Kunita, H.
Osaka J. Math.
21 (1984), 883–911

ON THE CONVERGENCE OF SOLUTIONS OF STOCHASTIC ORDINARY DIFFERENTIAL EQUATIONS AS STOCHASTIC FLOWS OF DIFFEOMORPHISMS

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(Received November 28, 1983)

Introduction

Let $F(\tau, x, t, \omega)$ and $G(\tau, x, t, \omega)$ be random vector fields on R^d with time parameters τ, t satisfying certain statistical properties. Let ψ_t^ε be a solution of the stochastic ordinary differential equation

$$\frac{dx}{dt} = \varepsilon F(\varepsilon^2 t, x, t, \omega) + \varepsilon^2 G(\varepsilon^2 t, x, t, \omega)\,.$$

A lot of attention has been shown to the limiting behavior of the solution ψ_t^ε as $\varepsilon \to 0$ and $t \to \infty$ with $\varepsilon^2 t$ remaining fixed, since the work of Khasminskii [8]. See also [3], [7], [13], [14], [16]. In these woiks, it is proved that $\phi_t^\varepsilon \equiv \psi_{t/\varepsilon^2}^\varepsilon$ converges weakly to a diffusion process ϕ_t with local characteristics a^{ij} and b^i which are determined from random vector fields $F(\tau, x, t, \omega)$ and $G(\tau, x, t, \omega)$ in a suitable way. (See (1.6)–(1.8) of Section 1). Note that ϕ_t^ε satisfies

$$\frac{d}{dt}\phi_t^\varepsilon = F_\varepsilon(t, \phi_t^\varepsilon)\,,$$

where

$$F_\varepsilon(t, x) = \frac{1}{\varepsilon} F(t, x, \frac{t}{\varepsilon^2}) + G(t, x, \frac{t}{\varepsilon^2})\,.$$

The purpose of this paper is to show that the weak limit of ϕ_t^ε satisfies a suitable Itô's stochastic differential equation, which can be regarded as the weak limit of the above stochastic ordinary equation. Indeed, we will see in Theorem 1 that setting $X_t^\varepsilon(x) = \int_0^t F_\varepsilon(s, x)ds$, the pair $(\phi_t^\varepsilon, X_t^\varepsilon)$ converges weakly to (ϕ_t, X_t), where ϕ_t is a diffusion process mentioned above and X_t is a Brownian motion with values in the space of vector fields. Furthermore, these two processes are linked by Itô's stochastic differential equation

$$d\phi_t = dX_t(\phi_t) + c(t, \phi_t)dt\,.$$

Here $c(t, \phi_t)dt$ is the "correction term" caused partly by Itô's stochastic integral and partly by non-symmetry of $F(\tau, x, t)$. When the Stratonovich differential $\circ dX_t(\phi_t)$ is well defined and a suitable symmetric hypothesis is satisfied, the above equation is written simply as (Theorem 2)

$$d\phi_t = \circ dX_t(\phi_t)\,.$$

Itô's stochastic integral by the Brownian vector field X_t was introduced by Le Jan [11] and Le Jan–Watanabe [12] for the study of Brownian motion in diffeomorphisms group $G=\text{Diffeo}\,(R^d)$.

Our limit theorem is formulated as a convergence of measures on the space $C([0, T]; G)\times C([0, T]; V)$, where V is the space of vector fields. In fact, the pair $(\phi_t^\varepsilon, X_t^\varepsilon)$ can be regarded as a continuous process with values in $G\times V$ and it converges weakly to a continuous process with values in $G\times V$. Moreover, the limiting process has independent increments. Thus it can be considered as a Brownian motion in $G\times V$. The relation between ϕ_t and X_t is that X_t or dX_t is the pathwise infinitesimal generator of ϕ_t.

In his paper [17], S. Watanabe pointed out that some limit theorems related to random ODE could be formulated naturally in the framework of stochastic flows of diffeomorphisms. Our result might be considered as a partial answer to the problem.

1. Formulation and statement of the theorem

We begin by introducing some assumptions to random vector fields $F(\tau, x, t)$ and $G(\tau, x, t)$ together with the mixing condition. Our hypothesis is close to that of Papanicolaou–Kohler [13].

