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ON THE CENTRALIZER OF THE LAPLACIAN OF $P_n(C)$ AND THE SPECTRUM OF COMPLEX GRASSMANN MANIFOLD $G_{2,n-1}(C)$

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0. Introduction. The purpose of the present paper is two-fold. The first half of which is to determine the centralizer of the Laplacian Δ of the complex projective space $\boldsymbol{P}_n(\boldsymbol{C})$ with the Fubini-Study metric g_0 and the other is to calculate explicitly the spectrum of the Grassmann manifold $\boldsymbol{G}_{2,n-1}(\boldsymbol{C})$ with the canonically normalized invariant metric g_1, as well as to give an explicit eigenspace decomposition of the Laplacian $\Delta^\wedge$ on $C^\infty(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$ as a complex analogue of our previous paper [5].

For this purpose we begin with some preliminaries on the algebra $\mathfrak{D}^*(\boldsymbol{P}_n(\boldsymbol{C}))$ of complex linear differential operators as well as the graded algebra $\boldsymbol{S}^*(\boldsymbol{P}_n(\boldsymbol{C}))$ (resp. bigraded algebra $\boldsymbol{S}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$) of complex contravariant symmetric tensor fields on $\boldsymbol{P}_n(\boldsymbol{C})$.

The centralizer of Δ in $\mathfrak{D}^*(\boldsymbol{P}_n(\boldsymbol{C}))$ is determined in **2**. Theorem 2.1 asserts that it coincides with the subalgebra of $\mathfrak{D}^*(\boldsymbol{P}_n(\boldsymbol{C}))$ generated by all Killing vector fields. The Killing algebra $\boldsymbol{K}^*(\boldsymbol{P}_n(\boldsymbol{C}))$ is introduced as the graded subalgebra of $\boldsymbol{S}^*(\boldsymbol{P}_n(\boldsymbol{C}))$ generated by all Killing vector fields. We also define the Plücker algebra: $\boldsymbol{K}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))=\boldsymbol{K}^*(\boldsymbol{P}_n(\boldsymbol{C}))\cap \boldsymbol{S}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$. In **3** the Radon-Michel transform $^\wedge$: $\boldsymbol{S}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))\to C^\infty(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$ is introduced. It has the following remarkable properties:

(i) $^\wedge$ commutes with the Lichnerowicz operator in the sense of Theorem 3.2.

(ii) The Plücker algebra $\boldsymbol{K}^{**}(\boldsymbol{P}_n(\boldsymbol{C})$ is transformed by $^\wedge$ onto the subalgebra of $C^\infty(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$ generated by normalized Plücker coordinates.

Theorem 2.1 enables us to obtain an eigenspace decomposition of the Lichnerowicz operator restricted to $\boldsymbol{K}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ (Theorem 4.1). In virtue of Theorem 3.2 the eigenspace decomposition of $\Delta^\wedge$ is obtained by transferring that of the Lichnerowicz operator in $\boldsymbol{S}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ to $C^\infty(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$ by means of the Radon-Michel transform (Theorem 4.2).

Finally, in the appendix we give a sequence of the differential opreators, annihilating eigenfunctions of the Laplacian Δ of $\boldsymbol{P}_n(\boldsymbol{C})$.

1. Let N be the ordered monoid of non-negative integers and let $\boldsymbol{N}^2 = \boldsymbol{N}\times \boldsymbol{N}$ be the product monoid of two copies of $\boldsymbol{N}$. Let $K=(k, l)\in \boldsymbol{N}^2$ and $K'=(k', l')\in \boldsymbol{N}^2$. An order $\geqq$ is defined in $\boldsymbol{N}^2$ by letting $K\geqq K'$ if and only if $k\geqq k'$ and $l\geqq l'$. Define two order preserving maps $|\ |$ and $!$: $\boldsymbol{N}^2\to \boldsymbol{N}$ by $\boldsymbol{N}^2\ni K\mapsto |K| \overset{\text{def.}}{=} k+l\in \boldsymbol{N}$ and $\boldsymbol{N}^2\in K\mapsto K! \overset{\text{def.}}{=} k!\cdot l!\in \boldsymbol{N}$ for $K=(k, l)$. Put $I=(1, \cdots, n)$ and $\bar{I}=(\bar{1}, \cdots, \bar{n})$. $\Gamma=(A, \bar{B})=(\alpha_1\cdots\alpha_k, \bar{\beta}_1\cdots\bar{\beta}_l)\in I^k\times \bar{I}^l$ is called a *multi-index* of bidegree $K=(k, l)$. K is also denoted by $\sharp\Gamma$. Henceforth the convention of dummy indices will be adopted.

Let M be a complex manifold of complex dimension n and let $C^\infty(M)$ be the algebra of complex valued C^∞-functions on M. A complex linear differential operator of order p on M is locally expressed in terms of multi-indices as

$$(1.1)\qquad D=\sum_{|\sharp\Gamma|\leqq p}(1/(\sharp\Gamma)!)\xi^\Gamma\partial^{|K|}/(\partial z)^\Gamma (=\sum_{0\leqq k+l\leqq p}(1/(k!\cdot l!)) \sum_{\alpha_1,\cdots,\alpha_k,\beta_1,\cdots,\beta_l=1}^{n} \xi^{\alpha_1\cdots\alpha_k,\bar{\beta}_1\cdots\bar{\beta}_l}\partial^{k+l}/\partial x^{\alpha_1}\cdots\partial z^{\alpha_k}\partial\bar{z}^{\beta_1}\cdots\partial\bar{z}^{\beta_l})$$

with respect to the local coordinates $z^1, \cdots, z^n$. Notice that β_i and $\bar{\beta}_i$ are confused in (1.1). Let $\mathfrak{D}^p(M))$ be the $C^\infty(M)$-module of complex linear differential operators of order p on M. Put $\mathfrak{D}^*(M)=\bigcup_{p\geqq 0}\mathfrak{D}^p(M)$. Let $\boldsymbol{S}^K(M)$ $(=\boldsymbol{S}^{k,l}(M))$ be the $C^\infty(M)$-module of contravariant complex symmetric tensor fields of bidegree $K=(k, l)$ on M. $\xi\in \boldsymbol{S}^K(M)$ is locally expressed in terms of multi-indices as

$$(1.2)\qquad \xi=(1(/K!)\sum_{\sharp\Gamma=K}\xi^\Gamma(\partial/\partial z)^\Gamma$$
$$(=(1/(k!\cdot l!))\sum_{\alpha_1\cdots\alpha_k,\beta_1\cdots\beta_l=1}^{n}\xi^{\alpha_1\cdots\alpha_k,\bar{\beta}_1\cdots\bar{\beta}_l}\partial/\partial z^{\alpha_1}\circ\cdots\circ\partial/\partial z^{\alpha_k}\circ\cdots$$
$$\circ\partial/\partial\bar{z}^{\beta_1}\circ\cdots\circ\partial/\partial\bar{z}^{\beta_l})=(1/(k!\cdot l!))\sum_{\sharp(A,\bar{B})=(k,l)}\xi^{A,\bar{B}}(\partial/z)^A\circ(\partial/\partial\bar{z})^B$$

with respect to the symmetric tensor product $\circ$. Henceforth we will use the notation of summations in (1.1) and (1.2).

Put

$$(1.3)\qquad \boldsymbol{S}^p(M)=\sum_{|K|=p}\boldsymbol{S}^K(M)\qquad\text{(direct sum)}.$$

Define a map σ^p: $\mathfrak{D}^p(M)\to \boldsymbol{S}^p(M)$ by $\sigma^p D=\sum_{|K|=p}\sum_{\sharp\Gamma=K}\xi^\Gamma(\partial/\partial z)^\Gamma$, where $D\in\mathfrak{D}^p(M)$ is as in (1.1). Let ι^p: $\mathfrak{D}^{p-1}(M)\to\mathfrak{D}^p(M)$ be the canonical inclusion for $p\geqq 0$. Put $f^p=\sigma^{p+1}$, $g^p=\iota^{p+1}$ and $L^*(M)=\bigcup_{p\geqq 0}L^p(M)$, where $L^p(M)=\mathfrak{D}^{p+1}(M)$ for $p\geqq -1$, $L^p(M)=\{0\}$ for $p\leqq -2$. Then the sequence of $C^\infty(M)$-modules:

$$0\to L^{p-1}(M)\xrightarrow{g^p}L^p(M)\xrightarrow{f^p}\boldsymbol{S}^{p+1}(M)\to 0$$

is exact. As $L^*(M)$ is a filtered Lie algebra (cf. [5], 2), $S^*(M) \overset{\text{def.}}{=} \sum_{p \geqq 0} S^p(M)$ (direct sum) is viewed as the graded Lie algebra assoicated with $L^*(M)$. $S^*(M)$ is also regarded as a bigraded algebra $S^{**}(M) \overset{\text{def.}}{=} \sum_{K \geqq 0} S^K(M)$ (direct sum) with respect to the symmetric tensor product, where $K=0$ means $(k, l)=(0, 0)$. Notice that $\xi \circ \eta = \sigma^{p+q}(D_1 D_2)$ for $\xi \in S^p(M)$ and $\eta \in S^q(M)$ with $D_1 \in \mathfrak{D}^p(M)$ and $D_2 \in \mathfrak{D}^q(M)$ such that $\sigma^p D_1 = \xi$ and $\sigma^q D_2 = \eta$. The bracket product of $S^*(M)$ inherited from that of $L^*(M)$ is given by $[\xi, \eta] = \sigma^{p+q-1}[D_1, D_2]$ for $\xi \in S^p(M)$, $\eta \in S^q(M)$ and $D_i (i=1, 2)$ as above (cf. [5], 1). The componentwise expression of $\xi \circ \eta$ for $\xi \in S^K(M)$ and $\eta \in S^{K'}(M)$ ($K=(k, l)$ and $K'=(k', l')$) is

$$\xi \circ \eta = \sum_{\sharp \Gamma = K''} (1/K'')(\xi \circ \eta)^{\Gamma} \circ (\partial/\partial z)^{\Gamma}, \tag{1.4}$$

where the summation is as in (1.2) for $K''=K+K'=(k+k', l+l')$ and $(\xi \circ \eta)^{\Gamma} = (1/(K! \cdot K'!)) \sum_{\pi \in \mathfrak{S}_{k''}} \sum_{\pi \in \mathfrak{S}_{l''}} \xi^{\alpha_{\pi(1)} \cdots \alpha_{\pi(k)}, \bar{\beta}_{\pi'(1)} \cdots \bar{\beta}_{\pi'(l)}} \cdot \eta^{\alpha_{\pi(k+1)} \cdots \alpha_{\pi(k+k')}, \bar{\beta}_{\pi'(l+1)} \cdots \bar{\beta}_{\pi'(l+l')}}$, denoted by $\mathfrak{S}_{k''}$ (resp. $\mathfrak{S}_{l''}$) the symmetric group of order k'' (resp. l'').

The componentwise loacl expression of $[\xi, \eta]$ for ξ and η as in (1.4) is

$$[\xi, \eta] = (1/K_1!) \sum_{\sharp \mu_1 = K_1} [\xi, \eta]_1^{\Gamma_1} (\partial/\partial z)^{\Gamma_1} + (1/K_2!) \sum_{\sharp \Gamma_2 = K_2} [\xi, \eta]_2^{\Gamma_2} (\partial/\partial z)^{\Gamma_2},$$

where the summation is as in (1.2) for $K_1=(k+k'-1, l+l')$, $K_2=(k+k', l+l'-1)$ and

$$\begin{aligned}[\xi, \eta^{\Gamma_1}]_1 = \sum_{\gamma=1}^{n} \Big[\sum_{\pi \in \mathfrak{S}_{k+k'-1}} \sum_{\pi' \in \mathfrak{S}_{l+l'}} \{ &(k/(K! \cdot K'!)) \xi^{\gamma \alpha_{\pi(1)} \cdots \alpha_{\pi(k-1)}, \bar{\beta}_{\pi'(1)} \cdots \bar{\beta}_{\pi'(l)}} \\ &\frac{\partial}{\partial z^{\alpha}} \eta^{\alpha_{\pi(k)} \cdots \alpha_{\pi(k'-1)}, \bar{\beta}_{\pi'(l+1)} \cdots \bar{\beta}_{\pi'(l+l')}} - (k'/(K! \cdot K'!)) \\ &\eta^{\gamma \alpha_{\pi(1)} \cdots \alpha_{\pi(k'-1)}, \bar{\beta}_{\pi'(1)} \cdots \bar{\beta}_{\pi'(l')}} \frac{\partial}{\partial z^{\gamma}} \xi^{\alpha_{\pi(k')} \cdots \alpha_{\pi(k'+k-1)} \cdots \bar{\beta}_{\pi'(l'+1)} \cdots \bar{\beta}_{\pi'(l+l')}} \} \Big].\end{aligned}$$

The local expression of $[\xi, \eta]_2^{\Gamma_2}$ is similar.

Notice that

$$\begin{aligned}&[S^p(M), S^q(M)] \subseteqq S^{p+q-1}(M), \\ &[S^K(M), S^{K'}(M)] \subseteqq S^{K+K'-(1,0)}(M) \oplus S^{K+K'-(0,1)}(M).\end{aligned}$$

From now on we assume that (M, g) is a compact Kählerian manifold with the Kählerian metric $ds^2 = 2 g_{\alpha\bar{\beta}} dz^{\alpha} d\bar{z}^{\beta}$.

Put

$$g^* = g^{\alpha\beta} \partial/\partial z^{\alpha} \circ \partial/\partial \bar{z}^{\beta},$$

where $(g^{\alpha\beta})$ is the inverse matrix of $(g_{\alpha\bar{\beta}})$. Define a linear operator T^*: $S^{**}(M) \rightarrow S^{**}(M)$ of bidegree (1.1) by

$$(1.5) \qquad T^*\xi = g^* \circ \xi \in \boldsymbol{S}^K(M)$$

for $\xi \in \boldsymbol{S}^{K-(1,1)}(M)$. $\boldsymbol{S}^{**}(M)$ is equipped with the Hermitian inner product of symmetric tensor fields:

$$(1.6) \qquad (\xi, \eta) = K! \int_M \langle \xi, \eta \rangle d\sigma \,,$$

where $\langle \ , \ \rangle$ is the pointwise Hermitian inner product of $\boldsymbol{S}^{**}(M)$ and $d\sigma$ is the volume element of (M, g). Let T be the adjoint operator of T^* with respect to the Hermitian inner product above. T is a linear operator: $\boldsymbol{S}^{**}(M) \to \boldsymbol{S}^{**}(M)$ of bidegree $(-1, -1)$. The componentwise local expression of $T^*\xi \in \boldsymbol{S}^K(M)$ for $\xi \in \boldsymbol{S}^{K-(1,1)}(M)$ and that of $T\xi \in \boldsymbol{S}^K(M)$ for $\xi \in \boldsymbol{S}^{K+(1,1)}(M)$ are

$$(1.7) \qquad T^*\xi = \sum_{\sharp\Gamma = K = (k,l)} (1/K!)(T^*\xi)^\Gamma (\partial/\partial z)^\Gamma \,,$$

where the summation is as in (1.2) with

$$(T^*\xi)^\Gamma = \sum_{(0,0) \leqq (i,j) \leqq K} g^{\alpha_i \bar{\beta}_j} \xi^{\alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_k, \bar{\beta}_1 \cdots \hat{\bar{\beta}}_j \cdots \bar{\beta}_l}$$

for $\Gamma = (\alpha_1 \cdots \alpha_k, \bar{\beta}_1 \cdots \bar{\beta}_l)$
and

$$(1.8) \qquad T\xi = \sum_{\sharp\Gamma = K = (k,l)} (T\xi)^\Gamma (\partial/\partial z)^\Gamma \,,$$

where the summation is as in (1.2) with

$$(T\xi)^\Gamma = g_{\bar{\delta}\gamma} \xi^{\gamma\alpha_1 \cdots \alpha_k, \bar{\delta}\bar{\beta}_1 \cdots \bar{\beta}_l}$$

for $\Gamma = (\alpha_1 \cdots \alpha_k, \bar{\beta}_1 \cdots \bar{\beta}_l)$.

Lemma 1.1. (i) *$[T, T^*] = (n+p)\mathbf{1}_p$ on $\boldsymbol{S}^p(M)$, where $\mathbf{1}_p$ is the identity operator on $\boldsymbol{S}^p(M)$.* (ii) *$[T^m, T^*] = m(n+p-m+1)\ T^{m-1}$ on $\boldsymbol{S}^{p+2m-2}(M)$.*

Proof. Let $\xi \in \boldsymbol{S}^K(M)$ be as in (1.2). Connecting (1.7) and (1.8), we have

$$(TT^*\xi)^\Gamma = g_{\gamma\bar{\delta}} \Big(\sum_{1 \leqq i \leqq k} \sum_{1 \leqq j \leqq l} g^{\alpha_i \bar{\beta}_j \gamma\alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_k, \bar{\delta}\bar{\beta}_1 \cdots \hat{\bar{\beta}}_j \cdots \bar{\beta}_l} + \sum_{1 \leqq i \leqq k} g^{\alpha_i \bar{\delta}} \xi^{\gamma\alpha_1 \cdots \hat{\alpha}_i \cdots \alpha, \bar{B}}$$

$$+ \sum_{1 \leqq j \leqq l} g^{\gamma\bar{\beta}_j} \xi^{\bar{\delta}\bar{\beta}_1 \cdots \hat{\bar{\beta}}_j \cdots \bar{\beta}_l} + g^{\gamma\bar{\delta}} \xi^{A, \bar{B}} \Big) \,,$$

where A and $\bar{B}$ are partial multi-indices as

$$\xi^{A,\bar{B}} = \xi^{\alpha_1 \cdots \alpha_k, \bar{B}} = \xi^{A, \bar{\beta}_1 \cdots \bar{\beta}_l} = \xi^{\alpha_1 \cdots \alpha_k, \bar{\beta}_1 \cdots \bar{\beta}_l} \,.$$

Then

$$TT^*\xi = T^*T\xi + (|K| + n)\xi \,,$$

from which (i) follows. (ii) is proved by induction on m. Q.E.D.

Define a linear differential operator δ^*: $S^{p-1}(M)\to S^p(M)$ of degree 1 by

$$\delta^*\xi = [g^*, \xi] = (1/2)\sigma^p[D, \Delta], \tag{1.9}$$

where $\sigma^{p-1}(D)=\xi\in S^{p-1}(M), D\in\mathfrak{D}^{p-1}(M)$ and $\Delta=-2g^{\alpha\bar\beta}\nabla_\alpha\nabla_{\bar\beta}$ is the Laplacian of (M, g) expressed in terms of the Kählerian connection ∇. $\delta^*\xi$ is independent of the choice of D. Let δ: $S^{p+1}(M)\to S^p(M)$ be the adjoint operator of δ^* with respect to the Hermitian inner product (1.6). δ is a linear differential operator: $S^{p+1}(M)\to S^p(M)$ of degree -1.

