} S^1 & \dashrightarrow & S^{2n+1} \\ & & \downarrow \\ & & \boldsymbol{C}P^n \quad n\geq 2 \end{array}$$

b) $$\begin{array}{ccc} S^3 & \dashrightarrow & S^{4n+3} \\ & & \downarrow \\ & & \boldsymbol{H}P^n \quad \text{for } n\geq 2 \end{array}$$

c) $$\begin{array}{ccc} S^1 & \dashrightarrow & S^3 \\ & & \downarrow \\ & & S^2(\tfrac{1}{2}) \end{array}$$

d) $$\begin{array}{ccc} S^3 & \dashrightarrow & S^7 \\ & & \downarrow \\ & & S^4(\tfrac{1}{2}) \end{array}$$

e) $$\begin{array}{ccc} S^7 & \dashrightarrow & S^{15} \\ & & \downarrow \\ & & S^8(\tfrac{1}{2}) \end{array}$$

Further in cases (a) and (b), $B$ is isometric to complex and quaternionic projective space respectively of sectional curvature $K^*$ with $1\leq K^*\leq 4$. In cases (c), (d), (e), $B$ is isometric to a sphere of curvature 4 as indicated in the diagram.

He also proved uniqueness in the cases (a), (b), and (c) but left the cases (d) and (e) unsettled.

The aim of this note is to prove uniqueness in all the cases. In fact we will give an intrinsic proof which will cover all the cases simultaneously. We will also classify base manifolds without using Berger's pinching theorem (see [4] as done in [7]). That is to say that we shall directly check that the base manifold is locally Riemannian symmetric, simply connected and compact of rank 1 and then use our method again to exclude the Cayley projective plane $Ca(P^2)\approx F_4/\mathrm{Spin}\ (9)$ This way we will also avoid using Adam's result [1].

The paper is organized as follows:
In §1 we collect some general results on Riemannian submersions which are more or less well known in some form or other. In §2 we quote the crucial theorem of Escobales (Theorem 2.2 [7]) concerning equivalence of the Riemannian submersions with totally geodesic fibres and also another theorem which gives a criterion for a $\pi^{-1}(S)\subseteq M$ to be totally geodesic where $S\subseteq B$ is a submanifold. This result is proved in [8] (Theorem 2.5). In §3 we classify the possible base manifolds $B$. In §4 we prove the uniqueness and finally in §5, the nonexistence of any Riemannian submersion of $S^n(1)\rightarrow Ca(p^2)$.

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**1.** In this section we collect some general results on Riemannian submersions which we shall use later. We will use the notations mainly used by O'Niell in his fundamental paper [12] on Riemannian submersions. The only difference is that we use lower case letters to denote tangent vectors at a fixed point and the corresponding Roman capitals to denote their appropriate extensions to vector fields in a neighbourhood. Any Levi-Civita connection will be denoted by $\nabla$.

For a Riemannian submersion $\pi$: $M\rightarrow B$, we denote by $\underline{V}$ the vector subbundle of $TM$ consisting of vectors tangential to the fibres and call it "the vertical bundle". Similarly, its orthogonal complement is denoted $\underline{H}$ and called "the horizontal distribution".

Following O'Niell we also define tensors $T$ and $A$ associated to $\pi$ and defined on $M$. For arbitrary vector fields $E$, $F$ in $M$, put

$$T_E F=\mathcal{H}(\nabla_{\mathcal{V}(E)}\mathcal{V}(F))+\mathcal{V}(\nabla_{\mathcal{V}(E)}\mathcal{H}(F))$$

where $\mathcal{V}(E)$, $\mathcal{H}(E)$ etc. denote the vertical and horizontal components respectively of $E$ etc.

We also put

$$A_E F=\mathcal{H}(\nabla_{\mathcal{H}(E)}\mathcal{V}(F))+\mathcal{V}(\nabla_{\mathcal{H}(E)}\mathcal{H}(F))$$

$T$ and $A$ satisfy following properties as proved in [12].

(1) $T_E$ and $A_E$ are skew symmetric endomorphisms of $TM$ which interchange horizontal and vertical distributions.
(2) $T_E = T_{\mathcal{V}(E)}$ while $A_E = A_{\mathcal{H}(E)}$
(3) For $U$, $V$ vertical, $T_U\, V$ is symmetric namely $T_U V = T_V U$. On the other hand for $X$, $Y$ horizontal, $A_X Y$ is skew symmetric i.e. $A_X Y = -A_Y X$.

DEFINITION. A vector field $X$ on $M$ is called basic if

(i) $X$ is horizontal
(ii) For $p, q \in M$ with $\pi(p) = \pi(q)$, we have $\pi_{*p}(X) = \pi_{*q}(X)$. Thus we get a vector field $X_*$ on $B$ such that $X$ is $\pi$-related to $X_*$.

**Lemma 1.1** (O'Niell [12]). *For $X$, $Y$ basic fields on $M$, the following are true*:

(i) $\langle X, Y\rangle = \langle X_*, Y_*\rangle \circ \pi$
(ii) *$\mathcal{H}([X, Y])$ is basic and $\pi$related to $[X_*, Y_*]$*
(iii) *$\mathcal{H}(\nabla_X Y)$ is basic and $\pi$-related to $\nabla_{X_*}(Y_*)$*
*and* (iv) *For $V$ vertical, $[X, V]$ is vertical.*

**Lemma 1.2** (Escobales [7], O'Niell [12]). *Let $X$, $Y$ be horizontal fields and $V$, $W$ be vertical. Then each of the following holds*:

(i) $A_X Y = \frac{1}{2}\mathcal{V}([X, Y])$

(ii) $\nabla_V W = T_V W + \mathcal{V}(\nabla_V W)$

(iii) (a) $\nabla_V X = \mathcal{H}(\nabla_V X) + T_V X$

(b) *If $X$ is basic,* $\mathcal{H}(\nabla_V X) = A_X V$

(iv) $\nabla_X V = A_X V + \mathcal{V}(\nabla_X V)$

$\nabla_X Y = \mathcal{H}(\nabla_X Y) + A_X Y$

Since $\underline{H}$ and $\underline{V}$ are subbundles of $TM$, they are equipped with canonical connections coming from $TM$. We denote these connections by $\nabla^h$ and $\nabla^v$ respectively. i.e. $\nabla^h_E X = \mathcal{H}(\nabla_E X)$ for $X$ horizontal and $E$ any arbitrary vector field on $M$

Likewise $$\nabla^v_E V = \mathcal{V}(\nabla_E V)$$

Both of these connections are Riemannian.

On the other hand $\underline{H}$ can also be viewed as the pull back bundle $\pi^{-1}(TB)$ and it can be furnished with the pull back connection coming from below. We will denote the pull back connection by $\nabla^*$.