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F}_{s,t}$, $0\leqq s\leqq t\leqq +\infty$ be a family of σ-fields in $\mathcal{F}$ and such that $\mathcal{F}_{s_1,t_1}\subset\mathcal{F}_{s_2,t_2}$ if $0\leqq s_2\leqq s_1\leqq t_1\leqq t_2$. We assume that σ-fields $\mathcal{F}_{s,t}$ are mixing relative to P in the following sense:

(A.I) The *mixing rate*

(1.1) $$\rho(t)\equiv \sup_{s>0}\ \sup_{A\in\mathcal{F}_{s+t,\infty},B\in\mathcal{F}_{0,s}} |P(A|B)-P(A)|$$

decreases to 0 as $t\uparrow\infty$ and satisfies

(1.2) $$\int_0^\infty \rho(s)^{1/2}ds<\infty\,.$$

Let $F(\tau, x, t, \omega)$ and $G(\tau, x, t, \omega)$ be *random vector fields*; measurable mappings from $[0, T]\times R^d\times[0, \infty)\times\Omega$ into R^d, where T is a positive number. We assume the following hypotheses (A.II) and (A.III).

(A.II) (i) For fixed τ, x, t, $F(\tau, x, t, \omega)$ and $G(\tau, x, t, \omega)$ are $\mathcal{F}_{t,t}$-measurable.
(ii) For almost all ω, $F(\tau, x, t, \omega)$ and $G(\tau, x, t, \omega)$ are continuous in three variables. Furthermore, $F(\tau, x, t, \omega)$ is twice continuously differentiable in x and $G(\tau, x, t, \omega)$ is continuously differentiable in x.
(iii) There is a constant C independent of τ, x, t and ω, ω' such that

$$(1.3) \qquad |F(\tau, x, t, \omega)| \leqq C(1+|x|),\ |G(\tau, x, t, \omega)| \leqq C(1+|x|),$$

$$(1.4) \qquad \left|\frac{\partial}{\partial x^j} F^i(\tau, x, t, \omega)\right| \leqq C,\quad \left|\frac{\partial}{\partial x^j} G^i(\tau, x, t, \omega)\right| \leqq C,$$

$$(1.5) \qquad \left|\frac{\partial}{\partial x^k} \sum_i F^i(\tau, x, t, \omega) \frac{\partial}{\partial x^i} F^j(\tau', x, t', \omega')\right| \leqq C,$$

(iv) $E[F(\tau, x, t)] = 0$.

(A.III) There are continuous functions $A^{ij}(\tau, x, y)$, $b^j(\tau, x)$ and $c^j(\tau, x)$ such that

$$(1.6) \qquad \left|A^{ij}(\tau, x, y) - \frac{1}{\varepsilon^3}\int_\tau^{\tau+\varepsilon}\int_\tau^\sigma E[F^i(s, x, \frac{s}{\varepsilon^2})F^j(\sigma, y, \frac{\sigma}{\varepsilon^2})]dsd\sigma\right| \leqq C\varepsilon(1+|x|)(1+|y|),$$

$$(1.7) \qquad \left|b^j(\tau, x) - \frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon} E[G^j(s, x, \frac{s}{\varepsilon^2})]ds\right| \leqq C\varepsilon(1+|x|),$$

$$(1.8) \qquad \left|c^j(\tau, x) - \frac{1}{\varepsilon^3}\int_\tau^{\tau+\varepsilon}\int_\tau^\sigma \sum_{i=1}^d E[F^i(s, x, \frac{s}{\varepsilon^2}) \frac{\partial}{\partial x^i} F^j(\sigma, x, \frac{\sigma}{\varepsilon^2})]dsd\sigma\right| \leqq C\varepsilon(1+|x|)$$

hold for any $\varepsilon > 0$.

We set

$$(1.9) \qquad a^{ij}(\tau, x, y) = A^{ij}(\tau, y, x) + A^{ji}(\tau, x, y).$$

Then it holds $a^{ji}(\tau, x, y) = a^{ij}(\tau, y, x)$.

REMARK. We will see in Section 4 that (A.I)~(A.III) imply that A^{ij}, b^i, c^i are uniformly Lipschitz continuous in the following sense. There is a constant L such that

$$(1.10) \qquad |A^{ij}(\tau, x, x) - A^{ij}(\tau, x, y) - A^{ij}(\tau, y, x) + A^{ij}(\tau, y, y)| \leqq L|x-y|^2,$$

$$(1.11) \qquad |b^i(\tau, x) - b^i(\tau, y)| + |c^i(\tau, x) - c^i(\tau, y)| \leqq L|x-y|$$

hold for any i, j and τ, x, y.