Lemma 1.2. *Decompose as $\delta^*=\partial^*+\bar\partial^*$ and $\delta=\partial+\bar\partial$, where ∂^*, $\bar\partial^*$, ∂ and $\bar\partial$ are linear differential operators: $S^{**}(M)\to S^{**}(M)$ of bidegree $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$, respectively. ∂ (resp. $\bar\partial$) is the adjoint operator of ∂^* (resp. $\bar\partial^*$). They have the following componentwise local expressions*

$$\partial^*\xi = \sum_\Gamma \sum_{|K|=p} \frac{1}{K!}\sum_{i=1}^{k} g^{\alpha_i\bar\delta}\nabla_{\bar\delta}\xi^{\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k,\bar B}(\partial/\partial z)^A\circ(\partial/\partial\bar z)^B,$$

$$\bar\partial^*\xi = \sum_\Gamma \sum_{|K|=p} \frac{1}{K!}\sum_{j=1}^{l} g^{\gamma\bar\beta_j}\nabla_\gamma\xi^{A,\bar\beta_1\cdots\hat{\bar\beta}_j\cdots\bar\beta_l}(\partial/\partial z)^A\circ(\partial/\partial\bar z)^B$$

for $\xi\in S^{p-1}(M)$, rdespectively.

$$\partial\xi = \sum_\Gamma \sum_{|K|=p} \frac{-1}{K!}\nabla_\gamma\xi^{\gamma\alpha_1\cdots\alpha_k,\bar B}(\partial/\partial z)^A\circ(\partial/\partial\bar z)^B,$$

$$\bar\partial\xi = \sum_\Gamma \sum_{|K|=p} \frac{-1}{K!}\nabla_{\bar\delta}\xi^{A,\bar\delta\bar\beta_1\cdots\bar\beta_l}(\partial/\partial z)^A\circ(\partial/\partial\bar z)^B$$

for $\xi\in S^{p-1}(M)$, respectively and $\sum_\Gamma$ is as in (1.4) $(\Gamma=(A, \bar B)=(\alpha_1\cdots\alpha_k, \bar\beta_1\cdots\bar\beta_l))$.

The proof of Lemma 1.2 is easy and is omitted.

Lemma 1.3. (i) $[T, \delta]=0$, (i)* $[T^*, \delta^*]=0$,
(ii) $[T, \partial]=0$, (ii)* $[T^*, \partial^*]=0$,
$\overline{\text{(ii)}}$ $[T, \bar\partial]=0$, $\overline{\text{(ii)}}$* $[T^*, \bar\partial^*]=0$,
(iii) $[T, \delta^*]=-\delta$, (iii)* $[T^*, \delta]=\delta^*$,
(iv) $[T, \partial^*]=-\bar\partial$, (iv)* $[T^*, \partial]=\bar\partial^*$,
$\overline{\text{(iv)}}$ $[T, \bar\partial^*]=-\partial$, $\overline{\text{(iv)}}$* $[T^*, \bar\partial]=\partial^*$,
(v) $[\bar\partial, \partial^*]=0$, (v)* $[\bar\partial^*, \partial]=0$.

Proof. (i), (i)*, (iii) and (iii)* were proved in [5]. (ii)$\sim\overline{\text{(ii)}}$* follow from (i) or (i)*. (iv)$\sim\overline{\text{(iv)}}$* follow from (iii) or (iii)*. (v) (resp. (v)*) follows from the identity $R_{\alpha\beta\gamma\bar\delta}=0$ (resp. $R_{\bar\alpha\bar\beta\bar\gamma\delta}=0$) for the Kählerian connection. Q.E.D.

Lemma 1.4. *δ^*, ∂^* and $\bar\partial^*$ are derivations on $S^*(M)$ with respect to the symmetric tensor product.*

Proof. As for δ^* the assertion was proved in our previous paper [5]. It follows easily that ∂^*(resp. $\bar{\partial}^*$) is also a derivation on $S^*(M)$. Q.E.D.

Define

$$(1.10) \qquad \square = [\delta, \delta^*], \quad \overline{\square}' = [\partial, \partial^*], \quad \square' = [\bar{\partial}, \bar{\partial}^*].$$

Evidently $\square$ is a linear differential operator: $S^*(M)\to S^*(M)$ of degree 0. $\square'$ and $\overline{\square}'$ are linear differential operators: $S^{**}(M)\to S^{**}(M)$ of bidegree (0, 0).

Lemma 1.5. $\square=\square'+\overline{\square}'$. *$\square$ is a linear differential operator on $S^{**}(M)$ of bidegree* (0, 0).

Proof. From Lemma 1.2 and Lemma 1.3 (v), (v)*

$$[\delta, \delta^*] = (\partial+\bar{\partial})(\partial^*+\bar{\partial}^*)-(\partial^*+\bar{\partial}^*)(\partial+\bar{\partial})=\square'+\overline{\square}'.$$

Thus the first assertion is proved. The second one follows from the first one immediately. Q.E.D.

Put

$$(1.11) \qquad \Delta = 2\Delta_R-\square, \quad \Delta_R = -(\nabla^{\alpha}\nabla_{\alpha}+\nabla^{\bar{\alpha}}\nabla_{\bar{\alpha}})$$

with $\nabla^{\alpha}=g^{\alpha\bar{\gamma}}\nabla_{\bar{\gamma}}$ and $\nabla^{\bar{\alpha}}=g^{\gamma\bar{\alpha}}\nabla_{\gamma}$. Δ and Δ_R are called the *Lichnerowicz operator* and the *rough Laplacian* of (M, g), respectively. The componentwise local expresseion of $\square$ and Δ satisfy the formulae:

$$(1.12) \qquad \square = \Delta_R-\kappa, \quad \Delta = \Delta_R+\kappa,$$

where κ is the linear operator on $S^{**}(M)$ of bidegree (0, 0) given in terms of components of curvature tensor and of Ricci tensor by

$$\begin{aligned}(\kappa\xi)^{\Gamma} = {} & \sum_{i=1}^{k} R^{\alpha}{}_{i\gamma}\xi^{\gamma\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k,\bar{B}}+\sum_{j=1}^{l} R^{\bar{\beta}}{}_{j\bar{\delta}}\xi^{A,\bar{\delta}\bar{\beta}_1\cdots\hat{\bar{\beta}}_j\cdots\bar{\beta}_l} \\ & -2\sum_{(i,j)\leqq(k,l)} R^{\alpha}{}_{i\gamma\bar{\delta}}{}^{\bar{\beta}}{}_{j}\xi^{\gamma\alpha_1\cdots\hat{\alpha}_i\cdots\alpha_k,\delta\bar{\beta}_1\cdots\hat{\bar{\beta}}_j\cdots\bar{\beta}_l} \\ & -2\sum_{1\leqq i<j\leqq k} R^{\alpha}{}_{i\gamma\delta}{}^{\alpha}{}_{j}\xi^{\gamma\delta\alpha_1\cdots\hat{\alpha}_i\cdots\hat{\alpha}_j\cdots\alpha_k,\bar{B}} \\ & -2\sum_{1\leqq i<j\leqq l} R^{\bar{\beta}}{}_{i\bar{\gamma}\bar{\delta}}{}^{\bar{\beta}}{}_{j}\xi^{A,\bar{\gamma}\bar{\delta}\bar{\beta}_1\cdots\hat{\bar{\beta}}_i\cdots\hat{\bar{\beta}}_j\cdots\bar{\beta}_l}\end{aligned}$$

for $\xi\in S^K(M)$ as in (1.2) ($K=(k, l)$).

Theorem 1.1 (i) $[\square, T]=0$, (ii) $[\square, T^*]=0$, (iii) $[\Delta, T]=0$, (iv) $[\Delta, T^*]=0$.

If (M, g) is locally symmetric, then

(v) $[\Delta, \delta^*]=0$, (vi) $[\Delta, \delta]=0$,
(vii) $[\Delta, \partial^*]=0$, $(\overline{\text{vii}})$ $[\Delta, \bar\partial^*]=0$,
(viii) $[\Delta, \partial]=0$, $(\overline{\text{viii}})$ $[\Delta, \bar\partial]=0$.

Proof. (i)~(vi) were proved in our previous paper for a Riemannian manifold [5]. (vii)~$(\overline{\text{viii}})$ follow from the decompositions in Lemma 1.2. Q.E.D.

Let $(\boldsymbol{P}_n(\boldsymbol{C}), g_o)$ be the complex projective space of complex dimension n with the Fubini-Study metric g_o.

Theroem 1.2.

(i)
$$\begin{aligned}\Delta &= 4(2kn+3k^2+l^2-2kl-p)\mathbf{1}_{k,l}-8T^*T+2\Box' \\ &= 4(2ln+3l^2+k^2-2kl-p)\mathbf{1}_{k,l}-8T^*T+2\overline{\Box}' \\ &= 4(pn+2k^2+2l^2-2kl-p)\mathbf{1}_{k,l}-8T^*T+\Box\end{aligned}$$

on $\boldsymbol{S}^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}))$ $(k+l=p)$, *where* $\mathbf{1}_{k,l}$ *is the identity operator on* $\boldsymbol{S}^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}))$.
(ii) $[\partial, \bar\partial]=4(k-l)T$, (ii)* $[\partial^*, \bar\partial^*]=4(l-k)T^*$ *on* $\boldsymbol{S}^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}))$.

Proof. If (M, g) is of constant holomorphic sectional curvature with scalar curvature $2(n+1)$, then

$$R_{\alpha\bar\beta\gamma\bar\delta} = 2(g_{\alpha\bar\beta}g_{\gamma\bar\delta}+g_{\alpha\bar\delta}g_{\gamma\bar\beta})\,. \tag{1.13}$$

On the other hand, (1.12) gives rise to

$$\Delta\xi = 2\kappa\xi+\Box\xi\,. \tag{1.14}$$

Subsituting (1.13) into (1.14) we obtain the third equality of (i). The first and the second ones of (i) will be obtained if we express $\Box'$ and $\overline{\Box}'$ in terms of the curvature tensor, respectively. (ii) and (ii)′ are also easily obtained making use of (1.13). Q.E.D.

2. Let $\tilde g_0$ be the canonical Hermitian metric on the punctured complex space $\boldsymbol{C}^{n+1}-\{0\}$ of complex dimension n. Let π: $\boldsymbol{C}^{n+1}-\{0\}\to\boldsymbol{P}_n(\boldsymbol{C})$ be the Hopf fibering. Then with respect to the conformally related metric $r^{-2}\tilde g_0$ $(r^2=\sum_{a=0}^{n} Z^a\bar Z^a$ for $Z=(Z^0, \cdots, Z^n)\in\boldsymbol{C}^{n+1}-\{0\})$ and the Fubini-Study metric g_o on $\boldsymbol{P}_n(\boldsymbol{C})$ π is a Riemannian submersion. Let $P=P(M, G)$ be a principal bundle on a manifold M with a Lie group G as its fibre. Denote by $\mathfrak{D}^G(P)$ the subalgebra of $\mathfrak{D}(P)$ consisting of all G-invariant differential operators on P.

Lemma 2.1 (cf. [5], 2). *$\mathfrak{D}(M)\cong\mathfrak{D}^G(P)/\mathcal{J}$, more precisely $\mathfrak{D}^p(M)\cong\mathfrak{D}^G(P)\cap\mathfrak{D}^p(P)/(\mathcal{J}\cap\mathfrak{D}^p(P))$, where $\mathcal{J}$ is the two-sided ideal in $\mathfrak{D}^G(P)$ generated by G-invariant vertical vector fields on P.*

Applying Lemma 2.1 to the Hopf fibering, we see that

$$\mathfrak{D}^{C^*}(C^{n+1}-\{0\})/(\zeta, \bar{\zeta}) \cong \mathfrak{D}(P_n(C)), \tag{2.1}$$

where $C^* = C - \{0\}$ and $\mathfrak{D}^{C^*}(C^{n+1}-\{0\}) = \{D \in \mathfrak{D}(C^{n+1}-\{0\} \mid [D, \zeta] = [D, \bar{\zeta}] = 0\}$ for $\zeta = \sum_{a=0}^{n} Z^a \partial/\partial Z^a$ and $\bar{\zeta} = \sum_{a=0}^{n} \bar{Z}^a \partial/\partial \bar{Z}^a$ with the bracket as in **1.** Here $(\zeta, \bar{\zeta})$ denotes the two-sided ideal in $\mathfrak{D}^{C^*}(C^{n+1}-\{0\})$ generated by ζ and $\bar{\zeta}$. From (2.1) we have also

$$(S^{C^*})^*(C^{n+1}-\{0\})/(\zeta, \bar{\zeta}) \cong S^*(P_n(C)), \tag{2.1$'$}$$

where $(S^{C^*})^*(C^{n+1}-\{0\})$ is the graded subalgebra of $S^*(C^{n+1}-\{0\})$ consisting of all C^* invariant symmetric tensor fields. Here we denote also by $(\zeta, \bar{\zeta})$ the ideal in $(S^{C^*})^*(C^{n+1}-\{0\})$ generated by ζ and $\bar{\zeta}$ by abuse of notation.

Let ι; $S^{2n+1} \to C^{n+1}-\{0\}$ be the canonical imbedding whose image is the unit sphere $\{Z \in C^{n+1}-\{0\} \mid \sum_{a=0}^{n} |Z^a|^2 = 1\}$. $C^{n+1}-\{0\}$ is regarded as a product bundle on S^{2n+1} with R^* as its fibre. We have an isomorphism:

$$\mathfrak{D}'(S^{2n+1}) \cong \mathfrak{D}(S^{2n+1}), \tag{2.2}$$

where $\mathfrak{D}'(S^{2n+1})$ is given by $\{D \in \mathfrak{D}(C^{n+1}-\{0\} \mid [D, r^2] = 0$ and $[D, \partial/\partial r] = 0\}$ (cf. [4] Lemma 1 and (1.4), p. 651). Notice that $r\partial/\partial r = \zeta + \bar{\zeta}$. On the other hand, applying Lemma 2.1 to the principal bundle $S^{2n+1} \to P_n(C)$ with the fibre S^1, we obtain the isomorphism:

$$\text{(2.3)} \quad \begin{aligned} &\text{(i)} \quad \mathfrak{D}^{S^1}(S^{2n+1})/\mathcal{J} \cong \mathfrak{D}(P_n(C)), \\ &\text{(ii)} \quad (S^{S^1})^*(S^{2n+1})/\mathcal{J} \cong S^*(P_n(C)). \end{aligned}$$

Connecting the two isomorphism (2.3) (i) and (2.2), we obtain an isomorphism:

$$\mathfrak{D}^{\dagger}(P_n(C))/(\tau) \cong \mathfrak{D}(P_n(C)), \tag{2.4}$$

where $\mathfrak{D}^{\dagger}(P_n(C))$ is the subalgebra of $\mathfrak{D}(C^{n+1}-\{0\})$ corresponding to $\mathfrak{D}^{S^1}(S^{n+1})$ via (2.2) and (τ) is the two-sided principal ideal in $\mathfrak{D}^{\dagger}(P_n(C))$ generated by the invariant vertical vector field

$$\tau = \sqrt{-1}(\zeta - \bar{\zeta}) \in \mathfrak{D}^{\dagger}(P_n(C)). \tag{2.5}$$

Notice that τ is tangent to fibres of the S^1-bundle: $S^{2n+1} \to (P_n(C))$ (under the identification (2.2)) and is an element of the center in $\mathfrak{D}^{\dagger}(P_n(C))$. Moreover, $\mathfrak{D}^{\dagger}(P_n(C)) = \{D \in \mathfrak{D}(C^{n+1}-\{0\}) \mid [D, r^2] = 0, [D, \zeta] = 0$, and $[D, \bar{\zeta}] = 0\}$.

REMARK, Let $D_1^{\dagger}$(resp. $D_2^{\dagger}$) be a representative of D_1(resp. D_2)$\in \mathfrak{D}(P_n(C))$. If $D_1^{\dagger}$ is the adjoint operator in $\mathfrak{D}(C^{n+1}-\{0\})$ to $D_2^{\dagger}$, then D_1 is adjoint to D_2 in $\mathfrak{D}(P_n(C))$.

Put

$$(S^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C})) = \sum_{p\geqq 0} (\sigma^\dagger)^p(\mathfrak{D}^\dagger)^p(\boldsymbol{P}_n(\boldsymbol{C})) \text{ (direct sum)},$$

where $(\sigma^\dagger)^p$ is the restriction of the symbol map σ^p of $\mathfrak{D}^p(\boldsymbol{C}^{n+1}-\{0\})$ to $(\mathfrak{D}^\dagger)^p(\boldsymbol{P}_n(\boldsymbol{C}))$. Then we obtain

$$(S^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C}))/(\tau) \cong S^*(\boldsymbol{P}_n(\boldsymbol{C})), \tag{2.4$'$}$$

where (τ) is the principal ideal in $(S^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C}))$ generated by τ.

Notice that $S^*(\boldsymbol{P}_n(\boldsymbol{C}))$ and $(S^{C^*})^*(\boldsymbol{C}^{n+1}-\{0\})$ have natural bigradations, while $(S^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C}))$ has no natural one.

Put

$$(S^\dagger)^K(\boldsymbol{P}_n(\boldsymbol{C})) = (S^\dagger)^{|K|}(\boldsymbol{P}_n(\boldsymbol{C})) \cap S^K(\boldsymbol{C}^{n+1}-\{0\})$$

for bidegree K and put

$$(S^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C})) = \sum_{K\geqq 0} (S^\dagger)^K(\boldsymbol{P}_n(\boldsymbol{C})) \text{ (direct sum)}.$$

A representative in $\mathfrak{D}^\dagger(\boldsymbol{P}_n(\boldsymbol{C}))$ of $D\in\mathfrak{D}(\boldsymbol{P}_n(\boldsymbol{C}))$ under the identification (2.4) (resp. a representative in $(S^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C}))$ of $\xi\in S^*(\boldsymbol{P}_n(\boldsymbol{C}))$ under (2.4)$'$) will be designated as $D^\dagger$ (resp. $\xi^\dagger$) in the following. From the construction of $(S^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C}))$ it follows that

$$[\xi^\dagger, r^2] = 0, \quad [\xi^\dagger, \zeta] = 0, \quad [\xi^\dagger, \bar{\zeta}] = 0 \tag{2.6}$$

for $\xi^\dagger\in(S^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C}))$.

Lemma 2.2. *Let $\xi^\dagger\in(S^\dagger)^p(\boldsymbol{P}_n(\boldsymbol{C}))$. Its components $(\xi^\dagger)^{A,\bar{B}}$ are bi-homogeneous functions of bidegree $K=(k, l)=\#(A, \bar{B})$ for each $K(p=|K|)$ with respect to the variables $(Z^0, \cdots, Z^n, \bar{Z}^0, \cdots, \bar{Z}^n)$. Moreover,*

$$\sum_{k+l=p-1} \sum_{c=0}^{n} (\bar{Z}^c(\xi^\dagger)^{ca_1\cdots a_k,\bar{B}} + Z^c(\xi^\dagger)^{A,\bar{c}\bar{b}_1\cdots\bar{b}_l}) = 0. \tag{2.7}$$

Proof. The first assertion follows from the second and the third equalities of (2.6) directly, while the second assertion (2.7) follows from the first equality of (2.6) Q.E.D.

Corollary. *Let $\xi^\dagger\in(S^\dagger)^K(\boldsymbol{P}_n(\boldsymbol{C}))$. Then*

$$\sum_{c=0}^{n} \bar{Z}^c(\xi^\dagger)^{ca_1\cdots a_{k-1},\bar{B}} = 0, \quad \sum_{c=0}^{n} Z^c(\xi^\dagger)^{A,\bar{c}\bar{b}_1\cdots\bar{b}_{l-1}} = 0$$

for $K=(k, l)$.

Notice that $C^\infty(\boldsymbol{P}_n(\boldsymbol{C}))=\mathfrak{D}^0(\boldsymbol{P}_n(\boldsymbol{C}))$ is isomorphic to $(\mathfrak{D}^\dagger)^0(\boldsymbol{P}_n(\boldsymbol{C}))$ which we also denote $(C^\infty)^\dagger(\boldsymbol{P}_n(\boldsymbol{C}))$. It consists of homogeneous functions of degree 0.