We then have the following lemma:

**Lemma 1.3.** *For $E$ an arbitrary vector field and $X$ horizontal*

$$\nabla^h_E X - \nabla^*_E X = A_X \mathcal{V}(E) .$$

Proof. Let $p \in M$ and $\pi(p) = b \in B$. Let $U$ be an open set around $b$ and $\{S^i\}$ an orthonormal framing of the tangent bundle $TB|_U$. Let $(\theta^i_j)$ be the connection matrix of one-forms

$$\text{i.e. } \nabla S^i = \sum_j \theta^i_j S^j \text{ on } U .$$

Let $\{\tilde{S}^i\}$ denote the corresponding basic framing of $\underline{H}|_{\pi^{-1}(U)}$. Then the connection matrix for $\nabla^*$ is $(\pi^*\theta^i_j)$, i.e. $\nabla^* \tilde{S}^i = \sum_j \pi^*(\theta^i_j) \tilde{S}^j$. Hence if $V$ is a vertical vector field defined on $\pi^{-1}(U)$

then $$\nabla^*_V \tilde{S}^i = 0$$

On the other hand $\nabla^h_V \tilde{S}^i = \mathcal{H}(\nabla_V \tilde{S}^i) = A_{\tilde{S}^i} V$ by Lemma 1.2 (iii) (b).

Therefore, $$\nabla^h_V \tilde{S}^i - \nabla^*_V \tilde{S}^i = A_{\tilde{S}^i} V \quad \forall i .$$

Since the left hand side of the above equation is tensorial (being the difference to two connections) $\tilde{S}^i$ can be replaced by an arbitrary section $X$ of $\underline{H}$. Hence for a vertical field $V$ and a horizontal field $X$,

$$\nabla^h_V X - \nabla^*_V X = A_X V .$$

Secondly, if $y$ is a horizontal vector at $p$, then

$$\nabla^*_y \tilde{S}^i = \sum_j \theta^i_j(\pi_* y) \tilde{S}^j$$

and $$\nabla^h_y \tilde{S}^i = \mathcal{H}(\nabla_y \tilde{S}^i)$$

Extending $y$ to a basic field $Y$ we get

$$\begin{aligned} \nabla^h_Y \tilde{S}^i = \mathcal{H}(\nabla_Y \tilde{S}^i) &= \text{basic field } \pi\text{-related to } \nabla_{\pi_* Y} S^i \\ &= \text{basic field } \pi\text{-related to } \sum_j \theta^i_j(\pi_* Y) S^j \\ &= \sum_j \theta^i_j(\pi_* Y) \tilde{S}^j \\ &= \nabla^*_Y \tilde{S}^i \end{aligned}$$

or $\nabla^h_Y \tilde{S}^i - \nabla^*_Y \tilde{S}^i = 0$

Again, the lefthand side being tensorial, the basic field $Y$ can be replaced by an arbitrary horizontal field $Y$ and equality will still hold.

In short we can write

$$\nabla^h_E X - \nabla^*_E X = A_X \mathcal{V}(E)$$

This proves the lemma.

**Corollary 1.1** (O'Niell [13]). *Let $\gamma: J \to B$ be a geodesic in $B$ where $J$ is an open interval around* 0 *in* $\boldsymbol{R}$. *Let $\gamma(0)=b$ and $p \in \pi^{-1}(b)$ be an arbitrary point in the fibre above $b$. If $\tilde{\gamma}$ is the horizontal lift of $\gamma$ passing through $p$, then $\tilde{\gamma}$ is also a geodesic in $M$.*

Proof. By definition of horizontal lifts, $\tilde{\gamma}'(t)$ is horizontal for every $t \in J$.

$$\begin{aligned} \text{Now } \nabla_{\tilde{\gamma}'(t)}\tilde{\gamma}'(t) &= \mathcal{H}(\nabla_{\tilde{\gamma}'(t)}\tilde{\gamma}'(t)) + \mathcal{V}(\nabla_{\tilde{\gamma}'(t)}\tilde{\gamma}'(t)) \\ &= \overset{h}{\nabla}_{\tilde{\gamma}'(t)}\tilde{\gamma}'(t) + A_{\tilde{\gamma}'(t)}\tilde{\gamma}'(t) \\ &= \overset{*}{\nabla}_{\tilde{\gamma}'(t)}\tilde{\gamma}'(t) + 0 \\ &= \text{Horizontal lift of } \nabla_{\gamma'(t)}\gamma'(t) \\ &= 0 \end{aligned}$$

Hence $\tilde{\gamma}$ is a geodesic.

**Corollary 1.2** (Hermann [11], O'Niell [13]). *If $\gamma$ is a geodesic in $M$ which is horizontal at a point, then $\gamma$ is horizontal throughout and $\pi \circ \gamma$ is a geodesic in $B$.*

Proof. Let $\gamma(0)=p$ be the point where $\gamma$ is given to be horizontal. Put $\gamma'(0)=x$. Let $\pi(p)=b$, $\pi_{*_q}(x)=x_*$ and $\tau$ be the geodesic in $B$ such that $\tau(0)=b$ and $\tau'(0)=x_*$.

Let $\tilde{\tau}$ be the horizontal lift of $\tau$ passing through $p$. By above corollary, $\tilde{\tau}$ is a geodesic satisfying

$$\tilde{\tau}(0)=p \quad \text{and} \quad \tilde{\tau}'(0)=x\,.$$

Hence $\tilde{\gamma}=\gamma$ i.e. $\gamma$ is horizontal everywhere and $\pi \circ \gamma = \pi \circ \tilde{\gamma} = \tau$ is a geodesic below.

We finish this section by collecting together the five basic equations of submersions derived by O'Niell in [12]. His curvature differs from ours by a sign. We denote by $R$ the curvature tensor of $M$, by $R$ that of fibres and by $R^*$ that of $B$. The horizontal lift of $R^*$ to $M$ will also be denoted $R^*$.