Given $\varepsilon > 0$, consider the stochastic ordinary differential equation

$$(1.12) \qquad \frac{dx}{dt} = \varepsilon F(\varepsilon^2 t, x, t, \omega) + \varepsilon^2 G(\varepsilon^2 t, x, t, \omega), \qquad t \geqq 0.$$

The solution starting from x at time s is denoted by $\psi^{\varepsilon}_{s,t}(x)$ or simply ψ^{ε}_{t}. We are interested in the behavior of the solution as $\varepsilon\to 0$ and $t\to\infty$ with $\varepsilon^2 t$ remaining fixed. For this purpose, we consider the process where the time scale is changed. Define $\phi^{\varepsilon}_{s,t}(x)=\psi^{\varepsilon}_{s/\varepsilon^2,t/\varepsilon^2}(x)$. Then it satisfies

$$(1.13)\qquad \frac{d}{dt}\phi^{\varepsilon}_{s,t}(x)=F_{\varepsilon}(t,\phi^{\varepsilon}_{s,t}(x)),$$

where

$$(1.14)\qquad F_{\varepsilon}(t,x)=\frac{1}{\varepsilon}F(t,x,\frac{t}{\varepsilon^2})+G(t,x,\frac{t}{\varepsilon^2}).$$

In the following discussion, we shall regard that $\phi^{\varepsilon}_{s,t}(x)$, s, $t\in[0, T]$, $x\in R^d$ is a random field. Obviously it has the following property. For almost all ω, the map $\phi^{\varepsilon}_{s,t}$: $R^d\to R^d$ is a homeomorphism and satisfies multiplicative property $\phi^{\varepsilon}_{s,u}=\phi^{\varepsilon}_{t,u}\circ\phi^{\varepsilon}_{s,t}$ for any s, t, u of $[0, T]$. It is often called a *stochastic flow of homeomorphisms*. Now define another random field

$$(1.15)\qquad X^{\varepsilon}_{s,t}(x)=\int_s^t F_{\varepsilon}(\tau,x)d\tau.$$

It can be regarded as a (random) vector field for each s, t, satisfying the additive property $X^{\varepsilon}_{s,u}=X^{\varepsilon}_{s,t}+X^{\varepsilon}_{t,u}$.

We shall introduce the law of the random field $(\phi^{\varepsilon}_{s,t}, X^{\varepsilon}_{s,t})$. A two parameter family of homeomorphisms $\phi_{s,t}$, s, $t\in[0, T]$ of R^d is called a *flow of homeomorphisms* if $\phi_{s,t}(x)$ is continuous in three variables and has the multiplicative property $\phi_{t,u}\circ\phi_{s,t}=\phi_{s,u}$ for any s, t, $u\in[0, T]$ and $\phi_{s,s}=$identity for any s. We denote by W_1 the set of all flows of homeomorphisms. Now let $X_{s,t}(x)$ be a two parameter family of vector fields continuous in three variables (s, t, x), satisfying the additive property $X_{s,t}+X_{t,u}=X_{s,u}$. The set of all these two parameter families of vector fields is denoted by W_2. For $\phi_{s,t}$, $\psi_{s,t}$ of W_1, we define the metric by $d(\phi,\psi)=\sup_{s,t\in[0,T]}\rho(\phi_{s,t},\psi_{s,t})$, where ρ is the compact uniform topology of $C(R^d; R^d)$. Note the relation $\phi_{s,t}=\phi^{-1}_{t,s}$. We see that (W_1, d) is a complete separable metric space. The metric of W_2 is defined in the same way. Then the product space $W=W_1\otimes W_2$ is also a complete separable metric space. Denote by $\mathcal{B}_W$ the topological Borel field of W. The pair of random fields $(\phi^{\varepsilon}_{s,t}(x), X^{\varepsilon}_{s,t}(x))$ induces a law on $(W, \mathcal{B}_W)$ which we denote by $P^{(\varepsilon)}$. The expectation by the measure $P^{(\varepsilon)}$ is denoted by $E^{(\varepsilon)}$.

We are ready to present the main result of this paper.