Denote by π_0 the canonical projection: $(\boldsymbol{S}^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C}))\to\boldsymbol{S}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ defined by (2.4)$'$ and denote by π its restriction to $(\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$.

Lemma 2.3. *π: $(\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))\to\boldsymbol{S}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ is an isomorphism of bigraded algebras.*

Proof. In order to prove the surjectivity of π it is sufficient to show that for any $\xi^\dagger\in(\boldsymbol{S}^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C}))$ there is $\eta^\dagger\in(\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ such that $\xi^\dagger-\eta^\dagger\in(\tau)$. Let $\xi^\dagger\in(\boldsymbol{S}^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C}))$ be a representative of $\xi\in\boldsymbol{S}^*(\boldsymbol{P}_n(\boldsymbol{C}))$, which can be rewritten as $\xi=\sum_{i_1,\cdots,i_p=0}^{n} f_{i_1\cdots i_p}\xi_{i_1}\circ\cdots\circ\xi_{i_p}$ by $f_{i_1\cdots i_p}\in C^\infty(\boldsymbol{P}_n(\boldsymbol{C}))$ and vector fields ξ_{i_j}'s $(j=1,\cdots p)$. If $(\xi_{i_j})^\dagger$ is a representative of ξ_{i_j} $(j=1,\cdots p)$, it is obvious that

$$\xi^\dagger-\sum_{i_1,\cdots,i_p=0}^{n} f^\dagger_{i_1\cdots i_p}\xi^\dagger_{i_1}\circ\cdots\circ\xi^\dagger_{i_p}\in(\tau)$$

for a representative $f^\dagger_{i_1\cdots i_p}\in(C^\infty)^\dagger(\boldsymbol{P}_n(\boldsymbol{C}))$ of $f_{i_1\cdots i_p}$. Hence the question is reduced to the case when $\xi^\dagger$ is a vector field. Put $\eta^\dagger=\xi^\dagger+\sqrt{-1}(\sum_{c=0}^{n}(\xi^\dagger)^c\bar{Z}^c/r^2)\tau$ for $\xi^\dagger=\sum_{c=0}^{n}((\xi^\dagger)^c\partial/\partial Z^c+(\xi^\dagger)^{\bar{c}}\partial/\partial\bar{Z}^c)$. By virtue of Corollary of Lemma 2.2 it is easily verified that $\eta^\dagger=\eta^\dagger_1+\eta^\dagger_2$, where $\eta^\dagger_1\in(\boldsymbol{S}^\dagger)^{1,0}(\boldsymbol{P}_n(\boldsymbol{C}))$ and $\eta^\dagger_2\in(\boldsymbol{S}^\dagger)^{0,1}(\boldsymbol{P}_n(\boldsymbol{C}))$. This proves the surjectivity of π.

Let $\xi^\dagger\in\sum_{|K|=p}(\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ be such that $\xi^\dagger=\tau\circ\theta^\dagger$ with $\theta^\dagger\in(\boldsymbol{S}^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C}))$. From the proof of the surjectivity above we can see that $\xi^\dagger$ can be expressed as $\xi^\dagger=\sum_{m=1}^{p}\tau^m\theta^\dagger_m$ with $\theta^\dagger_m\in(\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$. Then it follows easily that $\theta^\dagger=0$ $(m=1,\cdots p)$ by virtue of Lemma 2.2 and its Corollary. Thus the kernel of π is trivial. Q.E.D.

We introduce three linear differential operators $\delta^{*\prime}$, δ' and $\tilde{\partial}$: $\boldsymbol{S}^*(\boldsymbol{C}^{n+1}-\{0\})\to\boldsymbol{S}^*(\boldsymbol{C}^{n+1}-\{0\})$ and a linear differential operator $\exp(-\delta'/2)$: $\boldsymbol{S}^*(\boldsymbol{C}^{n+1}-\{0\})\to\mathfrak{D}(\boldsymbol{C}^{n+1}-\{0\})$ as follows.

$$\text{(2.8)}\qquad \delta^{*\prime}\Xi\overset{\text{def.}}{=}\sum_{|K|=p}(2r^2/K!)\left[\sum_{i=1}^{k}\frac{\partial\Xi^{a_1\cdots\hat{a}_i\cdots a_k,B}}{\partial\bar{Z}^{a_i}}+\sum_{j=1}^{l}\frac{\partial\Xi^{A,\bar{b}_1\cdots\hat{\bar{b}}_j\cdots\bar{b}_l}}{\partial Z^{b_j}}\right](\partial/\partial Z)^A\circ(\partial/\partial\bar{Z})^B\in\boldsymbol{S}^p(\boldsymbol{C}^{n+1}-\{0\})$$

for $\Xi\in\boldsymbol{S}^{p-1}(\boldsymbol{C}^{n+1}-\{0\})$.

$$\text{(2.9)}\qquad \delta'\Xi\overset{\text{def.}}{=}\sum_{|K|=p}(-1/K!)\left[\frac{\partial\Xi^{ca_1\cdots a_k,\bar{B}}}{\partial Z^c}+\frac{\partial\Xi^{A,c\bar{b}_1\cdots\bar{b}_l}}{\partial\bar{Z}^c}\right](\partial/\partial Z)^A\circ(\partial/\partial\bar{Z})^B\in\boldsymbol{S}^p(\boldsymbol{C}^{n+1}-\{0\})$$

for $\Xi\in\boldsymbol{S}^{p+1}(\boldsymbol{C}^{n+1}-\{0\})$.

$$(2.10)\quad \tilde{\partial}\Xi \overset{\text{def.}}{=} \sum_{|K|=p}(1/K!)\left[\frac{\partial \Xi^{ca_1\cdots a_k,\bar{B}}}{\partial Z^c}-\frac{\partial \Xi^{A,\bar{c}\bar{b}_1\cdots b_l}}{\partial \bar{Z}^c}\right]$$
$$(\partial/\partial Z)^A\circ(\partial/\partial\bar{Z})^B\in \boldsymbol{S}^p(\boldsymbol{C}^{n+1}-\{0\})$$

for $\Xi\in \boldsymbol{S}^{p+1}(\boldsymbol{C}^{n+1}-\{0\})$.

$$(2.11)\quad \exp(-\delta'/2)\Xi \overset{\text{def.}}{=} \sum_{q=0}^{p}(-1/2)^q\sum_{|K|=p-q}(1/K!)((\delta')^q\Xi)^{A,\bar{B}}$$
$$\cdot\partial^{p-q}/\partial Z^{a_1}\cdots\partial Z^{a_k}\partial\bar{Z}^{b_1}\cdots\partial\bar{Z}^{b_l}\in\mathfrak{D}^p(\boldsymbol{C}^{n+1}-\{0\})$$

for $\Xi\in \boldsymbol{S}^p(\boldsymbol{C}^{n+1}-\{0\})$. Recall that

$$(2.12)\quad 0\to\mathfrak{D}^{p-1}(\boldsymbol{C}^{n+1}-\{0\})\overset{\iota^p}{\to}\mathfrak{D}^p(\boldsymbol{C}^{n+1}-\{0\})$$
$$\overset{\sigma^p}{\to}\boldsymbol{S}^p(\boldsymbol{C}^{n+1}-\{0\})\to 0$$

is an exact sequence of $C^\infty(\boldsymbol{C}^{n+1}-\{0\})$-modules. $\exp(-\delta'/2)$ gives a splitting of (2.12) as a sequence of $\boldsymbol{C}$-modules.

Lemma 2.4. *Let* $\xi^\dagger\in(\boldsymbol{S}^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C}))$. (i) $\exp(-\delta'/2)\xi^\dagger\in\mathfrak{D}^\dagger(\boldsymbol{P}_n(\boldsymbol{C}))$. (ii) *If* $\delta^{*\prime}\xi^\dagger=0$, *then* $[\Delta^\dagger, \exp(-\delta'/2)\xi^\dagger]=0$ $(\Delta^\dagger=-4(r^2\delta^{ab}-Z^a\bar{Z}^b)\partial^2/\partial Z^a\partial\bar{Z}^b+2n(\zeta+\bar{\zeta}))$. (iii) $\exp(-\delta'/2)$ *induces a splitting of the short exact sequence*:

$$0\to(\mathfrak{D}^\dagger)^{p-1}(\boldsymbol{P}_n(\boldsymbol{C}))\xrightarrow{(\iota^\dagger)^p}(\mathfrak{D}^\dagger)^p(\boldsymbol{P}_n(\boldsymbol{C}))\xrightarrow{(\sigma^\dagger)^p}(\boldsymbol{S}^\dagger)^p(\boldsymbol{P}_n(\boldsymbol{C}))\to 0$$

as a sequence of $\boldsymbol{C}$*-modules.*

Proof. To prove (i) it is enough to show

$$[\exp(-\delta'/2)\xi^\dagger, r^2]=0\,,\quad [\exp(-\delta'/2)\xi^\dagger, \partial/\partial r]=0\,.$$

The first equality follows by straightforward calculations, while the other is clear from the homogeneity of components of $\exp(-\delta'/2)\xi^\dagger$. (ii) follows from the equality

$$[\Delta^\dagger, \exp(-\delta'/2)\xi^\dagger] = -4r^2[\sum_{a=0}^{n}\partial^2/\partial Z^a\partial\bar{Z}^a, \exp(-\delta'/2)\xi^\dagger]$$
$$= 8r^2(\exp(-\delta'/2)(\delta^{*\prime}\xi^\dagger/r^2)\,,$$

the proof of which is essentially the same as in our previous paper (cf. [5], **1**; the proof of Theorem 2.1 and Theorem 2.2). (iii) is clear from (i) and (2.12). Q.E.D.

$\exp(-\delta'/2)$ is a pseudo-connection introduced in our previous paper (cf. [5], 2).

Lemma 2.5. *Let* $\xi^\dagger\in(\boldsymbol{S}^\dagger)^{p-1}(\boldsymbol{P}_n(\boldsymbol{C}))$. *Assume* $\delta^{*\prime}\xi^\dagger\in(\tau)^s(s\geqq 1)$. *Then there exists* $\xi_0^\dagger\in(\boldsymbol{S}^\dagger)^{p-1}(\boldsymbol{P}_n(\boldsymbol{C}))$ *such that*

$$\text{(i) } \xi^\dagger - \xi_0^\dagger \in (\tau), \quad \text{(ii) } \delta^{*\prime}\xi_0^\dagger \in (\tau)^{s+1}.$$

Proof. The assumption is expressed as

$$\begin{aligned}
\text{(2.13)} \qquad & \sum_{|K|=p} (2r^2/K!)\Big[\sum_{i=1}^{k} \frac{\partial(\xi^\dagger)^{a_1\cdots \hat{a}_i \cdots a_k, \bar{B}}}{\partial \bar{Z}^{a_i}} + \sum_{j=1}^{l} \frac{\partial(\xi^\dagger)^{A, \bar{b}_1 \cdots \hat{\bar{b}}_j \cdots \bar{b}_l}}{\partial Z^{b_j}}\Big] \\
&= \sum_{|K|=p} \sum_{c+d=s} \sum_{1 \leqq i_1 \leqq \cdots \leqq i_c \leqq k} \sum_{1 \leqq j_1 \leqq \cdots \leqq j_d \leqq l} (1/K!) \\
&\quad \cdot \eta^{a_1 \cdots \hat{a}_{i_1} \cdots \hat{a}_{i_2} \cdots \hat{a}_{i_{c-1}} \cdots \hat{a}_{i_c} \cdots a_k, \bar{b}_1 \cdots \hat{\bar{b}}_{j_1} \cdots \hat{\bar{b}}_{j_2} \cdots \check{\bar{b}}_{j_{d-1}} \cdots \hat{\bar{b}}_{j_d} \cdots \bar{b}_l} \\
&\quad \cdot (\sqrt{-1}\, Z)^{a_{i_1}} \cdots (\sqrt{-1}\, Z^{a_{i_c}})(-\sqrt{-1}\, \bar{Z}^{b_{j_1}}) \cdots (-\sqrt{-1}\, \bar{Z}^{b_{j_d}})
\end{aligned}$$

for some $\eta \in (\boldsymbol{S}^\dagger)^{p-s}(\boldsymbol{P}_n(\boldsymbol{C}))$.

Applying $\tilde{\partial}$ to (2.13), by direct calculations we see that there exists $\eta' \in (\boldsymbol{S}^\dagger)^{p-s-1}(\boldsymbol{P}_n(\boldsymbol{C}))$ such that

$$\begin{aligned}
& \sum_{|K|=p-1} (2r^2/K!)\Big[\sum_{i=1}^{k} \frac{\partial(\tilde{\partial}\xi^\dagger)^{a_1\cdots \hat{a}_i \cdots a_k, \bar{B}}}{\partial \bar{Z}^{a_i}} + \sum_{j=1}^{l} \frac{\partial(\tilde{\partial}\xi^\dagger)^{A, \bar{b}_1 \cdots \hat{\bar{b}}_j \cdots \bar{b}_l}}{\partial Z^{b_j}}\Big] \\
&= \sum_{|K|=p-1} \sum_{c+d=s} \sum_{1 \leqq i_1 \leqq \cdots \leqq i_c \leqq k} \sum_{1 \leqq j_1 \leqq \cdots \leqq j_d \leqq l} (1/K!) \\
&\quad \cdot \eta^{a_1 \cdots \hat{a}_{i_1} \cdots \hat{a}_{i_2} \cdots \hat{a}_{i_{c-1}} \cdots \hat{a}_{i_c} \cdots \hat{a}_k, \bar{b}_1 \cdots \hat{\bar{b}}_{j_1} \cdots \hat{\bar{b}}_{j_2} \cdots \hat{\bar{b}}_{j_{d-1}} \cdots \hat{\bar{b}}_{j_d} \cdots \bar{b}_l} \\
&\quad \cdot (\sqrt{-1}\, Z^{a_{i_1}}) \cdots (\sqrt{-1}\, Z^{a_{i_c}})(-\sqrt{-1}\, \bar{Z}^{b_{j_1}}) \cdots (-\sqrt{-1}\sqrt{Z}^{b_{j_d}}) \\
&\quad + N \sum_{|K|=p} \sum_{c+d=s-1} \sum_{1 \leqq i_1 \leqq \cdots \leqq i_c \leqq k} \sum_{1 \leqq j_1 \leqq \cdots \leqq j_d \leqq l} \\
&\quad \cdot \eta^{a_1 \cdots \hat{a}_{i_1} \cdots \hat{a}_{i_{c-1}} \cdots \hat{a}_{i_c} \cdots a_k, \bar{b}_1 \cdots \hat{\bar{b}}_{j_1} \cdots \hat{\bar{b}}_{j_2} \cdots \hat{\bar{b}}_{j_{d-1}} \cdots \hat{\bar{b}}_{j_d} \cdots \bar{b}_l} \\
&\quad \cdot (\sqrt{-1}\, Z^{a_{i_1}}) \cdots (\sqrt{-1}\, Z^{a_{i_c}})(-\sqrt{-1}\, \bar{Z}^{b_{j_1}}) \cdots (-\sqrt{-1}\, \bar{Z}^{b_{j_d}})
\end{aligned}$$

for a certain integer N. We can conclude that

$$\delta^{*\prime}(\xi^\dagger - (1/N)(\tilde{\partial}\xi^\dagger \circ \tau)) \in (\tau)^{s+1}.$$

On the other hand, evidently we have $\tilde{\partial}\xi^\dagger \circ \tau \in (\tau)$. Thus $\xi_0^\dagger = \xi^\dagger - (1/N)(\tilde{\partial}\xi^\dagger \circ \tau)$ has the required preperties. Q.E.D.

Lemma 2.6. *Suppose that* $[\Delta^\dagger, D^\dagger] \equiv 0$ (*mod* (τ)) *for* $D^\dagger \in (\mathfrak{D}^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C}))$. *Then* (i) *there exists* $\xi_0^\dagger \in (\boldsymbol{S}^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C}))$ *such that* $\xi_0^\dagger - (\sigma^\dagger)^* D^\dagger \in (\tau)$ *and* $\delta^{*\prime}\xi_0^\dagger = 0$. (ii) *There exists* $D'^\dagger \in (\mathfrak{D}^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C}))$ *such that*

$$[\Delta^\dagger, D'^\dagger] = 0, \quad D^\dagger - D'^\dagger \equiv 0 \pmod{(\tau)}.$$

Proof. Applying Lemma 2.5 successively, we find $\xi_0^\dagger$ such that

$$\xi_0^\dagger - (\sigma^\dagger)^*(D^\dagger) \in (\tau) \quad \text{and} \quad \delta^{*\prime}\xi_0^\dagger \in (\tau)^q,$$

where q is strictly greater than the order of $D^\dagger$. It follows that $\delta^{*\prime}\xi_0^\dagger = 0$. (ii) Put $D_0^\dagger = \exp(-\delta'/2)\xi_0^\dagger$. By virtue of Lemma 2.4 (ii) we obtain $[\Delta^\dagger, D_0^\dagger] = 0$.

The existence of $D'^{\dagger}$ required is proved by induction on the order of D. Q.E.D.

Define

$$
(2.14)\qquad
\begin{aligned}
\underset{ab}{\xi^{\dagger}} &= \sqrt{-1}(Z^a\partial/\partial Z^b - \bar{Z}^b\partial/\partial\bar{Z}^a),\\
\underset{\bar{a}b}{\xi} &= Z^a\partial/\partial\bar{Z}^b - Z^b\partial/\partial\bar{Z}^a,\\
\underset{a\bar{b}}{\xi} &= \bar{Z}^a\partial/\partial Z^b - \bar{Z}^b\partial/\partial Z^a,\\
\underset{ab,cd}{\xi^{\dagger}} &= \underset{ad}{\xi^{\dagger}}\circ\underset{bc}{\xi^{\dagger}} - \underset{ca}{\xi^{\dagger}}\circ\underset{ad}{\xi^{\dagger}}
\end{aligned}
$$

Notice that $\underset{ab}{\xi^{\dagger}}\in(\boldsymbol{S}^{\dagger})^1(\boldsymbol{P}_n(\boldsymbol{C}))$ and $\underset{ab,cd}{\xi^{\dagger}}\in(\boldsymbol{S}^{\dagger})^{1,1}(\boldsymbol{P}_n(\boldsymbol{C}))$.

Lemma 2.7. $\underset{ab,cd}{\xi^{\dagger}} = \underset{a\bar{b}}{\xi}\circ\underset{\bar{c}d}{\xi}$.

Proof is obvious from the definition. Q.E.D.