Let $U$, $V$, $W$, $F$ be vertical vector fields and $X$, $Y$, $Z$, $H$ be horizontal. Then the equations of submersions are

$$\{0\} \quad -\langle R_{UV}W, F\rangle = -\langle R^v_{UV}W, F\rangle - \langle T_UW, T_VF\rangle + \langle T_VW, T_UF\rangle$$

$$\{1\} \quad -\langle R_{UV}W, Y\rangle = \langle (\nabla_VT)_UW, X\rangle - \langle (\nabla_UT)_VW, X\rangle$$

$$\begin{aligned} \{2\} \quad -\langle R_{XV}Y, W\rangle &= \langle (\nabla_XT)_VW, Y\rangle + \langle (\nabla_VA)_XY, W\rangle - \langle T_VX, T_WY\rangle \\ &\quad + \langle A_XV, A_YW\rangle \end{aligned}$$

$$\begin{aligned} \{3\} \quad -\langle R_{XY}Z, V\rangle &= \langle (\nabla_ZA)_XY, V\rangle + \langle A_XY, T_VZ\rangle \\ &\quad - \langle A_YZ, T_VX\rangle - \langle A_ZX, T_VY\rangle \end{aligned}$$

$$\begin{aligned} \{4\} \quad -\langle R_{XY}Z, H\rangle &= -\langle R^*_{X,Y}Z, H\rangle - 2\langle A_XY, A_ZH\rangle \\ &\quad + \langle A_YZ, A_XH\rangle + \langle A_ZX, A_YH\rangle \end{aligned}$$

Equation {4} also gives us the sectional curvatures of $B$ in terms of sectional curvatures of $M$ and the tensor $A$ namely

$$K(x, y) = K^*(x, y) - 3|A_x y|^2 \quad \text{for two}$$

orthonormal horizontal vectors $x$ and $y$.

Proof. See O'Niell [12].

**2.** In this section we will state two theorems proved by Escobales which we will use in the sequel and also rewrite O'Niell's equations in the case of $T\equiv 0$. In fact from this section onwards we will exclusively deal with Riemannian submersions with totally geodesic fibres henceforth simply referred to as Riemannian submersions.

The equations of Riemannian submersions in the case $T\equiv 0$ become

{a} $\langle R_{UV}W, F\rangle = \langle R^v_{UV}W, F\rangle$

{b} $\langle R_{UV}W, X\rangle = 0$

{c} $-\langle R_{XV}Y, W\rangle = \langle(\nabla_V A)_X Y, W\rangle + \langle A_X V, A_Y W\rangle$

{d} $-\langle R_{XY}Z, V\rangle = \langle(\nabla_Z A)_X Y, V\rangle$

{e} $\langle R_{XY}Z, H\rangle = \langle R^*_{XY}Z, H\rangle + 2\langle A_X Y, A_Z H\rangle - \langle A_Y Z, A_X H\rangle - \langle A_Z X, A_Y H\rangle$

Next we state the key result of Escobales which gives a criterion for equivalence of two Riemannian submersions with totally geodesic fibres.

**Theorem 2.1** (Theorem 2.2 [7]). *Let $\pi_i (i=1, 2)$ be Riemannian submersions from a connected complete Riemannian manifold $M$ onto another Riemannian manifold $B$. Assume that the fibres of $\pi$ are connected and totally geodesic in $M$. Suppose $\varphi$ is an isometry of $M$ which satisfies the following two properties at a given point $p\in M$.*

(1) *$\varphi_{*p}$: $T_pM \to T_{\varphi(p)}M$ maps $\underline{H}_{1p}$ onto $\underline{H}_{2\varphi(q)}$ where $\underline{H}_i$ denotes the horizontal distribution of $\pi_i$.*

(2) *For $e, f\in T_pM$, $\varphi_*(A_{1e}f) = A_{2\pi_*(e)}(\varphi_* f)$ where $A_i$ is integrability tensor of $\pi_i$. Then $\varphi$ induces an isometry $\underline{\varphi}$ of $B$ so that $\pi_1$ and $\pi_2$ become equivalent.*

Proof. See [7]

Finally in this section we state the following:

**Theorem 2.2** (Theorem 2.5 [8]). Let $\pi$: $M\to B$ a Riemannian submersion with totally geodesic fibres. Assume $P$ is a totally geodesic submanifold of $B$. Then $\pi^{-1}(P)$ is totally geodesic provided $A_Y X=0$ whenever $X$ is horizontal and tangent to $\pi^{-1}(P)$ and $Y$ is normal to $\pi^{-1}(P)$.

Proof. See [8]

**3.** In this section we determine the possibilities for $B$ when $M$ is of constant sectional curvature and prove that $B$ is necessarily locally symmetric of rank 1.

First we derive two equations from the fundamental equation {d} cf. §2.

**Lemma 3.1.** (1) $\mathcal{V}((\nabla_x A)_y(z)) = -\mathcal{V}(R_{y,z}(x))$
(2) $\mathcal{H}((\nabla_x A)_y(v)) = \mathcal{H}(R_{x,v}(y))$

*Here as usual $x, y, z$ are horizontal vectors at a point $p \in M$ and $v$ is vertical at $p$.*

Proof. (1) It is just the equation {d}.
(2) Let $z$ be horizontal at $p$. Extend $x, y, z$ to basic fields $X$, $Y$, $Z$ such that $\mathcal{H}(\nabla_e X)=0$ etc. for every horizontal vector $e$ at $p$. (This means that the corresponding fields $X_*$, $T_*$, $Z_*$ on $B$ are covariant constant at $\pi(p)$). Also extend $v$ to a vertical $V$ covariant constant at $\pi(P)$ as a section of $\underline{V}$ i.e. $\nabla_f^v V = \mathcal{V}(\nabla_f V)=0$ $\forall f \in T_p M$. Now eqn {d} is

$$-\langle R_{YZ}X, V\rangle = \langle (\nabla_X A)_Y Z, V\rangle$$

or restricting our attention at point $p$

$$\begin{aligned}
-\langle R_{x,v}y, z\rangle &= \langle \nabla_x(A_Y Z) - A_{\nabla_x Y}(z) - A_y(\nabla_x Z), v\rangle \\
&= \langle \nabla_x(A_Y Z), v\rangle - 0 (\text{since } (\mathcal{H}(\nabla_x Y) = 0 = \mathcal{H}(\nabla_x Z)) \\
&= \langle \nabla_x^v(A_Y Z), v\rangle \\
&= x\langle A_Y Z, V\rangle - \langle A_y z, \nabla_x^v(V)\rangle \\
&= -x\langle Z, A_Y V\rangle - 0 \\
&= -\langle \nabla_x^h Z, A_y v\rangle - \langle z, \nabla_x^h(A_Y V)\rangle \\
&= 0 - \langle z, \nabla_x(A_Y V)\rangle \ (\text{since } z \text{ is horizontal}) \\
&= -\langle z, (\nabla_x A)_y(v) + A_{\nabla_x Y}(v) + A_y(\nabla_x V)\rangle \\
&= -\langle z, (\nabla_x A)_y(v)\rangle + 0 + 0 \\
&\qquad (\text{since } \mathcal{H}(\nabla_x Y) = 0 = \mathcal{V}(\nabla_x V))
\end{aligned}$$

Thus we get

$$\mathcal{H}((\nabla_x A)_y(v)) = \mathcal{H}(R_{x,v}(y)) \ \text{ which is equation (2).}$$

**Corollary.** *If $M$ is of constant sectional curvature*

$$\mathcal{V}((\nabla_x A)_y(z)) = 0 \tag{3.1}$$

$$\mathcal{H}((\nabla_x A)_y(v)) = 0 \tag{3.2}$$

Proof. Use $R_{x,y}z=c(\langle y, z\rangle x-\langle x, z\rangle y)$ $c$ a constant for any $x, y, z\in T_pM$.