Theorem 1. *Assume* (A.I), (A.II) *and* (A.III). *Let* $P^{(\varepsilon)}$ *be the law of* $(\phi^{\varepsilon}_{s,t}, X^{\varepsilon}_{s,t})$ *on* $(W, \mathcal{B}_W)$. *Then there exists a unique law* $P^{(0)}$ *on* $(W, \mathcal{B}_W)$ *such that* $(\mathcal{B}_W, P^{(\varepsilon)})$ *converges weakly to* $(\mathcal{B}_W, P^{(0)})$ *as* $\varepsilon\to 0$. *Furthermore,* $P^{(0)}$ *admits the following properties.*

(i) $(X_{s,t}, P^{(0)})$ *is a Brownian vector field, i.e., a Gaussian random field such that* $X_{t_i,t_{i+1}}$, $i=0, \cdots, n-1$ *are independent for any* $0 \leqq t_0 < t_1 < \cdots < t_n \leqq T$. *The mean and covariance are given by*

$$E^{(0)}(X_{s,t}(x)) = \int_s^t b(\tau, x)d\tau \qquad (= \overline{X_{s,t}(x)}), \tag{1.16}$$

$$E^{(0)}((X^i_{s,t}(x) - \overline{X^i_{s,t}(x)})(X^j_{s,t}(y) - \overline{X^j_{s,t}(y)})) = \int_s^t a^{ij}(\tau, x, y)d\tau . \tag{1.17}$$

(ii) $(\phi_{s,t}, P^{(0)})$ *is a Brownian motion in* $G = Homeo\ (R^d)$, *i.e.,* $\phi_{t_i,t_{i+1}}$, $i=0, \cdots, n-1$ *are independent for any* $0 \leqq t_1 < \cdots < t_n \leqq T$. *The infinitesimal mean and covariance (local characteristics) are given by*

$$\lim_{h \to 0+} \frac{1}{h} E[\phi_{\tau,\tau+h}(x) - x] = b(\tau, x) + c(\tau, x), \tag{1.18}$$

$$\lim_{h \to 0+} \frac{1}{h} E[(\phi^i_{\tau,\tau+h}(x) - x^i)(\phi^j_{\tau,\tau+h}(y) - y^j)] = a^{ij}(\tau, x, y). \tag{1.19}$$

(iii) $\phi_{s,t}$ *and* $X_{s,t}$ *are linked by the following Itô's stochastic differential equation*

$$\phi_{s,t}(x) = x + \int_s^t dX_\tau(\phi_{s,\tau}(x)) + \int_s^t c(\tau, \phi_{s,\tau}(x))d\tau, \qquad t > s. \tag{1.20}$$

Here, Itô integral by Brownian vector field $X_{s,t}(x)$ is defined as follows. See Le Jan [11] and Le Jan–Watanabe [12]. Let f_t, $t \geqq s$ be a continuous $\mathcal{F}_{s,t}$-adapted R^d-valued process where s is fixed. Let δ_n, $n=1, 2, \cdots$ be a sequence of partitions $\delta_n = \{s = t_0 < t_1 < \cdots < T\}$ such that $|\delta_n| \equiv \max |t_{k+1} - t_k| \downarrow 0$. Then the limit

$$\tilde{M}^i_t = \lim_{n \to \infty} \sum_k X^i_{t_k \wedge t, t_{k+1} \wedge t}(f_{t_k \wedge t}) \tag{1.21}$$

exists in probability and is a continuous $\mathcal{F}_{s,t}$-adapted local martingale. Further, the joint quadratic variation of $\tilde{M}^i_t$ and $\tilde{M}^j_t$ is given by

$$\langle \tilde{M}^i, \tilde{M}^j \rangle_t = \int_s^t a^{ij}(\tau, f_\tau, f_\tau)d\tau, \tag{1.22}$$

i.e., $\tilde{M}^i_t \tilde{M}^j_t - \int_s^t a^{ij}(\tau, f_\tau, f_\tau)d\tau$ is a continuous local martingale. We denote $\tilde{M}^i_t$ by $\int_s^t dX^i_\tau(f_\tau)$.

It might be an interesting problem to relax the mixing condition (A.I) to a weaker one such as Borodin [3] or Kesten–Papanicolaou [7]. Assumption (A.II) is also rather stringent since the constant C is taken independently of τ, x, t, ω. We have not succeeded in relaxing these hypotheses. A difficulty appears in proving the tightness of the measures $\{P^{(\varepsilon)}, \varepsilon > 0\}$.