Lemma 2.8. *The centralizer of $\Delta^{\dagger}$ in $\mathfrak{D}^{\dagger}(\boldsymbol{P}_n(\boldsymbol{C}))$ is generated by $\underset{ab}{\xi}$'s.*

Proof. Let $[\Delta^{\dagger}, D^{\dagger}]=0$ for $D^{\dagger}\in\mathfrak{D}^{\dagger}(\boldsymbol{P}_n(\boldsymbol{C}))$. Notice that for $D^{\dagger}\in(\mathfrak{D}^{\dagger})^p(\boldsymbol{P}_n(\boldsymbol{C}))$, $[\Delta^{\dagger}, D^{\dagger}]=0$ is equivalent to $[(\Delta_0)', D^{\dagger}]=0$, where Δ_0 is the Laplacian of the standard sphere with the canonical metric and $(\Delta_0)'$ is the image of Δ_0 by the isomorphism in (2.2). In virtue of the result of our previous paper (cf. [5], Theorem 2.3), the symbol tensor field $\xi^{\dagger}=(\sigma^{\dagger})^p D^{\dagger}$ can be expressed as a linear combination of symmetric tensor product of the vector fields $\underset{ab}{\xi^{\dagger}}$'s, $\underset{a\bar{b}}{\xi}$'s and $\underset{\bar{a}b}{\xi}$'s. Notice that $\underset{ab}{\xi^{\dagger}}$'s, $\underset{b\bar{a}}{\xi}$'s and $\underset{a\bar{b}}{\xi}$'s $(0\leqq a<b\leqq n)$ are regarded as the canonical basis of the vector space of Killing vector fields on the standard sphere. It follows that

$$
(2.15)\qquad
\begin{aligned}
\xi^{\dagger} = &\sum_{a_1,\cdots,a_c,b_1,\cdots,b_c=0}^{n}\ \sum_{a'_1,\cdots,a'_d,\bar{b}'_1,\cdots,\bar{b}'_d=0}^{n}\ \sum_{\bar{a}''_1,\cdots,\bar{a}''_e,b''_1,\cdots,b''_e=0}^{n}\\
&\cdot C^{a_1\cdots a_c,b_1\cdots b_c,a'_1\cdots a'_d,\bar{b}'_1\cdots\bar{b}'_d,\bar{a}''_1\cdots\bar{a}''_e,b''_1\cdots b''_e}\\
&\cdot \underset{a_1b_1}{\xi^{\dagger}}\circ\cdots\circ\underset{a_cb_c}{\xi^{\dagger}}\circ\underset{a'_1\bar{b}'_1}{\xi}\circ\cdots\underset{a'_d\bar{b}'_d}{\xi}\circ\underset{\bar{a}''_1b''_1}{\xi}\circ\cdots\circ\underset{\bar{a}''_eb''_e}{\xi}
\end{aligned}
$$

with coefficients $C^{a_1\cdots b''_e}\in\boldsymbol{C}(c+d+e=p,\ a'_i<b'_i$ and $a''_i<b''_i)$. From Lemma 2.2 we see that $b=c$ in (2.15). By Lemma 2.7 $\xi^{\dagger}$ must be a linear combination of symmetric tensor products of $\underset{ab}{\xi}$'s. From this follows the existence of $D'^{\dagger}\in(\mathfrak{D}^{\dagger})^p(\boldsymbol{P}_n(\boldsymbol{C}))$ such that $\xi^{\dagger}=(\sigma^{\dagger})(D'^{\dagger})$ and $[\Delta^{\dagger}, D'^{\dagger}]=0$. As $D^{\dagger}-D'^{\dagger}\in(\mathfrak{D}^{\dagger})^{p-1}(\boldsymbol{P}_n(\boldsymbol{C}))$ belongs to the centralizer of $\Delta^{\dagger}$, our assertion follows by induction on the order of $D^{\dagger}$. Q.E.D.

Theorem 2.1. *The centralizer of the Laplacian Δ in $\mathfrak{D}(\boldsymbol{P}_n(\boldsymbol{C}))$ is the subalgebra generated by Killing vector fields.*

Proof. From Lemma 2.3 the centralizer of Δ is generated by Killing vector fields represented by $\underset{ab}{\xi}^{\dagger}$'s. Q.E.D.

Theorem 2.2. *Let* $\xi \in S^p(P_n(C))$. *The following three conditions are equivalent*:

(i) ξ *is a linear combination of symmetric tensor products of Killing vector fields.* (ii) $\delta^*\xi=0$. (iii) *There exists* $D \in \mathfrak{D}^p(P_n(C))$ *such that* $[D, \Delta]=0$ *and* $\xi=\sigma^p(D)$.

Proof. The implication (i)→(ii) is obtained from (2.9). (ii)→(iii) follows from Lemma 2.4 (ii). (iii)→(i) is essentially contained in the proof of Theorem 2.1. Q.E.D.

A bigraded subalgebra $(K^{\dagger})^{**}(P_n(C))=\sum_{k,l\geq 0}(K^{\dagger})^{k,l}(P_n(C))$ (direct sum) of $(S^{\dagger})^{**}(P_n(C))$ is defined by

$$(K^{\dagger})^{k,l}(P_n(C))=(K^{\dagger})^{k+l}(P_n(C))\cap(S^{\dagger})^{k,l}(P_n(C)),$$

where $(K^{\dagger})^{*}(P_n(C))$ is the subalgebra of $(S^{\dagger})^{*}(P_n(C))$ generated by $\underset{ab}{\xi}^{\dagger}$'s.

Theorem 2.3. (i) $(K^{\dagger})^{k,l}(P_n(C))=0$ *for* $k\neq l$. (ii) $(K^{\dagger})^{**}(P_n(C))$ *is generated by* $\underset{ab,cd}{\xi}^{\dagger}$*'s.*

Proof. Let $\xi^{\dagger}\in(K^{\dagger})^{k,l}(P_n(C))$. There exists $\underset{ab,cd}{\eta}^{\dagger}\in(K^{\dagger})^{k-1,l-1}(P_n(C))$ $(a\neq b,\ c\neq d)$ such that

$$\xi^{\dagger}=(1/(4(k+1)(l+1)))\sum_{\substack{a,b,c,d=0\\ a\neq b,c\neq d}}^{n}\underset{ab,cd}{\eta}^{\dagger}\circ\underset{ab,cd}{\xi}^{\dagger}.$$

In fact $\underset{ab,cd}{\eta}^{\dagger}$'s defined by

$$(\underset{ab,cd}{\eta}^{\dagger})^{A,\bar{B}}=\frac{\partial^2(\xi^{\dagger})^{da_1\cdots a_{k-1},\bar{b}\bar{b}_1\cdots\bar{b}_{l-1}}}{\partial Z^a\partial\bar{Z}^c}+\frac{\partial^2(\xi^{\dagger})^{ca_1\cdots a_{k-1},\bar{a}\bar{b}_1\cdots\bar{b}_{l-1}}}{\partial Z^b\partial\bar{Z}^d}$$
$$-\frac{\partial^2(\xi^{\dagger})^{da_1\cdots a_{k-1},\bar{a}\bar{b}_1\cdots\bar{b}_{l-1}}}{\partial Z^b\partial\bar{Z}^c}-\frac{\partial^2(\xi^{\dagger})^{ca_1\cdots a_{k-1},\bar{b}\bar{b}_1\cdots\bar{b}_{l-1}}}{\partial Z^a\partial\bar{Z}^d}$$

satisfy the required properties. By the induction on the degree of $\xi^{\dagger}$ it can be proved that $\xi^{\dagger}$ vanishes unless $k=l$. $\xi^{\dagger}$ must be a linear combination of symmetric tensor products of $\underset{ab,cd}{\xi}^{\dagger}$*'s*. This proves (i) and (ii). Q.E.D.

On account of Lemma 2.3, we can confuse $(K^{\dagger})^{**}(P_n(C))$ with its image $K^{**}(P_n(C))\subset S^{**}(P_n(C))$ by π. An element of $K^{**}(P_n(C))$ is called a *Killing tensor field* on $P_n(C)$ and $K^{*}(P_n(C))$ is called the *Killing algebra*. An element of $K^{**}(P_n(C))$ is called a *Plücker tensor field* on $P_n(C)$ and $K^{**}(P_n(C))$ is called the *Plücker algebra*.

Let ρ: $S^{**}(P_n(C))\to S^{**}(P_n(C))$ be a linear differential operator of bidegree (i, j). A linear differential operator $\rho^\dagger$: $S^{**}(C^{n+1}-\{0\})\to S^{**}(C^{n+1}-\{0\})$ is called a *lift* of ρ if the following two conditions are satisfied.

(i) $\rho^\dagger$ preserves $(S^\dagger)^{**}(P_n(C))$,

(ii) $\rho^\dagger\xi^\dagger$ is a representative of $\rho\xi\in(S^\dagger)^{**}(P_n(C))$ for any representative $\xi^\dagger$ of $\xi\in S^{**}(P_n(C))$.

Let ρ: $S^*(P_n(C))\to S^*(P_n(C))$ be a linear differential operator of degree q. A linear differential operator $\rho^\dagger$ is a *lift* of ρ if

$$\rho^\dagger=\sum_{i+j=q}\rho^\dagger_{ij},$$

where $\rho^\dagger_{ij}$ are lifts of ρ_{ij}: $S^{**}(P_n(C))\to S^{**}(P_n(C))$ of bidegree (i, j) and $\rho=\sum_{i+j=q}\rho_{ij}$. Let $\Xi\in S^{p-1}(C^{n+1}-\{0\})$. Define a linear differential operator $(\delta^*)^\dagger=(\partial^*)^\dagger+(\bar\partial^*)^\dagger$: $S^*(C^{n+1}-\{0\})\to S^*(C^{n+1}-\{0\})$ of degree 1 by

$$((\partial^*)^\dagger\Xi)^{A,\bar B}=2\sum_{|K|=p}\sum_{i=1}^{k}r^2\frac{\partial\Xi^{a_1\cdots\hat{a}_i\cdots a_k,\bar B}}{\partial\bar Z^{a_i}}+(\zeta\circ((\zeta-\bar\zeta)\Xi))^{A,\bar B}$$

and

$$((\bar\partial^*)^\dagger\Xi)^{A,\bar B}=2\sum_{|K|=p}\sum_{j=1}^{l}r^2\frac{\partial\Xi^{A,\bar b_1\cdots\hat{\bar b}_j\cdots\bar b_l}}{\partial Z^{b_j}}-(\bar\zeta\circ((\zeta-\bar\zeta)\Xi))^{A,\bar B}.$$

Notice that $(\partial^*)^\dagger$ (resp. $(\bar\partial^*)^\dagger$) is a linear differential operator of bidegree (1, 0) (resp. (0, 1) on $(S^\dagger)^{**}(P_n(C))$.

Lemma 2.9. $(\delta^*)^\dagger$, $(\partial^*)^\dagger$ *and* $(\bar\partial^*)^\dagger$ *are lifts of* δ^*, ∂^* *and* $\bar\partial^*$, *respectively.*

Proof. We can easily verify that

$$[(\partial^*)^\dagger, r^2]=0,\ [(\bar\partial^*)^\dagger, r^2]=0,\ [(\partial^*)^\dagger, \zeta]=0,$$
$$[(\partial^*)^\dagger, \bar\zeta]=0,\ [(\bar\partial^*)^\dagger, \zeta]=0,\ [(\bar\partial^*)^\dagger, \bar\zeta]=0$$

as operators on $(S^\dagger)^{**}(P_n(C))$. This implies that $(S^\dagger)^{**}(P_n(C))$ is preserved by $(\partial^*)^\dagger$ and $(\bar\partial^*)^\dagger$. On the other hand, $(g^*)^\dagger=2(r^2\delta^{ab}-Z^a\bar Z^b)\partial/\partial Z^a\circ\partial/\partial\bar Z^b$ is a representative of g^*_0 in $(S^\dagger)^{1,1}(P_n(C))$ and $\Delta^\dagger=-4(r^2\delta^{ab}-Z^a\bar Z^b)\partial/\partial Z^a\circ\partial/\partial\bar Z^b+2n(\zeta+\bar\zeta)$ is a lift (resp. a representative) of the Laplacian Δ for the metric g_0. Let $\xi^\dagger\in(S^\dagger)^K(P_n(C))$ be a representative of $\xi\in(S^\dagger)^K(P_n(C))$ $(K=(k, l))$. Then

$$(\delta^*\xi)^\dagger\equiv[(g^*)^\dagger, \xi^\dagger]=(1/2)((\sigma^\dagger)^{K+(1,0)}[\xi^\dagger, \Delta^\dagger])+(1/2)(\sigma^\dagger)^{K+(0,1)}[\xi^\dagger, \Delta^\dagger])$$
$$=(\delta^*)^\dagger\xi^\dagger-(\zeta-\bar\zeta)\circ((\zeta-\bar\zeta)\xi^\dagger)\ (\mathrm{mod}\ (\tau)).$$

From this we can conclude that $(\delta^*)^\dagger\xi^\dagger$ is a representative of $\delta^*\xi$ and our assertion for $(\delta^*)^\dagger$, $(\partial^*)^\dagger$ and $(\bar\partial^*)^\dagger$ follows. Q.E.D.

Let $\Xi\in S^{K-(1,1)}(C^{n+1}-\{0\})$ $(K=(k, l))$. Define

$$(T^*)^\dagger\Xi = (2/K!)\sum_{1\leq i\leq k}\sum_{1\leq j\leq l}(r^2\delta^{a_i b_j} - Z^{a_i}\bar{Z}^{b_j})\Xi^{a_1\cdots\overset{a_i}{\wedge}\cdots a_k,\bar{b}_1\cdots\overset{\bar{b}_j}{\wedge}\cdots\bar{b}_l}$$
$$\cdot(\partial/\partial Z)^A\circ(\partial/\partial\bar{Z})^B \in \boldsymbol{S}^K(\boldsymbol{C}^{n+1}-\{0\}).$$

Let $\Xi\in\boldsymbol{S}^{K+(1,1)}(\boldsymbol{C}^{n+1}-\{0\})$ $(K=(k, l))$. Define

$$T^\dagger\Xi = (1/K!)\sum_{c,d=0}^{n}(1/2r^2)(\delta^{cd} - Z^c\bar{Z}^d/r^2)\Xi^{ca_1\cdots a_k,\bar{d}\bar{b}_1\cdots\bar{b}_l}(\partial/\partial Z)^A\circ(\partial/\partial\bar{Z})^B.$$

Lemma 2.10. *$(T^*)^\dagger$ and $T^\dagger$ are lifts of T^* and T, respectively.*

Proof. We can easily verify that

$$[(T^*)^\dagger\xi^\dagger, r^2] = 0, \quad [(T^*)^\dagger\xi^\dagger, \zeta] = 0, \quad [(T^*)^\dagger\xi^\dagger, \bar{\zeta}] = 0,$$
$$[T^\dagger\xi^\dagger, r^2] = 0, \quad [T^\dagger\xi^\dagger, \zeta] = 0, \quad [T^\dagger\xi^\dagger, \bar{\zeta}] = 0$$

for any $\xi^\dagger\in(\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$. From this we see that $(T^*)^\dagger$ and $T^\dagger$ preserve $(\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$. From the definition $(T^*)^\dagger\xi^\dagger$ is proved to be a representative of $T^*\xi$ for any $\in\boldsymbol{S}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$. Thus $(T^*)^\dagger$ is a lift of T^*. On the other hand, T is the adjoint operator of T^* in $\boldsymbol{S}^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ with respect to the Hermitian inner product (1.6) and $T^\dagger$ is easily verified to be the adjoint operator of $(T^*)^\dagger$ in $S^{**}(C^{n+1}-\{0\}), r^{-2}\tilde{g}_0)$. We conclude that $T^\dagger\xi^\dagger$ is a representative of $T\xi$ and that $T^\dagger$ is a lift of T. Q.E.D.

Define three defferential operators $\partial^\dagger$, $\bar{\partial}^\dagger$ and $\delta^\dagger=\partial^\dagger+\bar{\partial}^\dagger$: $\boldsymbol{S}^{**}(\boldsymbol{C}^{n+1}-\{0\})\to\boldsymbol{S}^{**}(\boldsymbol{C}^{n+1}-\{0\})$ by

$$\partial^\dagger = [(\bar{\partial}^*)^\dagger, T^\dagger], \quad \bar{\partial}^\dagger = [(\partial^*)^\dagger, T^\dagger].$$

Lemma 2.11. *Let $K=(k, l)$.* (i) *Let $\xi^\dagger\in(\boldsymbol{S}^\dagger)^{K+(1,0)}(\boldsymbol{P}_n(\boldsymbol{C}))$. Then the componentwise local expression of $\partial^\dagger$ is*

$$\partial^\dagger\xi^\dagger = (1/K!)\Big[-\frac{\partial(\xi^\dagger)^{ca_1\cdots a_k,\bar{B}}}{\partial Z^c} - \sum_{j=1}^{l}\sum_{c=\bar{c}=1}^{n}(\xi^\dagger)^{ca_1\cdots a_k,\bar{c}\bar{b}_1\cdots\overset{\bar{b}_j}{\wedge}\cdots\bar{b}_l}\bar{Z}^{b_j}/r^2\Big]$$
$$\cdot(\partial/\partial Z)^A\circ(\partial/\partial\bar{Z})^B\in(\boldsymbol{S}^\dagger)^K(\boldsymbol{P}_n(\boldsymbol{C})).$$

(ii) *Let $\xi^\dagger\in(\boldsymbol{S}^\dagger)^{K+(0,1)}(\boldsymbol{P}_n(\boldsymbol{C}))$. The componentwise local expression of $\bar{\partial}^\dagger$ is*

$$\bar{\partial}^\dagger\xi^\dagger = (1/K!)\Big[-\frac{\partial(\xi^\dagger)^{A,\bar{c}\bar{b}_1\cdots\bar{b}_l}}{\partial\bar{Z}^c} - \sum_{i=1}^{k}\sum_{c=\bar{c}=1}^{n}(\xi^\dagger)^{ca_1\cdots\overset{a_i}{\wedge}\cdots a_k,\bar{c}\bar{b}_1\cdots\bar{b}_l}Z^{a_i}/r^2\Big]$$
$$\cdot(\partial/\partial Z)^A\circ(\partial/\partial\bar{Z})^B\in(\boldsymbol{S}^\dagger)^K(\boldsymbol{P}_n(\boldsymbol{C})).$$

Proof. (i) and (ii) are obtained by direct calculations using Corollary to Lemma 2.2. Q.E.D.

Define a linear differential operator $\square^\dagger$: $\boldsymbol{S}^{**}(\boldsymbol{C}^{n-1}-\{0\})\to\boldsymbol{S}^{**}(\boldsymbol{C}^{n+1}-\{0\})$ by

$$\Box^\dagger = [\delta^\dagger, (\delta^*)^\dagger]\,. \tag{2.16}$$

Lemma 2.12. (i) *$\Box^\dagger$ is a lift of $\Box$.* (ii) *Let $\xi^\dagger \in (S^\dagger)^K(\boldsymbol{P}_n(\boldsymbol{C}))$ $(K=(k, l))$. Then*

$$\Box^\dagger\xi^\dagger = (1/K!)\Big[-4\sum_{c,d=0}^{n}(r^2\delta^{cd}-Z^c\bar{Z}^d)\frac{\partial^2(\xi^\dagger)^{A,\bar{B}}}{\partial Z^c\partial\bar{Z}^d}+2(n-1)((\zeta+\bar{\zeta})\xi^\dagger)^{A,\bar{B}}$$
$$-4\sum_{c=0}^{n}\sum_{i=1}^{k}Z^{a_i}\frac{\partial(\xi^\dagger)^{ca_1\cdots\hat{a}_i\cdots a_k,\bar{B}}}{\partial Z^c}-4\sum_{c=0}^{n}\sum_{j=1}^{l}\bar{Z}^{b_j}\frac{\partial(\xi^\dagger)^{A,\bar{c}\bar{b}_1\cdots\hat{\bar{b}}_j\cdots\bar{b}_l}}{\partial\bar{Z}^c}$$
$$-(\kappa^\dagger\xi^\dagger)^{A,\bar{B}}\Big](\partial/\partial Z)^A\circ(\partial/\partial\bar{Z})^B\,,$$

where $\kappa^\dagger$: $\boldsymbol{S}^{**}(\boldsymbol{C}^{n+1}-\{0\})\to\boldsymbol{S}^{**}(\boldsymbol{C}^{n+1}-\{0\})$ is given by

$$\kappa^\dagger = 2(n-1)(\zeta+\bar{\zeta}+2\zeta^2+2\bar{\zeta}^2-2\bar{\zeta}\zeta)-4(T^*)^\dagger T^\dagger$$

and $\kappa^\dagger$ is a lift of κ in (1.12).