**Theorem 3.1.** *If $M=S^n(1)$ and $\pi: M\to B$ has fibres connected and totally geodesic, then $B$ is necessarily compact, simply connected and locally symmetric of rank* 1. *Here $n\geqslant 3$.*

Proof. We set about differentiating the curvature tensor $R^*$ of $B$. Equivalently we can differentiate its horizontal lift to $M$ using the pull-back connection on $\underline{H}$ in the direction of horizontal vectors. Further for a horizontal vector $e\in T_pM$, $\nabla^h_e=\nabla^*_e$ on $\underline{H}$ as we know from Lemma 1.3.

Hence for $x, y, z, e\in \underline{H}_p$

$$(\nabla^*_e R^*)_{x,y}(z)=(\nabla^h_e R^*)_{x,y}(z)\,.$$

As usual we extend $x, y, z$ to basic vector fields $X, Y, Z$ $\mathcal{H}(\nabla_e X)=\mathcal{H}(\nabla_e Y)$ $=\mathcal{H}(\nabla_e Z)=0$ for any vector $e\in \underline{H}_p$.

From equation {e} of §2

$$\begin{aligned} R^*_{XY}Z &= \mathcal{H}(R_{XY}Z)+2A_ZA_XY-A_XA_YZ-A_YA_ZX \\ &= \langle Y, Z\rangle X-\langle X, Z\rangle Y+2A_ZA_XY-A_XA_YZ-A_YA_ZX\,. \end{aligned}$$

Also
$$\begin{aligned} (\nabla^*_e R^*)_{x,y}(z) &= (\nabla^h_e R^*)_{x,y}(z)=\nabla^h_e(R^*_{XY}Z) \\ &= [\nabla^h_e(\langle Y, Z\rangle X)-\nabla^h_e(\langle Z, X\rangle Y)]+2\nabla^h_e(A_Z(A_XY)) \\ &\quad -\nabla^h_e(A_X(A_YZ))-\nabla^h_e(A_Y(A_ZX)) \\ &= 0-0+2\mathcal{H}\{(\nabla_e A)_z(A_x y)+A_{\nabla_e z}(A_x y)+A_z(\nabla_e(A_XY)) \\ &\quad -\text{similar terms} \\ &= 2\,\mathcal{H}\{0+0+A_z(\nabla_e A_XY))\}-\text{similar terms} \\ &\quad \text{(from eqn. 3.2 and the fact that } \mathcal{H}(\nabla_e Z)=0) \\ &= 2\,A_z\mathcal{V}((\nabla_e A_XY))-\text{similar terms} \\ &= 2\,A_z\mathcal{V}((\nabla_e A)_x(y)+A_{\nabla_e x}(y)+A_x(\nabla_e Y))-\text{similar terms} \\ &= 2\,A_z(0+0+0)-\text{similar terms} \\ &\quad \text{(from eqn. 3.1 and the fact that } \mathcal{H}(\nabla_e X)=\mathcal{H}(\nabla_e Y)=0) \\ &= 0 \end{aligned}$$

This proves that $B$ is locally symmetric. Clearly $B$ is compact and connected and it is simply connected because the fibres are connected and dim $S^n(1)\geqslant 3$. (This follows from long exact sequence of homotopy. We recall that since $S^n(1)$ is complete $\pi: S^n(1)\to B$ is a fibration as proved by R. Hermann in [11].)

It remains to prove that $B$ is of rank 1. It is also easy. Take $\gamma: \boldsymbol{R}\to B$ any geodesic in $B$. Let $\tilde{\gamma}$ be its horizontal lift to $S^n(1)$. Then $\tilde{\gamma}$ is also a geodesic (see Cor. 1 to Lemma 1.3) and hence is periodic. Therefore, $\gamma$ is also periodic. Thus every geodesic of B is periodic which means that $B$ is of rank 1.

From classification of compact, simply connected, Riemannian symmetric spaces of rank 1, we conclude that B can only be one of the following:

(1) $S^m(r)$, $m \geqslant 2$

(2) $CP^m(k)$, $m \geqslant 2$

(3) $\boldsymbol{H}P^m(k)$, $m \geqslant 2$

(4) Cayley projective plane $Ca(P^2)$ $(k) \approx F_4/\mathrm{Spin}$ (9).

Here $S^m(r)$ means sphere of radius $r$

$CP^m(k)$ means that the sectional curvature lies between $k$ and $4k$ with both extremes attained. Similarly, for $\boldsymbol{H}P^m(k)$ and $Ca(P^2)$ $(k)$.

We discuss (1), (2) and (3) now. (4) will be discussed in the last section. First we prove an important equation which will be used again and again later also.

**Proposition 3.1.** *Let $p \in M$ and $x, y \in \underline{H}_p$ and $v \in \underline{V}_p$. Then*

$$(A_x A_y + A_y A_x)(v) = -\mathcal{V}(R_{v,x}y + R_{v,y}x)$$

*Here we continue to assume that $T \equiv 0$.*

Proof. Extend $x$, $y$ uniquely to basic fields $X$, $Y$ defined all along the fibre through $p$.

$$\begin{aligned}
\text{Then at } p\ \langle \nabla_v(A_X Y), v\rangle &= \langle (\nabla_v A)_x(y) + A_{\nabla_v x}(y) + A_x(\nabla_v Y), v\rangle \\
&= \langle (\nabla_v A)_x(y), v\rangle + \langle A_{A_x v}y, v\rangle + \langle A_x A_y v, v\rangle \\
&\qquad\qquad \text{(using Lemma 1.2 (3) (b))} \\
&= \langle (\nabla_v A)_x(y), v\rangle - \langle A_y A_x v, v\rangle + \langle A_x A_y v, v\rangle \\
&= \langle (\nabla_v A)_x(y), v\rangle + \langle A_x v, A_y v\rangle - \langle A_y v, A_x v\rangle \\
&= \langle (\nabla_v A)_x(y)), v\rangle \\
&= \langle R_{v,x}(y), v\rangle - \langle A_x v, A_y v\rangle \\
&\qquad\qquad \text{(using eqn. \{c\} of §1)}
\end{aligned}$$