Proof. (i) is immediately obtained from Lemma 2.9 and 2.11. (ii) is obtained by direct calculations. Q.E.D.

Put

$$\Delta^\dagger = \Box^\dagger + \kappa^\dagger\,. \tag{2.17}$$

Lemma 2.13. (i) *$\Delta^\dagger$ is a lift of the Lichnerowicz operator Δ.*
(ii) *Let $\xi^\dagger \in (S^\dagger)^K(\boldsymbol{P}_n(\boldsymbol{C}))$ $(K=(k, l))$. Then*

$$\Delta^\dagger\xi^\dagger = (1/K!)\,[-4(r^2\delta^{cd}-Z^c\bar{Z}^d)\partial^2(\xi^\dagger)^{A,\bar{B}}/\partial Z^c\partial\bar{Z}^d+2(n-1)((\zeta+\bar{\zeta})\xi^\dagger)^{A,\bar{B}}$$
$$-4\sum_{c=0}^{n}\sum_{i=1}^{k}Z^{a_i}\frac{\partial(\xi^\dagger)^{ca_1\cdots\hat{a}_i\cdots a_k,\bar{B}}}{\partial Z^c}-4\sum_{c=0}^{n}\sum_{j=1}^{l}\frac{\partial(\xi^\dagger)^{A,\bar{c}\bar{b}_1\cdots\hat{\bar{b}}_j\cdots\bar{b}_l}}{\partial\bar{Z}^c}Z^{b_j}$$
$$+(\kappa^\dagger\cdot\xi^\dagger)^{A,\bar{B}}](\partial/\partial Z)^A\circ(\partial/\partial\bar{Z})^B\,,$$

where $\kappa^\dagger$ is as in Lemma 2.12.

Proof. (i) is immediately obtained from Lemma 2.12. (ii) follows from (2.17). Q.E.D.

3. An ordered pair of linearly independent vectors $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ in $\boldsymbol{C}^{n+1}$ is called a *2-frame*. Denote by $\boldsymbol{W}_2(\boldsymbol{C}^{n+1})$ the manifold of *2-frames* in $\boldsymbol{C}^{n+1}$. $GL(n+1, \boldsymbol{C})$ acts canonically on $\boldsymbol{W}_2(\boldsymbol{C}^{n+1})$. Assume that $\boldsymbol{C}^{n+1}$ is equipped with the canonical flat metric $\tilde{g}_o$. The manifold of orthonormal *2-frames* in $(\boldsymbol{C}^{n+1}, \tilde{g}_o)$ is the *Stiefel manifold,* which we denote by $\boldsymbol{V}_2(\boldsymbol{C}^{n+1})$. $\boldsymbol{V}_2(\boldsymbol{C}^{n+1})$ is identified with the homogeneous space $U(n+1)/U(n-1)$. Let $\boldsymbol{G}_{2,n-1}(\boldsymbol{C})$ be the *Grassmann manifold* of complex linear subspaces of complex dimension 2 in $\boldsymbol{C}^{n+1}$. $\boldsymbol{V}_2(\boldsymbol{C}^{n+1})$

is regarded as a principal bundle over $G_{2,n-1}(C)$ with the typical fibre $U(2)$, the projection of which will be denoted by π_V. Let H_2^+ denote the manifold of positive definite Hermitian matrices of degree 2. Define a map: $W_2(C^{n+1}) \to H_2^+$ by

$$W_2(C^{n+1}) \ni q = (q_1, q_2) \mapsto \rho^2 = (\rho^2_{\alpha\beta}),$$

where

$$\rho^2_{\alpha\beta} = \langle q_\alpha, q_\beta \rangle. \tag{3.1}$$

Let $\rho = \rho(q)$ denote the positive definite Hermitian square root of the matrix ρ^2.

Lemma 3.1. *The mapping Φ: $W_2(C^{n+1}) \to H_2^+ \times V_2(C^{n+1})$ defined by $((q \mapsto (\rho(q), \pi_W(q))$ with $\pi_W(q) = (q_\alpha \rho^{1\alpha}, q_\alpha \rho^{2\alpha}) \in V_2(C^{n+1})$, $\rho_{\alpha\beta} = (\rho^{-1})_{\alpha\beta})$ is a diffeomorphism.*

The proof is obvious.

We assume that $C^{n+1} - \{0\}$ is equipped with the Hermitian metric $r^{-2}\tilde{g}_0$. Let $\Xi^* \in S^{k,l}(C^{n+1} - \{0\})$. The associated covariant tensor field with Ξ^* with respect to the metric $r^{-2}\tilde{g}_0$ is denoted by Ξ_*. The components of Ξ_* are

$$\Xi_{\bar{A},B} = r^{-2p} \Xi^{A,\bar{B}},$$

where $p = |^*(A, \bar{B})|$ by the notations as in **1**. Let $q \in W_2(C^{n+1})$. Then $\pi_W(q)$ is regarded as a linear isometric imbedding: $C^2 \to C^{n+1}$ and thus also be regarded as a linear isometric imbedding: $C^2 - \{0\} \to C^{n+1} - \{0\}$, which we denote by ι_q. The componentwise expression of π_q is given by $(u^0, u^1) \to (Z^0, \cdots, Z^n)$, where

$$Z^a = \sum_{\beta=0}^{1} q^a_\alpha \rho^{\beta\alpha} u^\beta \qquad (0 \leqq a \leqq n,\ \rho^{\alpha\beta} \text{ is as in Lemma 3.1}). \tag{3.2}$$

Here (u^0, u^1) is a fixed complex linear orthonormal coordinate system of C^2. Put $(r_1)^2 = u^0\bar{u}^0 + u^1\bar{u}^1$. Then

$$(\iota_q)^*\Xi_* = ((\iota_q)^* r^{-2p}) \sum_{a_1,\cdots,a_k,b_1,\cdots,b_l=0}^{n} \ \sum_{\alpha_1,\cdots,\alpha_k\beta_1,\cdots,\beta_l=0}^{1} \Xi^{A,\bar{B}} \bar{q}^{a_1}_{\gamma_1} \bar{\rho}^{\alpha_1\gamma_1}$$
$$\cdots \bar{q}^{a_k}_{\gamma_k} \bar{\rho}^{\alpha_k\gamma_k} q^{b_1}_{\delta_1} \rho^{\beta_1\delta_1} \cdots q^{b_l}_{\delta_l} \rho^{\beta_l\delta_l} d\bar{u}^{\alpha_1} \circ \cdots \circ d\bar{u}^{\alpha_k} \circ du^{\beta_1} \circ \cdots \circ du^{\beta_l} \qquad (k+l=p).$$

The contravariant tensor field associated with the covariant tensor field Ξ_* will be designated by the superscript *. Thus we have

$$((\iota_q)^*\Xi_*)^* = \sum_{\alpha_1=\bar{\alpha}_1=0}^{1} \cdots \sum_{\alpha_k=\bar{\alpha}_k=0}^{1} \ \sum_{\beta_1=\bar{\beta}_1=0}^{1} \cdots \sum_{\beta_l=\bar{\beta}_l=0}^{1} (r_1)^{2p} (\iota_q^*\Xi_*)_{\bar{\alpha}_1\cdots\bar{\alpha}_k,\beta_1\cdots\beta_l}$$
$$\cdot \frac{\partial}{\partial u^{\alpha_1}} \circ \cdots \circ \frac{\partial}{\partial u^{\alpha_k}} \circ \frac{\partial}{\partial \bar{u}^{\beta_1}} \circ \cdots \circ \frac{\partial}{\partial \bar{u}^{\beta_l}} = \sum_{a_1,\cdots,a_k,\bar{b}_1,\cdots,\bar{b}_l=0}^{n}$$
$$\cdot \Xi^{A,B} \bar{q}^{a_1}_{\gamma_1} \bar{\rho}^{\alpha_1\gamma_1} \cdots \bar{q}^{a_k}_{\gamma_k} \bar{\rho}^{\alpha_k\gamma_k} q^{b_1}_{\delta_1} \rho^{\beta_1\delta_1} \cdots q^{b_l}_{\delta_l} \rho^{\beta_l\delta_l}$$

$$\cdot\frac{\partial}{\partial u^{\alpha_1}}\circ\cdots\circ\frac{\partial}{\partial u^{\alpha_k}}\circ\frac{\partial}{\partial \bar{u}^{\beta_1}}\circ\cdots\circ\frac{\partial}{\partial \bar{u}^{\beta_l}}\in \boldsymbol{S}^{k,l}(\boldsymbol{C}^2-\{0\}).$$

Furthermore, if $\Xi^*\in(\boldsymbol{S}^\dagger)^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}))$, then $((\iota_q)^*\Xi_*)^*\in(\boldsymbol{S}^\dagger)^{k,l}(\boldsymbol{P}_1(\boldsymbol{C}))$.

Henceforth fundamental differential operators on $(\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_1(\boldsymbol{C}))$ are denoted by $T_1^\dagger$, $(\delta_1^*)^\dagger$, e.t.c..

Define the Radon-Michel transform$\hat{\ }$: $(\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))\rightarrow C^\infty(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$ by

$$(3.3)\qquad (\xi^\dagger)\hat{\ }(\Gamma)=\begin{cases}\dfrac{2^k}{\mathrm{Vol}(S^3)}\displaystyle\int_{S^3}(T_1^\dagger)^k((\iota_q)^*(\xi^\dagger)_*)^*d\sigma & (k=l)\\ 0 & (k\neq l)\end{cases}$$

for $\xi^\dagger\in(\boldsymbol{S}^\dagger)^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}))$, where $\Gamma=\pi_V\circ\pi_W(q)$ and S^3 is the standard unit sphere in $\boldsymbol{C}^2$ with the canonical volume element $d\sigma$. $(\xi^\dagger)\hat{\ }(\Gamma)$ is easily seen to be independent of the choice of q. Evidently $\hat{\ }$ is a $\boldsymbol{C}$-linear map.

Let $p=(\boldsymbol{p}_1, \boldsymbol{p}_2)\in \boldsymbol{V}_2(\boldsymbol{C}^{n+1})$. Put $P^{ab}=p_0^a p_1^b-p_1^a p_0^b$, where $\boldsymbol{p}_\alpha=\sum_{a=0}^{n} p_\alpha^a \boldsymbol{e}_a$ $(\alpha=0, 1)$ for a fixed orthonormal basis $(\boldsymbol{e}_0, \cdots, \boldsymbol{e}_n)$ in $(\boldsymbol{C}^{n+1}, \tilde{g}_o)$. We can easily verify that $\sum_{a,b=0}^{n} P^{ab}\bar{P}^{ab}=2$. (P^{ab}) is called the normalized *Plücker coordinates* of the complex subspace of dimension 2 determined by the frame p.

Theorem 3.1. *The image $((\boldsymbol{K}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C})))\hat{\ }$ of the Radon-Michel transform is the the subalgebra of $C^\infty(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$ generated by the products $P^{ab}P^{cd}$ $(0\leqq a<b\leqq n$, $0\leqq c<d\leqq n)$. It is uniformly dense in $C^\infty(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$.*

Proof. Let $w^\gamma=(-1)^\gamma\bar{u}^\delta/r_1$ $(\gamma=0, 1;\ \delta=1-\gamma)$. Then

$$(3.4)\qquad \delta^{ab}=w^\alpha\bar{w}^\beta+(u^\alpha/r_1)(\bar{u}^\beta/r_1).$$

Put $p_0^a=\sum_{\alpha=0}^{1} q_\gamma^a\rho^{\alpha\gamma}u^\alpha/r_1$, $p_1^a=\sum_{\beta=0}^{1} q_\delta^a\rho^{\beta\delta}w^\beta$. Then the 2-frame $P=(\boldsymbol{p}_0, \boldsymbol{p}_1)$ given by $\boldsymbol{p}_\alpha=\sum_{a=0}^{n} p_\alpha^a\boldsymbol{e}_a$ $(\alpha=0, 1)$ belongs to $\boldsymbol{V}_2(\boldsymbol{C}^{n+1})$. On the other hand, by Lemma 2.7 and (2.14) we have

$$\underset{ab,cd}{\xi^\dagger}=\underset{a\bar{b}}{\xi}\circ\underset{\bar{c}d}{\xi}=(Z^a\partial/\partial\bar{Z}^b-Z^b\partial/\partial\bar{Z}^a)(\bar{Z}^c\partial/\partial Z^d-\bar{Z}^d\partial/\partial Z^c).$$

In virtue of (3.2), (3.3) and (3.4) we can easily verify

$$(3.5)\qquad (\underset{ab,cd}{\xi^\dagger})\hat{\ }=P^{ab}\bar{P}^{cd}.$$

As $(\boldsymbol{K}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ is generated by $\underset{ab,cd}{\xi^\dagger}$'s, $((\boldsymbol{K}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C})))\hat{\ }$ coincides with the subalgebra of $C^\infty(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$ generated by the products $P^{ab}\bar{P}^{cd}$'s above. Thus the first part of the theorem is proved.

From the theorem of Stone-Weierstrass the latter half of the theorem follows. Q.E.D.

Applying Lemma 2.1 to the principal bundle: $\boldsymbol{V}_2(\boldsymbol{C}^{n+1}) \to \boldsymbol{G}_{2,n-1}(\boldsymbol{C})$ with $U(2)$ as fibre, we obtain

$$\mathfrak{D}^*(\boldsymbol{G}_{2,n-1}(\boldsymbol{C})) = (\mathfrak{D}^{U(2)})^*(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))/\mathscr{J}\,, \tag{3.6}$$

where $\mathscr{J}$ is the two sided ideal in $\mathfrak{D}^{U(2)}(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))$ generated by $U(2)$-invariant vertical vector fields. On the other hand, as $\boldsymbol{W}_2(\boldsymbol{C}^{n+1})$ is diffeomorphic to $\boldsymbol{H}_2^+ \times \boldsymbol{V}_2(\boldsymbol{C}^{n+1})$ (Lemma 3.1), there are subalgebras $\tilde{\mathfrak{D}}^*(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))$ and $\tilde{\mathfrak{D}}^*(\boldsymbol{H}_2^+)$ of $\mathfrak{D}^*(\boldsymbol{W}_2(\boldsymbol{C}^{n+1}))$, respectively and each one of $\tilde{\mathfrak{D}}^*(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))$ and $\tilde{\mathfrak{D}}^*(\boldsymbol{H}_2^+)$ is the centralizer of the other in $\mathfrak{D}^*(\boldsymbol{W}_2(\boldsymbol{C}^{n+1}))$ (cf. [4], pp. 651–652, Lemma 1).

Lemma 3.2. (i) $C^\infty(\boldsymbol{V}_2(\boldsymbol{C}^{n+1})) = D^0(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))$ *is canonically isomorphic to the subalgebra* $\tilde{\mathfrak{D}}^0(\boldsymbol{V}_2(\boldsymbol{C}^{n+1})) = \{f \in C^\infty(\boldsymbol{W}_2(\boldsymbol{C}^{n+1}\})) \mid f$ *is constant along each fibre* $\pi_W^{-1}(p)$ *for* $p \in \boldsymbol{V}_2(\boldsymbol{C}^{n+1})\}$ *of* $C^\infty(\boldsymbol{W}_2(\boldsymbol{C}^{n+1}))$. $\mathfrak{D}^*(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))$ *is canonically isomorphic to the subalgebra* $\tilde{\mathfrak{D}}^*(\boldsymbol{V}_2(\boldsymbol{C}^{n+1})) \overset{\text{def.}}{=} \{D \in \mathfrak{D}^*(\boldsymbol{W}_2(\boldsymbol{C}^{n+1})) \mid [\rho_{\alpha\beta}, D] = 0.$ $[\partial/\partial\rho_{\alpha\beta}, D] = 0$ $(\alpha, \beta = 0, 1)\}$ *of* $\mathfrak{D}^*(\boldsymbol{W}_2(\boldsymbol{C}^{n+1}))$. (ii) $\mathfrak{D}^*(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$ *is isomorphic to the quotient algebra* $\widetilde{(\mathfrak{D}^{U(2)})^*(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))}/\tilde{\mathscr{J}}$, *where* $\widetilde{(\mathfrak{D}^{U(2)})^*(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))}$ *(resp.* $\tilde{\mathscr{J}}$) *is the subalgebra of* $\mathfrak{D}^*(\boldsymbol{W}_2(\boldsymbol{C}^{n+1}))$ *(resp. the two-sided ideal in* $\widetilde{(\mathfrak{D}^{U(2)})^*(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))}$ *corresponding to* $(\mathfrak{D}^{U(2)})^*(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))$ *(resp.* $\mathscr{J}$ *in* $(\mathfrak{D}^{U(2)})^*(\boldsymbol{V}_2(\boldsymbol{C}^{n+1}))$) *by the canonical isomorphism in* (i).

Proof. (i) follows from Lemma 1 in [4] cited above. (ii) is immediately obtained from (2.6) and (i). Q.E.D.

Put $g_1 = \sum_{a<b} dP^{ab} d\bar{P}^{ab}$, which is an $U(n+1)$-invariant Riemannian metric on $\boldsymbol{G}_{2,n-1}(\boldsymbol{C})$. $\boldsymbol{S}^{**}(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$ is equipped with the Hermitian inner product corresponding to the volume element of g_1 by (1.6).

Lemma 3.3. (i) $(\pi_V)^*(g_1) = \sum_{a,b=0}^{n} (\delta^{ab} - \bar{p}_\alpha^a p_\beta^b \delta^{\alpha\beta}) \delta^{\gamma\delta} dp_\gamma^a d\bar{p}_\delta^b$, *where* $p_\alpha^a = q_\gamma^a \rho^{\alpha\gamma}$ $(\alpha = 0, 1)$ *are components of* $p = (\boldsymbol{p}_1, \boldsymbol{p}_2) \in \boldsymbol{V}_2(\boldsymbol{C}^{n+1})$.

(ii) $(\pi_V \cdot \pi_W)^*(g_1) = \sum_{a,b=0}^{n} (\delta^{ab} - \bar{q}_\alpha^a q_\beta^b (\rho^2)^{\alpha\beta})(\rho^2)^{\gamma\delta} dq_\gamma^a \bar{q}_\delta^b$, *where* $(\rho^2)^{\alpha\beta}(\rho^2)_{\beta\gamma} = \delta_\gamma^\alpha$.

Proof. In virtue of the identities $\sum_{a,b=0}^{n} P^{ab}\bar{P}^{ab} = 2$ and $\langle \boldsymbol{p}_\alpha\ \boldsymbol{p}_\beta \rangle = 1$, $(\pi_V)^* g_1 = \sum_{a<b} d(p_0^a p_1^b - p_1^a p_0^b) d(\bar{p}_0^a \bar{p}_1^b - \bar{p}_0^a \bar{p}_1^b)$ is reduced to the required expression in (i). (ii) follows from (i) with the aid of the identity

$$\delta^{ab} - \bar{q}_\alpha^a q_\beta^b (\rho^2)^{\alpha\beta} = \delta^{ab} - \bar{p}_\alpha^a p_\beta^b \delta^{\alpha\beta}\,. \tag{3.7}$$

Q.E.D.

REMARK. The space $C^\infty(\boldsymbol{W}_2(\boldsymbol{C}^{n+1}))$ of smooth functions with compact support is equipped with the Hermitian inner product associated with the volume element of the metric $g_W=\sum_{a=0}^{n} d\bar{q}^a_\alpha dq^a_\beta(\rho^2)^{\alpha\beta}$. Let $\tilde{D}_1$ (resp. $\tilde{D}_2$) be a representative of D_1 (resp. $D_2)\in\mathfrak{D}^*(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$. If $\tilde{D}_2$ is the adjoint operator of $\tilde{D}_1$ in $\mathfrak{D}^*(V_2(\boldsymbol{C}^{n+1}))$, then D_2 is the adjoint operator of D_1 in $\mathfrak{D}^*(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$.