Now lefthand side is skew symmetric in $X$ and $Y$ while the righthand side is symmetric. Hence both vanish simultaneously. Thus

$$\langle \nabla_v(A_X Y), v\rangle = 0 \text{ for } X,\ Y \text{ basic and } v \text{ vertical} \qquad (*)$$

$$\begin{aligned}
\text{and} \quad & \langle R_{v,x}y, v\rangle = \langle A_x v, A_y v\rangle \\
\Rightarrow \quad & \langle R_{v,x}y, w\rangle + \langle R_{w,x}y, v\rangle = \langle A_x v, A_y w\rangle + \langle A_x w, A_y v\rangle \\
& \qquad\qquad \text{for every } x, y \in \underline{H}_p \text{ and } v, w \in \underline{V}_p \\
\text{or} \quad & \langle R_{v,x}y, w\rangle + \langle R_{v,y}x, w\rangle = -\langle (A_x A_y + A_y A_x)(v), w\rangle \\
\text{Hence} \quad & (A_x A_y + A_y A_x)(v) = -\mathcal{V}(R_{v,x}(y) + R_{v,y}(x))
\end{aligned}$$

**Corollary 3.1** (Bishop [5]). *If $X$, $Y$ are basic fields on $M$ then $A_X Y$ is a* *killing field along the fibres.*

Proof. The equality (*) in the proof of the above theorem.

**Corollary 3.2.** *For* $M=S^n(1)$

$$(A_x A_y + A_y A_x)(v) = -2\langle x, y\rangle v$$

*and* $A_x^2 v = -|x|^2 v$

Proof. Obvious.

Now we go back to our classification of base manifolds.

(1) *The case* $S^n(1)\to S^m(c)$. Using the long exact sequence of homotopy we find that the fibre is forced to be $S^{m-1}(\pi_{m-1}(\text{Fibre})\approx \boldsymbol{Z})$. So that $n=2m-1$.

Now let $p\in S^{2m-1}$ and $x\in \underline{H}_p$ be of unit norm. Thus for $v\in \underline{H}_p$, $A_x^2 v=-v$ (by Cor. 2 to Prop. 3.1). Hence if we consider

$$A_x\colon \underline{V}_p \to \underline{H}_p \text{ we see that}$$
$$|A_x v|^2 = \langle A_x v, A_x v\rangle = -\langle A_x^2 v, v\rangle = |v|^2$$

So $A_x$ is an isometric embedding of $\underline{V}_p$ in $\underline{H}_p$. Since $\dim \underline{H}_p = \dim \underline{V}_p + 1$ $(m=(m-1)+1)$ and $A_x(\underline{V}_p)$ is orthogonal to $x$, we see that $A_x\colon \underline{V}_p \to x^\perp$ is an isometric isomorphism.

(Here and elsewhere if $x\in \underline{H}_p$, then $x^\perp$ is its orthogonal complement in $\underline{H}_p$ only)

Also $A_x\colon x^\perp \to \underline{V}_p$ is s.t. the composite

$\underline{V}_p \xrightarrow{A_x} x^\perp \xrightarrow{A_x} \underline{V}_p$ is equal to $-1$. Hence

$A_x\colon x^\perp \to \underline{V}_p$ is also an isometric isomorphism.

Now put $\pi(p)=b\in S^m$ and choose an orthonormal basis $\{x_1, \cdots, x_m\}$ of $T_b S^m$. These give rise to basic fields $\{X_1, \cdots, X_m\}$ along the fibre $\pi^{-1}(b)$ and trivialization of the bundle $\underline{H}|_{\pi^{-1}(b)}$

Put $$V_i = A_{X_m} X_i \quad i=1, 2, \cdots, m-1$$

Clearly, $\{V_1, \cdots, V_{m-1}\}$ is an orthonormal framing of $\underline{V}|_{\pi^{-1}(b)} \approx T(\pi^{-1}(\mathrm{b}))$.

Hence the tangent bundle of $\pi^{-1}(b)\approx S^{m-1}$ is trivial. This implies that "$m-1=1$, 3 or 7". See [2].

Thus the only Riemannian submersions of the spheres onto spheres are

$$S^3(1)\to S^2(c),\quad S^7(1)\to S^4(c) \text{ and } S^{15}(1)\to S^8(c)$$

where $c$ denotes the radius.

In each case $c=\frac{1}{2}$ as can be computed by O'Niell's formula for sectional curvatures of base. This has been done in [7]. One simply uses the fact that if $x \perp y$ are horizontal and of unit norm, then $|A_x y|^2 = \langle A_x y, A_x y\rangle = -\langle A_x^2 y, y\rangle = \langle y, y\rangle = 1$. (Here $A_x^2 y = -y$ since any $y \in x$ can be written as $A_x v$ for a unique $v \in \underline{V}_p$ of same norm as that of $y$.)

(2) *The case* $S^n(1) \to CP^m(k)$. Again by long exact sequence we get $n = 2m+1$ and using O'Neill's formula $k=1$ See [7].

(3) Same analysis holds for $HP^m$ and we get $n = 4m+3$

(4) For Cayley plane $Ca(P^2)$, dimension of the fibre is 7 and that of total space is 23.

In both of the above categories also $k=1$ i.e. the sectional curvature lies between 1 and 4 with both extreme values attained.

REMARK.

(1) $S^3(1) \to S^2\left(\frac{1}{2}\right)$ falls in $S^{2n+1} \to CP^n$ category, $n \geq 1$

(2) $S^7(1) \to S^4\left(\frac{1}{2}\right)$ falls in $S^{4n+3} \to HP^n$ category, $n \geq 1$

and (3) $S^{15}(1) \to S^8\left(\frac{1}{2}\right)$ falls in $S^{8n+7}(1) \to Ca(P^n)$ category $n = 1, 2$

**4.** We have put Riemannian submersions of spheres with connected totally geodesic fibres into three categories

$$\text{(a)}\quad \begin{array}{l} S^1 \to S^{2n+1} \\ \qquad\quad \downarrow \pi \\ \qquad\quad CP^n,\ n \geq 1 \end{array}$$

$$\text{(b)}\quad \begin{array}{l} S^3 \to S^{4n+3} \\ \qquad\quad \downarrow \pi \\ \qquad\quad HP^n,\ n \geq 1 \end{array}$$

$$\text{(c)}\quad \begin{array}{l} S^7 \to S^{8n+7} \\ \qquad\quad \downarrow \\ \qquad\quad Ca(p^n),\ n = 1, 2\,. \end{array}$$

We now prove that each member of the above three categories is unique upto equivalence. We also know that except for the category (c), $n=2$ each of the above Riemannian submersions exists. Later we shall prove the nonexistence in the above exceptional case. But first we come to uniqueness.

**Theorem 4.1.** *For each allowable n in each of the above category there exists at most one Riemannian submersion upto equivalence.*

Proof. For definiteness we will prove the uniqueness in the case of $S^{4n+3} \to HP^n$, $n \geq 1$ and the method will make it obvious how to proceed for $S^{2n+1} \to CP^n$ and $S^{8n+7} \to Ca(p^n)$ $(n=1, 2)$.