Denote by $\Delta^\wedge$ the Laplacian of $(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}), g_1)$ and put $\tilde{\Delta}^\wedge=-4\sum_{a,b=0}^{n}(\delta^{ab}-\bar{q}^a_\alpha q^b_\beta(\rho^2)^{\alpha\beta})(\rho^2)_{\gamma\delta}\partial^2/\partial\bar{q}^a_\gamma\partial q^b_\delta+2(n-1)(q^a_\alpha\partial/\partial q^a_\alpha+\bar{q}^a_\alpha\partial/\partial\bar{q}^a_\alpha)$, where $(\rho^2)^{\alpha\beta}=(\rho^{-2})_{\alpha\beta}$.

Lemma 3.4. *$\tilde{\Delta}^\wedge$ is a representative of $\Delta^\wedge$ in $\tilde{\mathfrak{D}}^2(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$.*

Proof. We can easily verify that $[\tilde{\Delta}^\wedge, \rho_{\alpha\beta}]=0$ and $[\tilde{\Delta}^\wedge, \partial/\partial\rho_{\alpha\beta}]=0$ $(0\leqq\alpha, \beta\leqq n)$. Thus $\tilde{\Delta}^\wedge\in\tilde{\mathfrak{D}}^2(V_2(\boldsymbol{C}^{n+1}))$. Moreover, it is seen to be $GL(2, \boldsymbol{C})$-invariant and consequently it represents some linear differential operator belonging tc $\mathfrak{D}^2(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$. The following three conditions are easily verified: (i) $\tilde{\Delta}^\wedge$ is a self-adjoint operator in $\tilde{\mathfrak{D}}^2(V_2(\boldsymbol{C}^{n+1}))$. (ii) $\sigma^2(\tilde{\Delta})^\wedge\in\widetilde{\boldsymbol{S}^{1,1}}(V_2(\boldsymbol{C}^{n+1}))$ represents $(g_1)^*$, where $(g_1)^*$ is the contravariant Riemannian metric associated with g_1. (iii) $\tilde{\Delta}^\wedge$ annihilates constants. Thus we can conclude that $\tilde{\Delta}^\wedge$ is a representative of $\Delta^\wedge$.

Lemma 3.5. (i) $\tilde{\Delta}^\wedge(\rho^{\alpha\beta}f)=\rho^{\alpha\beta}\tilde{\Delta}^\wedge f$ *for* $f\in C^\infty(\boldsymbol{W}_2(\boldsymbol{C}^{n+1}))$,

(ii) $\tilde{\Delta}^\wedge(q^a_\gamma\rho^{\alpha\gamma})=2(n-1)q^a_\gamma\rho^{\alpha\gamma}$,

$\overline{\text{(ii)}}$ $\tilde{\Delta}^\wedge(\bar{q}^a_\gamma\bar{\rho}^{\alpha\gamma})=2(n-1)\bar{q}^a_\gamma\bar{\rho}^{\alpha\gamma}$,

(iii) $\tilde{\Delta}^\wedge(q^a_\gamma q^b_\delta\rho^{\alpha\gamma}\rho^{\beta\delta})=4(n-1)q^a_\gamma q^b_\delta\rho^{\alpha\gamma}\rho^{\beta\delta}$,

$\overline{\text{(iii)}}$ $\tilde{\Delta}^\wedge(\bar{q}^a_\gamma\bar{q}^b_\delta\bar{\rho}^{\alpha\gamma}\bar{\rho}^{\beta\delta})=4(n-1)\bar{q}^a_\gamma\bar{q}^b_\delta\bar{\rho}^{\alpha\gamma}\bar{\rho}^{\beta\delta}$,

(iii′) $\tilde{\Delta}^\wedge(\bar{q}^a_\gamma q^b_\delta\bar{\rho}^{\alpha\gamma}\rho^{\beta\delta})=4\delta^{\alpha\beta}(-\delta^{ab}+\bar{q}^a_\gamma q^b_\delta(\rho^2)^{\gamma\delta})+4(n-1)\bar{q}^a_\gamma q^b_\delta\bar{\rho}^{\alpha\gamma}\rho^{\beta\delta}$.

(iv) $4(-\delta^{ab}+\bar{q}^a_\alpha q^b_\beta(\rho^2)^{\alpha\beta})(\rho^2)_{\gamma\delta}\left(\frac{\partial(q^c_\nu\rho^{\tau\nu})}{\partial\bar{q}^a_\gamma}\frac{\partial(q^d_\mu\rho^{\varepsilon\mu})}{\partial q^b_\delta}+\frac{\partial(q^c_\nu\rho^{\tau\nu})}{\partial q^b_\delta}\frac{\partial(q^d_\mu\rho^{\varepsilon\mu})}{\partial\bar{q}^a_\gamma}\right)=0$,

$\overline{\text{(iv)}}$ $4(-\delta^{ab}+\bar{q}^a_\alpha q^b_\beta(\rho^2)^{\alpha\beta})(\rho^2)_{\gamma\delta}\left(\frac{\partial(\bar{q}^c_\nu\bar{\rho}^{\tau\nu})}{\partial\bar{q}^a_\gamma}\frac{\partial(\bar{q}^d_\mu\bar{\rho}^{\varepsilon\mu})}{\partial q^b_\delta}+\frac{\partial(\bar{q}^c_\nu\bar{\rho}^{\tau\nu})}{\partial q^b_\delta}\frac{\partial(\bar{q}^d_\mu\bar{\rho}^{\varepsilon\mu})}{\partial\bar{q}^a_\beta}\right)=0$,

(iv′) $4(-\delta^{ab}+\bar{q}^a_\alpha q^b_\beta(\rho^2)^{\alpha\beta})(\rho^2)_{\gamma\delta}\left(\frac{\partial(q^c_\nu\rho^{\tau\nu})}{\partial\bar{q}^a_\gamma}\frac{\partial(\bar{q}^d_\mu\bar{\rho}^{\varepsilon\mu})}{\partial q^b_\delta}+\frac{\partial(q^c_\nu\rho^{\tau\nu})}{\partial q^b_\delta}\frac{\partial(\bar{q}^d_\mu\bar{\rho}^{\varepsilon\mu})}{\partial\bar{q}^a_\gamma}\right)$

$=4(-\delta^{cd}+\bar{q}^c_\alpha q^d_\beta(\rho^2)^{\alpha\beta})\delta^{\tau\varepsilon}$.

Proof. These are verified by direct calculations. Q.E.D.

Theorem 3.2. *Let $\xi^\dagger\in(\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$. Then*

$$\Delta^\wedge(\xi^\dagger)^\wedge=(\Delta^\dagger\xi^\dagger)^\wedge,$$

where $\Delta^\wedge$ is as in Lemma 3.4.

Proof. Let $\xi^\dagger\in(\boldsymbol{S}^\dagger)^{k,k}(\boldsymbol{P}_n(\boldsymbol{C}))$. Consider a representative $\widetilde{(\xi^\dagger)^\wedge}$ in

$C^\infty(\boldsymbol{W}_2(\boldsymbol{C}^{n+1}))$ of $(\xi^\dagger)^\wedge \in C^\infty(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$. Let $\Gamma \in \boldsymbol{G}_{2,n-1}(\boldsymbol{C})$ and choose $q \in \boldsymbol{W}_2(\boldsymbol{C}^{n+1})$ so that $(\pi_V \circ \pi_W)(q) = \Gamma$. Then we have

$$
\begin{aligned}
\Delta(\widetilde{(\xi^\dagger)^\wedge})(q) = & \frac{2^k}{\mathrm{vol.}\,(S^3)} \int_{S^3} \sum_{a_1,\cdots,a_k,b_1,\cdots,b_k=0}^{n} -4(\delta^{ab} - \bar{q}^a_\alpha q^b_\beta (\rho^2)^{\alpha\beta})(\rho^2)_{\gamma\delta}) \\
& \cdot \Big[\frac{\partial^2 Z^c}{\partial \bar{q}^a_\gamma \partial q^b_\delta} \frac{\partial(\xi^{\dagger A,\bar{B}})}{\partial Z^c} + \frac{\partial^2 \bar{Z}^c}{\partial \bar{q}^a_\gamma \partial q^b_\delta} \frac{\partial(\xi^{\dagger A,\bar{B}})}{\partial \bar{Z}^c} + \frac{\partial Z^c}{\partial \bar{q}^a_\gamma} \frac{\partial Z^d}{\partial q^b_\delta} \frac{\partial^2(\xi^{\dagger A,\bar{B}})}{\partial Z^c \partial Z^d} \\
& + \Big(\frac{\partial Z^c}{\partial \bar{q}^a_\gamma} \frac{\partial \bar{Z}^d}{\partial q^b_\delta} + \frac{\partial \bar{Z}^d}{\partial \bar{q}^a_\gamma} \frac{\partial Z^c}{\partial q^b_\delta} \Big) \frac{\partial^2(\xi^{\dagger A,\bar{B}})}{\partial Z^c \partial \bar{Z}^d} + \frac{\partial \bar{Z}^c}{\partial \bar{q}^a_\gamma} \frac{\partial \bar{Z}^d}{\partial q^b_\delta} \frac{\partial^2(\xi^{\dagger A,B})}{\partial \bar{Z}^c \partial \bar{Z}^d} \Big) \bar{W}^A W^B \\
& + k \Big\{ \Big(\frac{\partial \bar{W}^{a_k}}{\partial \bar{q}^a_\gamma} \frac{\partial Z^c}{\partial q^b_\delta} + \frac{\partial Z^c}{\partial \bar{q}^a_\gamma} \frac{\partial \bar{W}^{a_k}}{\partial q^b_\delta} \Big) \frac{\partial(\xi^{\dagger A,\bar{B}})}{\partial Z^c} + \Big(\frac{\partial \bar{W}^{a_k}}{\partial \bar{q}^a_\gamma} \frac{\partial \bar{Z}^c}{\partial q^b_\delta} + \frac{\partial \bar{Z}^c}{\bar{\partial} q^a_\gamma} \frac{\partial \bar{W}^{a_k}}{\partial q^b_\delta} \Big) \\
& \cdot \frac{\partial(\xi^{\dagger A,\bar{B}})}{\partial \bar{Z}^c} + \frac{\partial^2 \bar{W}^{a_k}}{\partial \bar{q}^a_\gamma \partial q^b_\delta} \xi^{\dagger A,B} \Big\} \bar{W}^{a_1} \cdots \bar{W}^{a_{k-1}} W^B + k \Big\{ \Big(\frac{\partial W^{b_k}}{\partial \bar{q}^a_\gamma} \frac{\partial Z^c}{\partial q^b_\delta} + \frac{\partial Z^c}{\partial \bar{q}^a_\gamma} \\
& \cdot \frac{\partial W^{b_k}}{\partial q^b_\delta} \Big) \frac{\partial(\xi^{\dagger A,\bar{B}})}{\partial Z^c} + \Big(\frac{\partial \bar{Z}^c}{\partial \bar{q}^a_\gamma} \frac{\partial W^{b_k}}{\partial q^b_\delta} + \frac{\partial W^{b_k}}{\partial \bar{q}^a_\gamma} \frac{\partial \bar{Z}^c}{\partial q^b_\delta} \Big) \frac{\partial(\xi^{\dagger A,\bar{B}})}{\partial \bar{Z}^c} + \frac{\partial^2 W^{b_k}}{\partial \bar{q}^a_\beta \partial q^b_\delta} \xi^{\dagger A,\bar{B}} \Big\} \\
& \cdot \bar{W}^A W^{b_1} \cdots W^{b_{k-1}} + \frac{k(k-1)}{2} \Big(\frac{\partial \bar{W}^{a_{k-1}}}{\partial \bar{q}^a_\gamma} \frac{\bar{W}^{a_k}}{\partial q^b_\delta} + \frac{\bar{W}^{a_k}}{\partial \bar{q}^a_\gamma} \frac{\bar{W}^{a_{k-1}}}{\partial q^b_\delta} \Big) \xi^{\dagger A,\bar{B}} \bar{W}^{a_1} \cdots \\
& \bar{W}^{a_{k-2}} + \frac{k(k-1)}{2} \Big(\frac{\partial W^{b_k}}{\partial \bar{q}^a_\gamma} \frac{\partial W^{b_{k-1}}}{\partial q^\delta_b} + \frac{\partial W^{b_k}}{\partial \bar{q}^a_\gamma} \frac{\partial W^{b_{k-1}}}{\partial q^b_\delta} \Big) \xi^{\dagger A,\bar{B}} \bar{W}^A W^{b_1} \cdots W^{b_{k-2}} \\
& + k^2 \Big(\frac{\partial \bar{W}^{a_k}}{\partial \bar{q}^a_\gamma} \frac{\partial W^{b_k}}{\partial q^b_\delta} + \frac{\partial W^{b_k}}{\partial \bar{q}^a_\gamma} \frac{\partial \bar{W}^{a_k}}{\partial q^b_\delta} \Big) \xi^{\dagger A,B} \bar{W}^{a_1} \cdots \bar{W}^{a_{k-1}} W^{b_1} \cdots W^{b_{k-1}} \Big] d\sigma \\
& + \frac{2(n-1)2^k}{\mathrm{Vol.}\,(S^3)} \int_{S^3} \sum_{a_1,\cdots a_k,b_1 \cdots b_k=0}^{n} q^a_\gamma \Big[\frac{\partial Z^c}{\partial q^a_\gamma} \frac{\partial(\xi^{\dagger A,\bar{B}})}{\partial Z^c} \bar{W}^A W^B \\
& + \frac{\partial \bar{Z}^c}{\partial q^a_\gamma} \frac{\partial(\xi^{\dagger A,\bar{B}})}{\partial \bar{Z}^c} \bar{W}^A W^B + k \xi^{\dagger A,\bar{B}} \frac{\partial \bar{W}^{a_k}}{\partial q^a_\gamma} \bar{W}^{a_1} \cdots \bar{W}^{a_{k-1}} W^B \\
& + k \xi^{\dagger A,\bar{B}} \frac{\partial W^{b_k}}{\partial q^a_\gamma} \bar{W}^A W^{b_1} \cdots W^{b_{k-1}} \Big] d\sigma + \frac{2(n-1)2^k}{\mathrm{vol.}\,(S^3)} \\
& \cdot \int_{S^3} \sum_{a_1 \cdots a_k,b_1 \cdots b_k=0}^{n} \bar{q}^a_\gamma \Big[\frac{\partial Z^c}{\partial \bar{q}^a_\gamma} \frac{\partial(\xi^{\dagger A,\bar{B}})}{\partial Z^c} \bar{W}^A W^B + \frac{\partial \bar{Z}^c}{\partial \bar{q}^a_\gamma} \frac{\partial(\xi^{\dagger A,\bar{B}})}{\partial \bar{Z}^c} \bar{W}^A W^B \\
& + k \xi^{\dagger A,\bar{B}} \frac{\partial \bar{W}^{a_k}}{\partial \bar{q}^a_\gamma} \bar{W}^{a_1} \cdots \bar{W}^{a_{k-1}} W^B + k \xi^{\dagger A,\bar{B}} \frac{\partial W^{b_k}}{\partial \bar{q}^a_\gamma} \bar{W}^A W^{b_1} \cdots W^{b_{k-1}} \Big] d\sigma,
\end{aligned}
$$

where $\bar{W}^A = \bar{W}^{a_1} \cdots \bar{W}^{a_k}$ and $W^B = W^{b_1} \cdots W^{b_k}$ with $\bar{W}^a = \sum_{\alpha=0}^{1} \bar{q}^a_\beta \bar{\rho}^{\alpha\beta} \bar{w}^\alpha$, $W^a = \sum_{\alpha=0}^{1} q^a_\beta \rho^{\alpha\beta} w^\alpha$, $Z^c = \sum_{\alpha=0}^{1} q^c_\beta \rho^{\alpha\beta} u^\alpha$ and $\bar{Z}^c = \sum_{\alpha=0}^{1} \bar{q}^c_\beta \bar{\rho}^{\alpha\beta} \bar{u}^\alpha$ (cf. (3.2)).

The first term of the first integral together with the first terms of the second and the third integrals becomes by virtue of Lemma 3.5 (ii)

$$\frac{2^k}{\mathrm{Vol.}\,(S^3)}\int_{S^3}\sum_{a_1\cdots a_k,b_1\cdots b_k=0}^{n}(\Delta^\wedge Z^c)\frac{\partial(\xi^{\dagger A,\bar{B}})}{\partial Z^c}\bar{W}^A W^B d\sigma$$
$$=2k(n-1)(\xi^\dagger)^\wedge(\Gamma)=2k(n-1)\widetilde{(\xi^\dagger)^\wedge}(q)\,.$$

Similarly the second term of the first integral together with the second terms of the second and the third integrals becomes

$$\frac{2^k}{\mathrm{vol.}\,(S^3)}\int_{S^3}\sum_{a_1,\cdots,a_k,b_1,\cdots,b_k=0}^{n}(\Delta^\wedge\bar{Z}^c)\frac{\partial(\xi^\dagger)^{A,\bar{B}}}{\partial\bar{Z}^c}\bar{W}^A W^B d\sigma$$
$$=2k(n-1)(\xi^\dagger)^\wedge(\Gamma)=2k(n-1)\widetilde{(\xi^\dagger)^\wedge}(q)\,.$$

The third (resp. fifth term) of the first integral vanishes by virtue of Lemma 3.5 (iv) (resp. of Lemma 3.5 $\overline{\text{(iv)}}$). The eighth term of the first integral together with the third ones of the second and the third integrals becomes with the aid of Lemma 3.5 (ii)

$$\frac{2^k}{\mathrm{vol.}\,(S^3)}\int_{S^3}\sum_{a_1,\cdots,a_k,b_1,\cdots,b_k=0}^{n}(\Delta^\wedge\bar{W}^{a_k})(\xi^\dagger)^{A,\bar{B}}\bar{W}^{a_1}\cdots\bar{W}^{a_{k-1}}W^B d\sigma$$
$$=2k(n-1)(\xi^\dagger)^\wedge(\Gamma)=2k(n-1)\widetilde{(\xi^\dagger)^\wedge}(q)\,.$$

Similarly the eleventh term of the first integral together with the fourth ones of the second and the third integrals respectively becomes

$$\frac{2^k}{\mathrm{vol.}\,(S^3)}\int_{S^3}\sum_{a_1,\cdots,a_k,b_1,\cdots,b_j=0}^{n}(\Delta^\wedge W^{b_k})(\xi^\dagger)^{A,\bar{B}}\bar{W}^A W^{b_1}\cdots W^{b_{k-1}}d\sigma$$
$$=2k(n-1)(\xi^\dagger)^\wedge(\Gamma)=2k(n-1)\widetilde{(\xi^\dagger)^\wedge}(q)\,.$$

by virtue of Lemma 3.5 (ii). The twelfth (resp. the thirteenth) term of the first integral vansihes by virtue of Lemma 3.5 $\overline{\text{(iv)}}$ (resp. by Lemma 3.5 (iv)). The sixth term of the first integral vanishes by virtue of Lemma 3.5 (iv′) and of the following formula.

$$\sum_{\varepsilon=0}^{1}u^\varepsilon\bar{w}^\varepsilon=0\qquad\text{(cf. the proof of Theorem 3.1)}\,.$$

Similarly, the tenth term of the first integral vanishes. The fourth term of the first integral is by virtue of Lemma 3.6 (iv′) rewritten as

$$\frac{2^k}{\mathrm{vol.}(S^3)}\int_{S^3}\sum_{a_1,\cdots,a_k,b_1,\cdots,b_k=0}^{n}4(-\delta^{cd}+q^c_\alpha\bar{q}^d_\beta(\rho^2)^{\alpha\beta})\frac{\partial^2(\xi^\dagger)^{A,\bar{B}}}{\partial Z^c\partial\bar{Z}^d}\bar{W}^A W^B d\sigma.$$

On the other hand, we have $-\delta^{cd}+q^c_\alpha\bar{q}^d_\beta(\rho^2)^{\alpha\beta}=-\delta^{cd}+Z^c\bar{Z}^d+W^c\bar{W}^d$ (cf. (3.4)). Thus the integral above is reduced to

$$\frac{2^k}{\mathrm{vol.}(S^3)}\int_{S^3}\sum_{a_1,\cdots,a_k,b_1,\cdots,b_k=0}^{n}4(-\delta^{cd}+Z^c\bar{Z}^d)\frac{\partial^2(\xi^\dagger)^{A,\bar{B}}}{\partial Z^c\partial\bar{Z}^d}\bar{W}^A W^B d\sigma\,,$$

because the identity

$$\int_{S^3} \sum_{a_1,\cdots,a_k,b_1,\cdots,b_k=0}^{n} \frac{\partial^2(\xi^\dagger)^{A,\bar{B}}}{\partial Z^c \partial \bar{Z}^d} \bar{W}^d \bar{W}^A W^c W^B d\sigma = 0$$

follows from Corollary to Lemma 2.2 and the Stoke's theorem. Similarly the fourteenth term of the first integral is rewritten as

$$k^2 \int_{S^3} \sum_{a_1,\cdots,a_k,b_1,\cdots,b_k=0}^{n} 4(-\delta^{a_k b_k} + Z^{a_k}\bar{Z}^{b_k} + W^{a_k}\bar{W}^{b_k})(\xi^\dagger)^{A,\bar{B}} \bar{W}^{a_1}\cdots \bar{W}^{a_{k-1}} W^{b_1}\cdots W^{b_{k-1}} d\sigma$$

$$= 4k^2(\xi^\dagger)^\wedge(\Gamma) - 4((T^*)^\dagger T^\dagger \xi^\dagger)^\wedge(\Gamma) = 2\widetilde{((\zeta^2+\bar{\zeta}^2-\zeta\circ\bar{\zeta})(\xi^\dagger))^\wedge}(q) - 4\widetilde{((T^*)^\dagger T^\dagger \xi^\dagger)^\wedge}(q)\,.$$

Comparing these results with the expression of $\Delta^\dagger$ in **2**, we obtain

$$\widetilde{\Delta^\wedge(\xi^\dagger)^\wedge} = \widetilde{(\Delta^\dagger \xi^\dagger)^\wedge}\,.$$

From this we conclude that

$$\Delta^\wedge(\xi^\dagger)^\wedge = (\Delta^\dagger \xi^\dagger)^\wedge\,. \qquad \text{Q.E.D.}$$

4. Eigenspace decomposition of Lichnerowicz operator on $K^{**}(P_n(C))$

Put $\lambda'_{k,l,m} = 4((2k-m)n + 3k^2 + l^2 - 2kl - (m+1)(k+l) + m^2 + 2m)$. Let S: $(S^\dagger)^{**}(P_n(C)) \to (S^\dagger)^{**}(P_n(C))$ be the linear differential operator of bidegree $(-1, -1)$ defined by

$$(4.1) \quad S = \Delta^\dagger T^\dagger - \lambda'_{k,l,1} T^\dagger + 6(T^*)^\dagger (T^\dagger)^2 - \partial^\dagger T^\dagger (\partial^*)^\dagger + (\partial^*)^\dagger T^\dagger \partial^\dagger$$

on $(S^\dagger)^{k,l}(P_n(C))$.

Lemma 4.1. $(2m/(m+1))(T^\dagger)^{m-1}S = \Delta^\dagger(T^\dagger)^m - \lambda'_{k,l,m}(T^\dagger)^m + 4(m+2)/(m+1) \cdot (T^*)^\dagger (T^\dagger)^{m+1} - (2/(m+1))\{\partial^\dagger (T^\dagger)^m (\partial^*)^\dagger - (\partial^*)^\dagger (T^\dagger)^m \partial^\dagger\}$.

Proof. From Lemma 1.1 (ii) we obtain

$$\begin{aligned}(4.2) \qquad (T^\dagger)^{m-1}S = {} & \Delta^\dagger (T^\dagger)^m - \lambda'_{k,l,1}(T^\dagger)^m + 6(T^*)^\dagger (T^\dagger)^{m+1} \\ & + ((m-1)(n+p) - (m-1)^2 + (m-1))(T^\dagger)^m \\ & - \partial^\dagger (T^\dagger)^m (\partial^*)^\dagger + (\partial^*)^\dagger (T^\dagger)^m \partial^\dagger - (m-1)(T^\dagger)^{m-1} \bar{\partial}^\dagger \partial^\dagger\,.\end{aligned}$$

In the same way as in the proof of the first equality of Theorem 1.2, we have

$$\Delta^\dagger = \lambda'_{k,l,0} \mathbf{1}_{k,l} - 8(T^*)^\dagger T^\dagger + 2(\Box')^\dagger\,,$$

where $\mathbf{1}_{k,l}$ is the identity opreator: $(S^\dagger)^{k,l}(P_n(C)) \to (S^\dagger)^{k,l}(P_n(C))$. Applying $(T^\dagger)^m$ to the identity above, we obtain

$$\Delta^\dagger(T^\dagger)^m = \lambda'_{k,l,0}(T^\dagger)^m - 8(T^*)^\dagger(T^\dagger)^{m+1} - 8(mn+mp-m^2+m)(T^\dagger)^m$$
$$-2(\partial^\dagger(T^\dagger)^m(\partial^*)^\dagger + (\partial^*)^\dagger(T^\dagger)^m\partial^\dagger - m\bar{\partial}^\dagger\partial^\dagger(T^\dagger)^{m-1})$$

on $(\boldsymbol{S}^\dagger)^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}))$ $(p=k+l)$.

Eliminating $\bar{\partial}^\dagger\partial^\dagger$ from (4.2) and the equality above, we can obtain the desired formula. Q.E.D.

We remark that the image of S restricted to $(\boldsymbol{K}^\dagger)^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}))$ lies in $(\boldsymbol{K}^\dagger)^{k-1,l-1}(\boldsymbol{P}_n(\boldsymbol{C}))$. This fact is deduced from the following formulae:

$$[(\partial^*)^\dagger, S] = 4(n+3k-l+2)T^\dagger(\partial^*)^\dagger ,$$
$$[(\bar{\partial}^*)^\dagger, S] = -4(k-l)\partial^\dagger + 4(n-k+l)T^\dagger(\bar{\partial}^*)^\dagger .$$

Put

$$B_m^* = 4m(m+1)(T^*)^\dagger + 2(\partial^*)^\dagger(\bar{\partial}^*)^\dagger ,$$
$$A_m^* = (\prod_{i=1}^m B_i^*)(T^\dagger)^m \tag{4.3}$$

for any integer $m \geqq 0$.

Lemma 4.2. (i) $(2m/(m+1))(T^\dagger)^{m-1}S = \Delta^\dagger(T^\dagger)^m - \lambda_{k,m}(T^\dagger)^m + (1/(m+1)^2) \cdot B_{m+1}^*(T^\dagger)^m$ *on* $(\boldsymbol{K}^\dagger)^{k,k}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ *for* $m \geqq 0$, *where*

$$\lambda_{k,m} = 4((2k-m)n + 2k^2 - 2(m+1)k + m^2 + 2m) = \lambda'_{k,k,m} .$$

(ii) A_k^* *leaves* $(\boldsymbol{K}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ *invariant.*

Proof. (i) follows from Lemma 4.1 immedately. We prove (ii) by induction on m. For $m=0$ (i) is reduced to

$$0 = \Delta^\dagger - \lambda_{k,0}\mathbf{1}_{k,k} + A_1^* ,$$

where $\mathbf{1}_{k,k}$ is the identity operator on $(\boldsymbol{K}^\dagger)^{k,k}(\boldsymbol{P}_n(\boldsymbol{C}))$. $\Delta^\dagger$ leaves $(\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))$ and $(\boldsymbol{K}^\dagger)^*(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ invariant and consequently leaves $(\boldsymbol{K}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ invariant. From the equality above we see that A_1^* leaves $(\boldsymbol{K}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ invariant. Applying $\prod_{i=1}^n B_i^*$ to the both sides of (i), we obtain

$$(2m/(m+1))B_m^*A_{m-1}^*S = \Delta^\dagger A_m^* - \lambda_{k,m}A_m^* + (1/(m+1)^2)A_{m+1}^* . \tag{4.4}$$

From the induction hypothesis and the remark to Lemma 4.1 we can conclude that A_{m+1}^* also preserves $(\boldsymbol{K}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$. Q.E.D.

Define $(\boldsymbol{P}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g) = \sum_{k,l \geqq 0} (\boldsymbol{P}^\dagger)^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ (direct sum) with

$$(\boldsymbol{P}^\dagger)^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}), g_0) = (\boldsymbol{K}^\dagger)^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}), g_0) \cap ((T^*)^\dagger((\boldsymbol{K}^\dagger)^{k-1,l-1}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)))^\perp ,$$

where $A^{\perp}$ denotes the orthogonal complement of A in $(\boldsymbol{S}^{\dagger})^{k,l}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$. Let Π_0: $(\boldsymbol{K}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0) \rightarrow (\boldsymbol{K}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ be the orthogonal projection with the image $(\boldsymbol{P}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$. Put

$$H_m = \Pi_0 A_m^* . \tag{4.5}$$

Lemma 4.3. $\Delta^{\dagger} H_m - \lambda_{k,m} H_m + (1/(m+1)^2) H_{m+1} = 0$ *on* $(\boldsymbol{P}^{\dagger})^{k,k}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ $(m \geqq 0)$.

Proof. From Lemma 4.1, 4.2 (ii) and the first equality of (4.3) the image of the left-hand side of the equality (4.4) restricted to $(\boldsymbol{K}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ is contained in $(T^*)^{\dagger}((\boldsymbol{K}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0))$. Applying Π_0 to (4.4), we obtain the lemma. Q.E.D.

Define a linear endomorphism $P_{k,m}$ of $(\boldsymbol{P}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ by

$$P_{k,m} = \frac{n+2k-2m-2}{m! \cdot (n+2k-m-2)!} \sum_{i=m}^{k} \frac{(-1)^{i-m}(n+2k-i-m-3)!}{2^{2i}(i!)^2 \cdot (i-m)!} \qquad (k \geqq m \geqq 0) .$$

Theorem 4.1. (i) $\Delta^{\dagger} = \sum_{m=0}^{n} \lambda_{k,m} P_{k,m}$ *on* $(\boldsymbol{P}^{\dagger})^{k,k}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$, *and this gives the eigenspace decomposition of* $\Delta^{\dagger}$ *restricted to* $(\boldsymbol{P}^{\dagger})^{k,k}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$. (ii) $P_{k,m} = \delta_{km} \mathbf{1}_{k,k}$ *on* $(\boldsymbol{P}^{\dagger})^{k,k}(\boldsymbol{P}_2(\boldsymbol{C}), g_0)$. (iii) $P_{km} \neq 0$ *for* $k \geqq m \geqq 0$, $n \geqq 3$. (iv) $(\boldsymbol{K}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0) = \sum_{h \geqq 0} \sum_{m=0}^{h} ((T^*)^{\dagger})^m \cdot ((\boldsymbol{P}^{\dagger})^{h-m,k-m}(\boldsymbol{P}_n(\boldsymbol{C}), g_0))$ (*direct sum*). *Thus the eigenspace decomposition* (i) *yields that of* $(\boldsymbol{K}^{\dagger})^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$.

The proof of Theorem 4.1 is divided into the following eight lemmas.

Lemma 4.4. $\displaystyle\sum_{m=0}^{n} \frac{(-1)^m \binom{n}{m}}{(a+m)\cdots(a+m+r)} = \frac{(n+r)!}{r! \cdot a(a+1)\cdots(a+n+r)}$.

Proof. For $r=0$ the left-hand side of the assertion is equal to

$$\sum_{m=}^{n} \frac{(-1)^m \binom{n}{m}}{a+m} = \int_0^1 (1-x)^n x^{a-1} dx = B(n+1, a) = \Gamma(n+1) \cdot \Gamma(a) / \Gamma(n+a+1) ,$$

which coincides with the right-hand side. The lemma follows by induction on r using elementary difference calculus. Q.E.D.

Lemma 4.5. $\displaystyle\sum_{m=0}^{j} \frac{(-1)^m \binom{j}{m}(x-2m)}{\prod_{i=0}^{j}(x-m-i)} = \delta_0^j$.

Proof. For $j=0$ the equality is immediately verified. If $j\geqq 1$ the left-hand side of the assertion is expressed as the sum of two terms:

$$2\sum_{m=0}^{j}\frac{(-1)^m\binom{j}{m}}{\prod_{i=1}^{j}(x-m-i)}+\sum_{m=0}^{j}\frac{(-1)^m\binom{j}{m}}{\prod_{i=0}^{j}(x-m-i)}.$$

Taking Lemma 4.4 into account, we see imemdiately that the quantity above vanishes for $j\geqq 1$. Q.E.D.

Lemma 4.6. $\Delta^\dagger P_{k,m}=\lambda_{k,m}P_{k,m}$ *on* $(\boldsymbol{P}^\dagger)^{k,k}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$.

Proof.

$$\Delta^\dagger\Big(\sum_{j=m}^{k}\frac{(-1)^{j-m}(n+2k-j-m-3)!}{2^{2j}\cdot(j!)^2\cdot(j-m)!}H_j\Big)=\sum_{j=m}^{k}\frac{(-1)^{j-m}\cdot(n+2k-j-m-3)!}{2^{2j}\cdot(j!)^2\cdot(j-m)!}$$
$$\cdot(\lambda_{k,j}H_j-(1/(j+1^2))H_{j+1})=\frac{(n+2k-2m-3)!}{2^{2m}\cdot(m!)^2}\lambda_{k,m}H_m$$
$$+\sum_{j=m+1}^{k}\Big(\frac{(-1)^{j-m}(n+2k-j-m-3)!}{2^{2j}\cdot(j!)^2\cdot(j-m)!}\lambda_{k,j}H_j-\frac{(-1)^{j-m-1}\cdot(n+2k-j-m-2)!}{2^{2(j-1)}\cdot(j!)^2\cdot(j-m-1)!}H_j\Big)$$
$$=\frac{(n+2k-2m-3)!}{2^{2m}(m!)^2}\lambda_{k,m}H_m=\sum_{j=m+1}^{k}\frac{(-1)^{j-m}(n+2k-j-m-3)!}{2^{2j}\cdot(j!)^2\cdot(j-m)!}$$
$$\cdot\{\lambda_{k,j}+4(j-m)(n+2k-j-m-2)\}H_j.$$

As $\lambda_{k,m}=\lambda_{k,j}+4(j-m)(n+2k-j-m-2)$, we obtained the lemma. Q.E.D.

Corollary. $P_{k,m}P_{k,m'}=0$ *for* $m\neq m'$.

This is a direct consequence of Lemma 4.6.

Lemma 4.7. $\sum_{m=0}^{k}P_{k,m}=\mathbf{1}_{k,k}$ *on* $(\boldsymbol{P}^\dagger)^{k,k}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$.

Proof. $\sum_{m=0}^{k}P_{k,m}=\sum_{m=0}^{k}\frac{n+2k-2m-2}{m!\cdot(n+2k-m-2)!}\sum_{j=m}^{k}\frac{(-1)^{j-m}\cdot(n+2k-j-m-3)!}{2^{2j}\cdot(j!)^2\cdot(j-m)!}$

$$=\sum_{m=0}^{k}\frac{(-1)^j}{2^{2j}\cdot(j!)^2}H_j\sum_{m=0}^{j}\frac{(-1)^m\cdot(n+2k-2m-2)\cdot(n+2k-j-m-3)!}{(n+2k-m-2)!}\binom{j}{m}$$

on $(\boldsymbol{P}^\dagger)^{k,k}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$. On the other hand, substituting $x=n+2k-2$ into the equality in Lemma 4.5, we have

$$\sum_{m=0}^{j}\binom{j}{m}\frac{(-1)^m\cdot(n+2k-2m-2)\cdot(n+2k-j-m-3)!}{(n+2k-m-2)!}=\delta_0^j.$$

Then

$$\sum_{m=0}^{k} P_{k,m} = \sum_{j=0}^{k} \frac{(-1)^j}{2^{2j}\cdot(j!)^3} H_j \delta_0^j = \mathbf{1}_{k,k}\,. \qquad \text{Q.E.D.}$$

From Lemma 4.6 and Lemma 4.7 follows Theorem 4.1 (i) immediately.

Let Φ be an eigenfunction of the Laplacian Δ of $(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ for the first eigenvalue $4(n+1)$. We can identify Φ with its representative $\Phi^\dagger$ in virtue of the isomorphism (2.4), restricted to $\mathfrak{D}^0(\boldsymbol{P}_n(\boldsymbol{C}))$.

Lemma 4.8. (i) $B_1^*(Z^a\bar{Z}^b/r^2)=8\sum_{c=0}^{n}\xi^\dagger_{ca,cb}$. (ii) $B_1^*\Phi^\dagger\in(\boldsymbol{K}^\dagger)^{1,1}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$.

Proof. It is enough to prove (i) because (ii) is a direct consequence of (i). We know that $\Phi^\dagger=c+\sum_{a,b=0}^{n} c_{ab}(Z^a\bar{Z}^b/r^2)$ for some $c\in\boldsymbol{C}$ and $c_{ab}\in\boldsymbol{C}$ (cf. [2] pp. 172–173). Then we have

$$\begin{aligned} B_1^*(Z^a\bar{Z}^b/r^2) &= 2(\partial^*)^\dagger(\bar{\partial}^*)^\dagger(Z^a\bar{Z}^b/r^2)+(T^*)^\dagger(Z^a\bar{Z}^b/r^2)\\ &= 8(r^2\delta^{ad}\delta^{bc}-Z^a\delta^{cd}\bar{Z}^b-Z^c\delta^{ad}\bar{Z}^b-Z^a\delta^{bc}\bar{Z}^d+2Z^a\bar{Z}^bZ^c\bar{Z}^d/r^2\\ &\quad+2(r^2\delta^{cd}-Z^c\bar{Z}^d)Z^a\bar{Z}^b/r^2)(\partial/\partial Z^c)\circ(\partial/\partial\bar{Z}^d)\,. \end{aligned}$$

On the other hand, as $\zeta-\bar{\zeta}\equiv 0\ (\mathrm{mod}\,(\tau))$

$$\begin{aligned} 8\sum_{c=0}^{n}\xi^\dagger_{ac}\circ\xi^\dagger_{cb} &= 8\sum_{c=0}^{n}(\sqrt{-1})^2(Z^a\partial/\partial Z^c-\bar{Z}^c\partial/\partial\bar{Z}^a)(Z^c\partial/\partial Z^b-\bar{Z}^b\partial/\partial Z^c)\\ &\equiv 8r^2\partial/\partial Z^b\circ\partial/\partial\bar{Z}^a-8\sum_{c=0}^{n}Z^a\bar{Z}^c\partial/\partial Z^b\circ\partial/\partial\bar{Z}^c-8\sum_{c=0}^{n}Z^c\bar{Z}^b\partial/\partial Z^c\circ\partial/\partial\bar{Z}^a\\ &\quad+8Z^a\bar{Z}^b\sum_{c=0}^{n}\partial/\partial Z^c\circ\partial/\partial\bar{Z}^c+8\sum_{c=0}^{n}\xi^\dagger_{cc}\circ\xi^\dagger_{ab} \qquad (\mathrm{mod}\,(\tau))\,. \end{aligned}$$

Comparing these two equalities we obtain

$$B_1^*(Z^a\bar{Z}^b/r^2)-8\sum_{c=0}^{n}\xi^\dagger_{ca,cb}\equiv 0 \qquad (\mathrm{mod}\,(\tau))\,. \tag{4.7}$$

By virtue of Lemma 2.3 and its proof we have

$$(\boldsymbol{K}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)\cap(\tau)\subset(\boldsymbol{S}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}))\cap(\tau)=\{0\}\,.$$

Hence the left-hand side of (4.7) vanishes. This proves (i). Q.E.D.