Let $p \in S^{4n+3}$. As usual $\underline{H}_p$ and $\underline{V}_p$ are horizontal and vertical subspaces of $T_pS^{4n+3}$. Let us define a map $\mathfrak{A}$: $\underline{V}_p \to \text{End } \underline{H}_p$ as follows:

$$\langle \mathfrak{A}(v)(x), y \rangle = \langle A_x v, y \rangle$$

we shall denote $\mathfrak{A}(v)$ by $A^v$

Thus $$A^v x = A_x v \quad \text{for } x \in \underline{H}_p\,.$$

It is trivial to check that $A^v$ is skew symmetric.

**Claim.** *$v \to A^v$ has the following property*:

$$A^v A^w + A^w A^v = -2\langle v, w \rangle Id\,.$$

Proof of the claim. Let $\langle x, y \rangle \in \underline{H}_p$. Then

$$\begin{aligned}
\langle (A^v A^w + A^w A^v)(x), y \rangle &= \langle A^v(A_x w) + A^w(A_x v), y \rangle \\
&= -\langle A_x w, A_y v \rangle - \langle A_x v, A_y w \rangle \\
&= \langle A_y A_x w, v \rangle + \langle A_x A_y w, v \rangle \\
&= -2\langle x, y \rangle \langle v, w \rangle \text{ by Corollary 3.2}\,. \\
&= \langle -2\langle v, w \rangle x, y \rangle
\end{aligned}$$

Hence $(A^v A^w + A^w A^v)(x) = -2\langle v, w \rangle x$

This proves the claim.

But this is precisely the condition for $\mathfrak{A}$ to extend to a representation of the Clifford algebra $C(\underline{V}_p)$ of the Euclidean space $\underline{V}_p$. Since $\underline{V}_p$ is 1, 3 or 7 dimensional $C(\underline{V}_p)$ has at most two types of irreducible representations depending on the action of the centre $v_1 v_2 v_3$.

(Here $\{v_1, v_2, v_3\}$ is an ordered orthonormal basis of $\underline{V}_p$) (See [6]). See also remarks at the end.

Also each of this simple module is of dim 4. Since $C(\underline{V}_p)$ is central simple and $\mathfrak{A}$ is a real representation it breaks $\underline{H}_p$ into a simple modules.

We will check that $\mathfrak{A}$ is an isotypical representation i.e. only one type of simple module occurs $n$ times in $\underline{H}_p$.

For this let $x \in \underline{H}_p$ with $|x| = 1$

The centre acts by $A^{v_1} \circ A^{v_2} \circ A^{v_3}$.

Consider the function $x \to \langle A^{v_1} A^{v_2} A^{v_3} x, x \rangle$ defined on the unit sphere in $\underline{H}_p$.

$$\begin{aligned}
\langle A^{v_1} A^{v_2} A^{v_3} x, x \rangle &= -\langle A^{v_2} A^{v_3} x, A^{v_1} x \rangle \\
&= -\langle A_{A_x v_3} v_2, A_x v_1 \rangle = \langle A_x A_{A_x v_3}(v_2), v_1 \rangle
\end{aligned}$$

Now $A_x A_{A_x} v_3(v_2) \in \underline{V}_p$

We check that it is a multiple of $v_1$

$$\text{(i)}\quad \langle A_x A_{A_x v_3}(v_2), v_2\rangle = -\langle A_{A_x v_3}(v_2), A_x v_2\rangle = -\langle A^{v_2}(A_x v_3), Ax^{v_2}x\rangle$$
$$= \langle A_x v_3, (A^{v_2})^2 x\rangle = \langle A_x v_3, -x\rangle = 0$$
$$\text{(ii)}\quad \langle A_x A_{A_x v_3}(v_2), v_3\rangle = -\langle A_{A_x v_3}(v_2), A_x v_3\rangle$$
$$= \langle v_2, A_{A_x v_3}(A_x v_3)\rangle = 0$$

Hence $A_2 A_{A_x v_3}(v_2)$ is a multiple of $v_1$

Also $A_x A_{A_x v_3}(v_2) = -A_{A_x v_3}(A_x v_2) - 2\langle x, A_x v_3\rangle v_2$ (Corollary 3.2)
$$= -A_{A_x v_3}(A_x v_2)\,.$$

We claim that $A_{A_x v_3}(A_x v_2)$ is of unit norm and relegate its proof to the next lemma. Assuming this conclude that

$$A_x A_{A_x v_3}(v_2) = +v_1 \text{ or } -v_1$$
$$\text{i.e. } \langle A_x A_{A_x v_3}(v_2), v_1\rangle = +1 \text{ or } -1$$
$$\text{or } \langle A^{v_1} A^{v_2} A^{v_3} x, x\rangle = +1 \text{ or } -1 \text{ for every } x$$

of unit norm in $\underline{H}_p$.

By continuity we conclude that $\langle A^{v_1} A^{v_2} A^{v_3} x, x\rangle$ is a constant which without loss of generality we can assume to be$+1$. In case it is $-1$ we can simply replace the ordered basis $\{v_1, v_2, v_3\}$ by $\{-v_1, v_2, v_3\}$.

Also $A^{v_1} A^{v_2} A^{v_3}$ is an isometry of $\underline{H}_p$, we conclude by Cauchy-Schwarz inequality that $A^{v_1} A^{v_2} A^{v_3} x = x$ or $A^{v_1} A^{v_2} A^{v_3} = +Id$

This proves that $\underline{H}_P$ is an isotypical $C(\underline{V}_p)$ module.

If $S^{4n+3} \xrightarrow{\pi'} HP^n$ is another Riemannian submersion then choose $q \in S^{4n+3}$ and consider horizontal and vertical spaces $\underline{H}'_q$ and $\underline{V}'_q$. Define $L_1$: $\underline{V}'_q \to \underline{V}_p$ an isometry such that $C(\underline{V}'_q)$ acts on $\underline{H}_p$ with its centre also acting as $+Id$. Thus we get the diagram

$$\begin{tikzcd} C(\underline{V}_p) \arrow[r, "\mathfrak{A}"] & \text{End}(\underline{H}_p) \\ C(\underline{V}'_p) \arrow[u, "C(L_1)"] \arrow[r, "\mathfrak{A}'"] & \text{End}(\underline{H}'_q) \end{tikzcd}$$

Since both $\underline{H}_p$ and $\underline{H}'_q$ become $C(\underline{V}'_q)$ modules of dimension $4n$ on which the centre acts by $+Id$. There exists an interwining isometry $L_2$: $\underline{H}'_q \to \underline{H}_p$ which makes the above diagram commute.

By definitions of $\mathfrak{A}$ and $\mathfrak{A}'$ we see that

$L_1 \oplus L_2$: $T_q S^{4n+3} \to T_p S^{4n+3}$ is an isometry which maps $\underline{H}'_q$ to $\underline{H}_p$ and maps $A$ onto $A'$. $L_1 \oplus L_2$ gives a unique isometry $\varphi$: $S^{4n+3} \to S^{4n+3}$ sending $q$ to $p$ s.t. $\varphi_{*q} = L_1 \oplus L_2$. Thus theorem 2.1 applies and we get the equivalence.

**Lemma 4.1.** $|A_{A_x v_3} A_x v_2|^2 = 1$ *where the notation is an explained in the* *proof of uniqueness.*

Proof. Put $S\subset \underline{H}_p$ the linear span of $\{x, A_x v_1, A_x v_2, A_x v_3\}$
For each $i=1, 2, 3$, $|A_x(A_x v_i)|^2=1$

Hence by O'Niell's formula we get

$$K^*(\pi_* x, \pi_* A_x v_i) = K(x, A_x v_i)+3|A_x(A_x v_i)|^2$$
$$= 1+3 = 4$$

On the other hand by geometry of $\boldsymbol{HP}^n$, there exists a unique totally geodesic projective line $\boldsymbol{HP}^1$ passing through $\pi(p)$ s.t.

(i) $\pi_* x\in T_{\pi(p)}\boldsymbol{HP}^1$

(ii) For orthonormal vectors $y, z\in T_{\pi(p)}\boldsymbol{HP}^1$

$$K^*(y, z) = 4$$

(iii) For $y\in T_{\pi(p)}(\boldsymbol{HP}^1)$ and $z\in(T_{\pi(p)}\boldsymbol{HP}^1)^{\perp}$

$$K^*(y, z) = 1\,.$$

All this together implies that $\pi_* S=T_{\pi(p)}\boldsymbol{HP}^1$
In particular $K^*(\pi_*(A_x v_2), \pi_*(A_x v_3))=4$
which gives $|A_{A_x v_3}(A_x v_2)|^2=1$

Some remarks are in order:

1) For $S^{(8n+7)}\to Ca(P^n)$ $(n=1, 2)$ same proof goes through. In the case of $S^{2n+1}\to \boldsymbol{C}P^n$, dim $\underline{V}_p=1$, and hence $C(\underline{V}_p)\approx \boldsymbol{C}$ therefore the reprn is automatically isotypical.

2) The cases $n=1$ for (a) (b) and (c) actually become simpler because the relevant Clifford modules are already irreducible so that one need not check isotypicality.

3) The method is intrinsic and does not need explicit calculations of the tensor $A$ in the standard cases done by Gray [9] and used by Escobales [7].

4) The uniqueness of the Riemannian submersion $S^7\to S^4$ also proves the uniqueness of the Riemannian submersion $\boldsymbol{C}P^1 \dashrightarrow \boldsymbol{C}P^3\to \boldsymbol{H}P^1$ left unsettled in [8].

## 5. Existence questions

**Proposition 5.1.** *The Riemannian submersion $\pi: S^{23}(1)\to Ca(P^2)$ does not exist.*

Proof. Assume that it exists. Take $p\in S^{23}$. As proved earlier $\underline{H}_p$ becomes $C(\underline{V}_p)$ module having two irreducible components each of dimension 8. Let $x\in\underline{H}_p$ be of unit norm. Then there exists a unique 8 dimensional subspace $S\subset\underline{H}_p$ with $x\in S$ such that $\pi_*(S)$ is tangential to the unique Cayley projective line $Ca(P^1)\approx S^8(\frac{1}{2})$ passing through $\pi(p)$ and containing $\pi_*(x)$ as its tangent vector.

If $y \in (\pi_* S)^\perp$ then by geometrical properties of $Ca(P^2)$, $K^*(\pi_* x, y) = 1$. This means that if $\tilde{y}$ is the horizontal lift of $y$ in $\underline{H}_p$, it belongs to $S^\perp$ in $\underline{H}_p$ and by O'Niell's equation for sectional curvatures

$$K^*(\pi_* x, y) = 1 = 1 + 3\ |A_x \tilde{y}|^2$$
$$\text{or } A_x \tilde{y} = 0$$

This implies on using theorem 2.2 that $\pi^{-1}(Ca(P^1))$ is totally geodesic in $S^{23}$ and hence is isomorphic to $S^{15}(1)$. Also $p \in S^{15}(1)$ and $T_p S^{15} = \underline{V}_p \oplus S$. The restricted submersion $S^{15} \to Ca(P^1)$ becomes the standard one with $S$ as its horizontal space at $p$. Therefore $S$ is a $C(\underline{V}_p)$ module and can easily be checked to be an invariant submodule of $\underline{H}_p$ under the original $C(\underline{V}_p)$ action. Hence we have proved that for any $x \in \underline{H}_p$, $x \neq 0$ there exists an irreducible submodule of $\underline{H}_p$ passing through $x$. This is a contradiction if dim $\underline{V}_p \geqslant 4$. Hence a submersion $S^{23} \to Ca(P^2)$ cannot exist.

## Addendum. Riemannian submersions of complex projective space

In [8] Escobales classified Riemannian submersions from complex projective spaces under the assumptions that the fibres are connected, complex totally geodesic submanifolds, he proved;

**Theorem 1** (Theorem 5.2 [8]). *Any submersion $\rho\colon CP^r \to B$ with connected complex totally geodesic fibres with $2 \leqslant$ dim fibre $\leqslant 2r-2$ must fall into one of the following two classes:*

$$\text{(i)}\quad \begin{array}{ccc} S^2 & \dashrightarrow & CP^{2n+1} \\ & & \downarrow \\ & & HP^n \end{array}$$

$$\text{(ii)}\quad \begin{array}{ccc} CP^3 & \dashrightarrow & CP^7 \\ & & \downarrow \\ & & S^8(\tfrac{1}{2}) \end{array}$$

If we assume that the sectional curvatures $K$ of the total spaces are normalized so that $1 \leqslant K \leqslant 4$ then the sectional curvatures $K^*$ of base also lie in the same limits in case (i) and $K^* = 4$ in case (ii). Moreover for $n \geqslant 2$ any two members of class (i) are equivalent.

The author left two question unsettled

1) Whether $\begin{array}{ccc} S^3 & \dashrightarrow & CP^3 \\ & & \downarrow \\ & & HP^1 \approx S^4(\tfrac{1}{2}) \end{array}$ is unique or not

2) Whether the class (ii) is nonempty or not.