Lemma 4.9. (i) $(T^\dagger)^i(\xi^\dagger_{ab,cd})^k=((k!)^2/((k-i)!)^2)(\xi^\dagger_{ab,cd})^{k-i}(1/2^i\cdot r^{2i})(Z^a\bar{Z}^c\delta^{bd}+Z^b\bar{Z}^d\delta^{ac}-Z^a\bar{Z}^d\delta^{bc}-Z^b\bar{Z}^c\delta^{ac})^i$.

(ii) $\Pi_0(\prod_{j=1}^{i}B_j^*(T^\dagger)^i(\xi^\dagger_{ab,cd})^k)=2^{2i}\dfrac{(k!)^2\cdot(i!)^2}{((k-i)!)^2}\Pi_0((\xi^\dagger_{ab,cd})^{k-i}\circ(\Xi(ab, cd))^i)$

with

$$\Xi(ab, cd)=\sum_{e=0}^{n}(\delta^{bd}\xi^\dagger_{ea,ec}+\delta^{ac}\xi^\dagger_{eb,ed}-\delta^{bc}\xi^\dagger_{ea,ed}-\delta^{ad}\xi^\dagger_{eb,ec})\in(\boldsymbol{K}^\dagger)^{1,1}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)\,.$$

Proof. From Lemma 2.7 we know that

$$T^\dagger(\underset{ab,cd}{\xi^\dagger}) = k^2(\underset{ab,cd}{\xi^\dagger})^{k-1}T(\underset{ab,cd}{\xi^\dagger}).$$

On the other other hand, we obtain by direct calculation

$$T^\dagger(\underset{ab,cd}{\xi}) = \frac{1}{2r^2}(Z^a\bar{Z}^c\delta^{bd}+Z^b\bar{Z}^d\delta^{ac}-Z^a\bar{Z}^d\delta^{bc}-Z^b\bar{Z}^c\delta^{ad}).$$

From these two equalities we can prove (i) by induction on i. To prove (ii) it is enough to verify that

(a) $\Pi_0(\prod_{j=1}^{i} B_j^*\xi^\dagger-(B_1^*)^i\xi^\dagger)=0$ for $\xi^\dagger\in(\boldsymbol{K}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$.

(b) $B_1^*T^\dagger(\underset{ab,cd}{\xi^\dagger})=2^3\Xi(ab, cd)$.

(a) is evident. (b) follows from Lemma 4.8. Q.E.D.

Notice that there are only following two cases in which $\Xi(ab, cd)$ are non-trivial.

Case 1. Among a,b,c and d, three and only three of them are distinct.
Case 2. $a=c$, $b=d$ and $a\neq b$ or $a=d$, $b=c$ and $a\neq b$.

Lemma 4.10. *In $(\boldsymbol{P}_2(\boldsymbol{C}), g_0)$ $P_{k,m}\neq 0$ if and only if $k=m$.*

Proof. In virtue of Lemma 4.9 and (4.6), the expression of $P_{k,k}$ is reduced to the following form:

$$P_{k,k}(\Pi_0\underset{ab,cd}{\xi^\dagger})^k = \Pi_0(\Xi(ab, cd))^k.$$

As $n=2$, in case 1, say $a=d=0$, $b=1$, $c=2$,

$$P_{k,k}(\Pi_0\underset{01,20}{\xi^\dagger})^k = \Pi_0(-\sum_{e=0}^{2}\underset{e1,e2}{\xi^\dagger}))^k = \Pi_0(\underset{01,20}{\xi^\dagger}+\underset{11,21}{\xi^\dagger}+\underset{21,22}{\xi^\dagger})^k = \Pi_0(\underset{01,20}{\xi^\dagger})^k.$$

In case 2, say $a=c=0$ and $b=d=1$,

$$P_{k,k}(\Pi_0\underset{01,10}{\xi^\dagger})^k = \Pi_0(-\sum_{e=0}^{2}(\underset{e0,e0}{\xi^\dagger}+\underset{e1,e1}{\xi^\dagger}))^k = \Pi_0(-\underset{01,01}{\xi^\dagger}+g_0^*)^k = \Pi_0(\underset{01,10}{\xi^\dagger})^k.$$

We can conclude that $P_{k,k}=\mathbf{1}_{k,k}$ on $(\boldsymbol{P}^\dagger)^{k,k}(\boldsymbol{P}_2(\boldsymbol{C}), g_0)$. Owing to Corollary of Lemma 4.6 and Lemma 4.7 we have $P_{k,m}=0$ if $m\neq k$. Q.E.D.

From Lemma 4.7 and Lemma 4.10 follows (ii) of Theorem 4.1 immediately.

In order to prove the rest of Theorem 4.1, we need the following two lemmas. Let X_{ij} $(0\leqq i<j\leqq n)$, Y_{kl} $(0\leqq k<l\leqq n)$ be intederminates.

We employ the following notations:

$\boldsymbol{C}[X, Y]$: the polynomial algebra generated by all of X_{ij}'s and Y_{kl}'s,

$\boldsymbol{C}[X\cdot Y]$: the subalgebra of $\boldsymbol{C}[X, Y]$ generated by products $X_{ij}\cdot Y_{kl}$,

I_0': the ideal of $\boldsymbol{C}[X, Y]$ generated by Π_{ijkl} $(0\leqq i<j<k<l\leqq n)$, Π_{pqrs} $(0\leqq p<q<r<s\leqq n)$,

I_1': the ideal of $\boldsymbol{C}[X, Y]$ generated by Π_{ijkl} $(0\leqq i<j<k<l\leqq n)$, $\overline{\Pi}_{pqrs}$ $(0\leqq p<q<r<s\leqq n)$ and $G-1$,

I_2': the ideal of $\boldsymbol{C}[X, Y]$ generated by Π_{ijkl} $(0\leqq i<j<k<l\leqq n)$, $\overline{\Pi}_{pqrs}$ $(0\leqq p<q<r<s\leqq n)$ and G,

$$I_i = I_i' \cap \boldsymbol{C}[X\cdot Y] \qquad (0\leqq i\leqq 2),$$

where $\Pi_{ijkl}=X_{ij}X_{kl}-X_{ik}X_{jl}+X_{il}X_{jk}$, $\overline{\Pi}_{pqrs}=Y_{pq}Y_{rs}-Y_{pr}Y_{qs}+Y_{ps}Y_{qr}$,

$$G = \sum_{0\leqq i<j\leqq n} X_{ij}Y_{ij}\,.$$

Lemma 4.11. (i) *The image of the Radon-Michel transform restricted to* $(\boldsymbol{K}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ *is the subalgebra* $\boldsymbol{C}[P^{ab}\bar{P}^{cd}\,(0\leqq a<b\leqq n;\ 0\leqq c<d\leqq n)]$ *in* $\tilde{C}^\infty(\boldsymbol{G}_{2,n-1}(\boldsymbol{C}))$, *where* P^{ab}*'s are normalized Plücker coordinates.* (ii) *The image of the Radon-Michel transform restricted to* $(\boldsymbol{K}^\dagger)^{**}\boldsymbol{P}_n((\boldsymbol{C}), g_0)$ *is isomorphic to the quotient algebra* $\boldsymbol{C}[X\cdot Y]/I_1$. (iii) *The kernel of the Radon-Michel transform restricted to* $(\boldsymbol{K}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ *is the principal ideal in* $(\boldsymbol{K}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$ *generated by* $(g_0^*/2)-1$.

Proof. (i) is essentially proved in **3**. (ii) is another expression of (i) based on the classically known result on complex normalized Plücker coordinates. (iii) As we have by direct calculation:

$$(g_0^*/2)-1 = (r^2\delta^{ab}-Z^a\bar{Z}^b)\partial/\partial Z^a\circ\partial/\partial\bar{Z}^b-1 = \sum_{0\leqq a<b\leqq n} \underset{\bar{a}b}{\xi}\circ\underset{a\bar{b}}{\xi}-1\,,$$

we obtain

$$((g_0^*/2)-1)^\wedge = \sum_{0\leqq i<j\leqq n} P^{ij}\bar{P}^{ij}-1\,.$$

Thus $(g_0^*/2)-1$ is contained in the kernel of the Radon-Michel transform restricted to $(\boldsymbol{K}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)$.

On the other hand, we have the following identities by direct calculations:

$$\underset{ij,mn}{\xi^\dagger}\circ\underset{kl,pq}{\xi^\dagger}-\underset{ik,mn}{\xi^\dagger}\circ\underset{jl,pq}{\xi^\dagger}+\underset{il,mn}{\xi^\dagger}\circ\underset{jk,pq}{\xi^\dagger} = (\underset{i\bar{j}}{\xi}\circ\underset{k\bar{l}}{\xi}-\underset{i\bar{k}}{\xi}\circ\underset{j\bar{l}}{\xi}+\underset{i\bar{l}}{\xi}\circ\underset{j\bar{k}}{\xi})\,\underset{\bar{m}n}{\xi}\circ\underset{\bar{p}q}{\xi}=0$$

$$(\text{resp. } \underset{mn,ij}{\xi^\dagger}\circ\underset{pq,kl}{\xi^\dagger}-\underset{mn,ik}{\xi^\dagger}\circ\underset{pq,jl}{\xi^\dagger}+\underset{mn,il}{\xi^\dagger}\circ\underset{pq,jk}{\xi^\dagger} = \underset{m\bar{n}}{\xi}\circ\underset{p\bar{q}}{\xi}(\underset{\bar{i}j}{\xi}\circ\underset{\bar{k}l}{\xi}-\underset{\bar{i}k}{\xi}\circ\underset{\bar{j}l}{\xi}+\underset{\bar{i}l}{\xi}\circ\underset{\bar{j}k}{\xi}=0)\,.$$

The left-hand sides of the identities above have

$$(\bar{P}^{ij}\bar{P}^{kl}-\bar{P}^{ik}\bar{P}^{jl}+\bar{P}^{il}\bar{P}^{jk})P^{mn}P^{pq}\,(\text{resp. } \bar{P}^{mn}\bar{P}^{pq}(P^{ij}P^{kl}-P^{ik}P^{jl}+P^{il}P^{jk}))$$

as their images of the Radon-Michel transform. Comparing with the results in (i) and (ii), we can easily conclude that the kernel of the Radon-Michel transform coincides with the principal ideal generated by $(g_0^*/2)-1$. Q.E.D.

Lemma 4.12. $(\boldsymbol{K}^\dagger)^{**}(\boldsymbol{P}_n(\boldsymbol{C}), g_0)\cong\boldsymbol{C}[X\cdot Y]/I_0$.

Proof. Let Φ: $C[X\cdot Y]\rightarrow(K^{\dagger})^{**}(P_n(C), g_0)$ be given by $X_{ij}Y_{kl}\mapsto\underset{i\bar{j}}{\xi}\circ\underset{k\bar{l}}{\xi}=\underset{ij,kl}{\xi^{\dagger}}$. Then Φ is a surjective homorphism of graded algebras. Obviuosly $I_0\subset\ker\Phi$. If we consider the homomorphism Φ followed by the Radon-Michel transform, Lemma 4.11 tells us that the kernel of Φ is exactly I_0. Q.E.D.

Proof. of Theorem 4.1 (iii). Let $n\geqq 3$. From Lemma 4.9 (ii) we have

$$(4.8)\quad P_{k,m}(\underset{01,01}{\xi^{\dagger}})^k=\sum_{i=m}^{k}\frac{(-1)^{i-m}2^{2i}\cdot(i!)^2\cdot(k!)^2(n+2k-2m-2)}{m!\cdot(n+2k-m-2)!\cdot((k-i)!)^2\cdot(i-m)!}\cdot(n+2k-m-i-3)!\cdot\Pi_0(\Xi(01,01)^i(\underset{01,01}{\xi^{\dagger}})^{k-i}).$$

If $P_{k,m}$ vanishes identically, then

$$\Pi_0\sum_{i=m}^{k}c_{k,m,i}(\sum_{e=0}^{n}(\underset{e0,e0}{\xi^{\dagger}}+\underset{e1,e1}{\xi^{\dagger}}))^i(\underset{01,01}{\xi^{\dagger}})^{k-i}=0,$$

where $c_{k,m,i}$ are numerical coefficients of $\Pi_0(\Xi(01,01))^i(\underset{01,01}{\xi^{\dagger}})^{k-i}$ in (4.8). From Lemma 4.11 it follows that

$$(4.9)\quad \sum_{i=m}^{k}C_{k,m,i}\sum_{e=1}^{n}[\sum_{e=2}^{n}(X_{0e}Y_{0e})+X_{01}Y_{01}+\sum X_{1e}Y_{1e}]^i(X_{01}Y_{01})^{k-i}$$

should be in the ideal I_2. However this is not the case. If we put $X_{01}=1$, $X_{23}=2\sqrt{-1}$, $X_{02}=X_{03}=1$, $X_{12}=-\sqrt{-1}(2+\sqrt{2})$, $X_{13}=-\sqrt{-2}$, $X_{ij}=0$ for $\mathrm{Max}\{i,j\}\geqq 4$, $Y_{01}=2$, $Y_{23}=\sqrt{-1}$, $Y_{02}=-Y_{03}=1$, $Y_{12}=-\sqrt{-2}$, $Y_{13}=\sqrt{-1}(2+\sqrt{-2})$, $Y_{pq}=0$ for $\mathrm{Max}\{p,q\}\geqq 4$, then X_{ij}'s and Y_{kq}'s satisfy $\Pi_{ijkl}=0$, $\overline{\Pi}_{pqrs}=0$, $\sum_{0\leqq i<j\leqq n}X_{ij}\cdot Y_{ij}=0$, and $\sum_{i=m}^{k}c_{k,m,i}(X_{01}Y_{01}-X_{23}Y_{23})^i(X_{01}Y_{01})^{-i}=0$, Consequently we can conclude $c_{k,m,i}=0$. This is a contradiction. Q.E.D.

(iv) of Theorem 4.1 is easily obtained from the definition of $(P^{\dagger})^{**}(P_n(C))$ and the properties of T^*. The proof of Theorem 4.1 is finished.

By virtue of Theorem 3.2 the eigenspace decomposition of $\Delta^{\wedge}$ is obtained by transferring the decomposition of $(K^{\dagger})^{**}(P_n(C))$.

Theorem 4.2. *The spectrum of* $(G_{2,n-1}(C), g_1)$ *is* $\lambda_{k,m}=4((2k-m)n+2k^2-2(m+1)k+m^2+2m)$ *for* $n\geqq 3$, $\lambda_{k,k}=4(k^2+2k)$ *for* $n=2$.

Proof. By virtue of the theorem of Stone-Weierstrass $C[P^{ab}, \bar{P}^{cd}$ $(0\leqq a<b\leqq n;\ 0\leqq c<d\leqq n)]$ is uniformly dense in $C^{\infty}(G_{2,n-1}(C))$. The image of an eigensubspace of $\Delta^{\dagger}$ in $(K^{\dagger})^{**}(P_n(C))$ by the Radon-Michel transform is nontrivial, which is essentially proved in the proof of (iii) in Theorem 4.1. Our assertion follows from Theorem 3.2 and Theorem 4.1. Q.E.D.

Appendix. Differential equation for eigenfunctions of the Laplacian Δ in $(P_n(C), g_0)$

Define a linear differential operator in $C^{n+1}-\{0\}$ of order $2h+1$ by $D_h^*=(\delta^*)^\dagger\cdot\prod_{i=1}^{h} B_i^*$ $(h\geqq 0)$. Note that $D_0=(\delta^*)^\dagger$.

Put

$$(\partial_0^*\Xi)^{A,\bar{B}} = 2\sum_{i=1}^{k+1}\frac{\partial\Xi^{a_1\cdots\overset{a_i}{\wedge}\cdots a_{k+1},\bar{B}}}{\partial\bar{Z}^{a_i}},$$

$$(\bar{\partial}_0^*\Xi)^{A,\bar{B}} = 2\sum_{j=1}^{l+1}\frac{\partial\Xi^{A,\bar{b}_1\cdots\overset{\bar{b}_j}{\wedge}\cdots\bar{b}_{l+1}}}{\partial Z^{b_j}}$$

for $\Xi\in S^{k,l}(C^{n+1}-\{0\})$ and put

$$\delta_0^* = \partial_0^*+\bar{\partial}_0^*.$$

Lemma A. *$D_h^*(\eta/r^{2h})=r^{2h+2}\delta_0^*(\partial_0^*)^h(\bar{\partial}_0^*)^h\eta$, if $\eta/r^{2h}\in(S^\dagger)^{h,h}(P_n(C))$.*

Proof. $D_0^*(\eta/r^0)=\delta^*\eta=r^2\delta_0^*\eta$. Suppose that the assertion be true for $i\geqq 0$. Then

$$D_{i+1}^*(\eta/r^{2i+2}) = D_i^*B_{i+1}^*(\eta/r^{2i+2}) = r^{2i+4}\delta_0^*(\partial_0^*)^{i+1}(\bar{\partial}_0^*)^{i+1}\eta$$

by virtue of Leibnitz's formula. Q.E.D.

Let E_i be the eigensubspace for the eigenvalue $4i(n+i)$ of the Laplacian Δ of $(P_n(C), g_0)$.

Theorem A. *Let $f\in(C^\infty)^\dagger(P_n(C))$. $D_h^*f=0$ if and only if $f\in\sum_{i=0}^{h} E_i$.*

Proof. Put $\Psi=r^{2h}f$. From Lemma A. $D_h^*f=0$ if and only if $r^{2h+2}\delta_0^*\cdot(\partial_0^*)^h(\bar{\partial}_0^*)^h\Psi=0$. Thus $D_h^*f=0$ if and only if Ψ is a homogeneous polynomial of bidegree (h, h) with respect to $(Z^a, \bar{Z}^b)$. The latter is known to be equivalent to $f\in\sum_{i=0}^{h} E_i$. Q.E.D.

References

[1] M. Berger, P. Graduchon et E. Mazet: Le spectre d'une veriété riemannienne, Lecture notes in Math. 194, Springer.

[2] M. Greub, S. Halperin, R. Vanstone: Connection, curvature and cohomology, Academic Press, 1973.

[3] A. Lichnerowicz: *Propagateurs et commutateurs en relativité génerale,* Inst. Hautes Études Publ. Math. **17** (1980), 293–344.

[4] K. Tandai, T. Sumitomo: *Killing tensor of degree* 2 *and spectrum of* $SO(n+1)/SO(n-1)\times SO(2)$: Osaka J. Math. **17** (1980), 649–675.

[5] T. Sumitomo, K. Tandai: *Killing tensor fields on the standard sphere and spectra of* $SO(n+1)/SO(n-1)\times SO(2)$ *and* $O(n+1)/O(n-1)\times O(2)$, Osaka J. Math. **20** (1983), 51–78.

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