We have already remarked in §4, Remark no. 4 uniqueness of the Riemannian submersion $S^7 \to S^4(\frac{1}{2})$ implies an affirmative answer to $Q1$.

Here we give an answer to the $Q2$ in the negative.

**Main Theorem.** *There can not exist a Riemannian submersion of $CP^7$ onto $S^8(\frac{1}{2})$ such that the fibres are connected totally geodesic and isomorphic to $CP^3 \subset CP^7$.*

Proof. Suppose the submersion in question exists. Take the composite $S^{15}(1) \xrightarrow{\pi} CP^7 \xrightarrow{\rho} S^8(\frac{1}{2})$. $\rho \cdot \pi$ is also a Riemannian submersion. If $b \in S^8(\frac{1}{2})$ then $\pi^{-1}\rho^{-1}(b) = \pi^{-1}(CP^3) = S^7(1)$ is totally geodesic. (See [8] Coroallry 2.6. Here $\pi$ is the natural submersion). Therefore $\rho \circ \pi$ is a Riemannian submersion of $S^{15}(1)$ to $S^8(\frac{1}{2})$ with connected totally geodesic fibres. Hence by uniqueness it must be the Hopf fibration.

Thus we have the following commutative diagram:

$$\begin{tikzcd} S^7 \arrow[r, dashed] \arrow[d, "\pi"] & S^{15} \arrow[r, "H"] \arrow[d, "\pi"] & S^8(\frac{1}{2}) \\ CP^3 \arrow[r, dashed] & CP^7 \arrow[ur, "\rho"'] & \end{tikzcd}$$

Equivalently, there exists a complex structure $J$ on $\boldsymbol{R}^{16}$ in which $S^{15}$ sits such that if $b \in S^8(\frac{1}{2})$, then $\pi^{-1}\rho^{-1}(b) \approx S^7$ generates an $\boldsymbol{R}^8$ which is stable under $J$.

We will prove that such a situation is not possible. For this we look at the geometry of the Hopf fibration $S^7 \longrightarrow S^{15} \xrightarrow{H} S^8$.

Let $\sigma$: Spin (9) $\to SO(16)$ be the spin representation of Spin(9). It is known that Spin (9) acts transitively on $S^{15}(1) \subset \boldsymbol{R}^{16}$. Also Spin (9) acts transitively on $S^8(\frac{1}{2})$ via the natural double covering map Spin (9)$\to SO(9)$.

For any $b \in S^8(\frac{1}{2})$, the isotropy $G_b$ in Spin (9) is conjugate to the standard embedding of Spin (8) in Spin (9). Hence if we restrict $\sigma | G_b$, then $\sigma | G_b$ breaks $\boldsymbol{R}^{16}$ into two $\frac{1}{2}$ spin representations. Let us call them $\boldsymbol{R}^8_{\pm}$. Further $\boldsymbol{R}^8_+ \cap S^{15} = H^{-1}(b)$.

Let $\boldsymbol{R}^9$ be the oriented Euchidean space in which $S^8(\frac{1}{2})$ sits. If we regard Spin (9) to be sitting inside the Clifford algebra $C(\boldsymbol{R}^9)$, then $b^\perp \subset \boldsymbol{R}^9$ of dimension 8 is such that $C(b^\perp) \cap \text{Spin}(9) = G_b$. For definiteness we give the unique orientation to $b^\perp$ which is such that on adjoining $+b$ we get back the original orientation on $\boldsymbol{R}^9$.

Now we have the following commutative diagram

$$\begin{tikzcd} G_b \arrow[r, hook] \arrow[d, hook] & C^+(b^\perp) \arrow[d, hook] \\ \text{Spin}(9) \arrow[r, hook] & C^+(\boldsymbol{R}^9) \end{tikzcd}$$

where $C^+$ denotes the even component of $C$.
Since $\sigma$ is the spin representation, it extends to a unique representation $C(\sigma)$ of $C^+(\boldsymbol{R}^9)\approx M(16, \boldsymbol{R})$ (See [3]) such that $C(\sigma)|_{C^+(b^\perp)}$ breaks $\boldsymbol{R}^{16}$ in the above mentioned manner. Let $\{e_1, \cdots, e_8\}$ be an oriented basis of $b^\perp$ then $z'=e_1, \cdots, e_8$ lies in the centre of $C^+(b)$ and $z'$ acts by $+Id$ on $\boldsymbol{R}^8_+$ and by $-Id$ on $\boldsymbol{R}^8_-$ (see Chevally: Theory of Spinors). More, precisely $C(\sigma)$ $(z')=\pm 1$ on $\boldsymbol{R}^8_\pm$.

Since $\boldsymbol{R}^8_+ \cap S^{15}=H^{-1}(b)$, $\boldsymbol{R}^8_+$ is invariant under $J$ and so is $\boldsymbol{R}^8_-$. Thus $C(\sigma)$ $(z')$ commutes with $J$.

Let $z\in C(\boldsymbol{R}^9)$ be the generator of the centre of $C(\boldsymbol{R}^9)$ which comes from the chosen orientation of $\boldsymbol{R}^9$,

$$\begin{aligned} &\text{then} \quad z = e_1\cdot\cdots\cdot e_8\, b \\ &\text{or} \qquad zb = -e_1\cdot\cdots\cdot e_8 = -z'\,. \end{aligned}$$

Therefore, $C(\sigma)$ $(zb)$ commutes with $J$ for every $b\in S^8(\frac{1}{2})$ and hence for every $b\in \boldsymbol{R}^9$.

Consider the linear map

$$\begin{aligned} \alpha\colon \boldsymbol{R}^9 &\to M(16, \boldsymbol{R}) \\ b &\mapsto C(\sigma)\,(zb) \end{aligned}$$

It has following properties:

i) It factors through $M(8, \boldsymbol{C})\subseteq M(16, \boldsymbol{R})$ where $(\boldsymbol{R}^{16}, J)$ has been regarded as $\boldsymbol{C}^8$

ii) $[C(\sigma)\,(zb)]^2=C(\sigma)\,((zb)^2)=C(\sigma)\,(|b|^2)=|b|^2.\ Id.$

Hence $\alpha$ extends to a homomorphism
$C'(\alpha)$: $C'(\boldsymbol{R}^9)\to M(8, \boldsymbol{C})$ where $C'(\boldsymbol{R}^9)$ is the Clifford algebra corresponding to the quadratic form $b\to -|b|^2$. (See [3]).

But from [3] again we know that $C'(\boldsymbol{R}^9)\approx M(16, \boldsymbol{R})\oplus M(16, \boldsymbol{R})$ so that the above homomorphism is impossible to exist. This gives us the required contradiction.

REMARK. The situation where $\boldsymbol{C}P^7\to S^8$ is topologically possible has not been ruled out yet. It may be of interest to study it.

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