In order to see the Brownian vector field more explicitly, we shall consider the case where the random vector fields $F(\tau, x, t)$ and $G(\tau, x, t)$ are of separate type:

$$(1.23)\qquad \frac{d\phi_t^\varepsilon}{dt} = \frac{1}{\varepsilon}\sum_k \tilde{F}^k(t, \phi_t^\varepsilon)\eta_k\left(\frac{t}{\varepsilon^2}\right) + \tilde{G}(t, \phi_t^\varepsilon)\,.$$

Here $\tilde{F}_k(\tau, x)$, $k=1, \cdots, n$, and $\tilde{G}(\tau, x)$ are deterministic vector fields satisfying conditions of (A.II) and continuously differentiable in t. $\eta_k(t)$, $k=1, \cdots, n$, are stationary zero-mean processes, $\mathcal{F}_{t,t}$-measurable and bounded. We define the noise intensity matrix (r_{kl}) by

$$(1.24)\qquad r_{kl} = \int_0^\infty R_{kl}(s)ds\,, \qquad R_{kl}(s) = E[\eta_k(s+t)\eta_l(t)]\,.$$

Then it holds

$$(1.25)\qquad A^{ij}(\tau, x, y) = \sum_{k,l}\lim_{\varepsilon\to 0}\left\{\frac{1}{\varepsilon^3}\int_\tau^{\tau+\varepsilon}\int_\tau^\sigma E\left[\eta_k\left(\frac{s}{\varepsilon^2}\right)\eta_l\left(\frac{\sigma}{\varepsilon^2}\right)\right]dsd\sigma\right\}\tilde{F}_k^i(\tau, x)\tilde{F}_l^j(\tau, y)$$
$$= \sum_{k,l} r_{kl}\tilde{F}_k^i(\tau, x)\tilde{F}_l^j(\tau, y)\,,$$

$$(1.26)\qquad b^i(\tau, x) = \tilde{G}^i(\tau, x)\,,$$

and similarly,

$$(1.27)\qquad c(\tau, x) = \sum_{k,l} r_{kl}\left(\sum_i \tilde{F}_k^i(\tau, x)\frac{\partial}{\partial x^i}\tilde{F}_l(\tau, x)\right).$$

For the study of the limiting behavior of $\phi_{s,t}^\varepsilon$, it is convenient to consider the law of triple $(\phi_{s,t}^\varepsilon(x), X_{s,t}^\varepsilon(x), (B_t^{1,\varepsilon} \cdots B_t^{n,\varepsilon}))$, where

$$(1.28)\qquad B_t^{k,\varepsilon} = \frac{1}{\varepsilon}\int_0^t \eta_k\left(\frac{\tau}{\varepsilon^2}\right)d\tau\,.$$

Let $W_3 = C([0, T]; R^n)$ and denote its element by $(B_t^1, \cdots, B_t^n)$. Let $\tilde{W} = W_1 \times W_2 \times W_3$ and $\mathcal{B}_{\tilde{W}}$, the topological Borel field.

Theorem 2. *Let $(\tilde{W}, \mathcal{B}_{\tilde{W}}, \tilde{P}^{(\varepsilon)})$ be the law of the triple $(\phi_{s,t}^\varepsilon, X_{s,t}^\varepsilon, (B_t^{1,\varepsilon}, \cdots, B_t^{n,\varepsilon}))$. Then there is a unique law $\tilde{P}^{(0)}$ on $(\tilde{W}, \mathcal{B}_{\tilde{W}})$ such that $(\mathcal{B}_{\tilde{W}}, \tilde{P}^{(\varepsilon)})$ converges weakly to $(\mathcal{B}_{\tilde{W}}, \tilde{P}^{(0)})$. Furthermore, the law $\tilde{P}^{(0)}$ admits the following properties.*

(i) *$(B_t^1, \cdots, B_t^n)$ is an n-dimensional Brownian motion with zero-mean and convariance $t(r_{kl}+r_{lk})$, $k, l=1, \cdots, n$.*

(ii) *Brownian vector field $(X_{s,t}, \tilde{P}^{(0)})$ is represented by*

$$(1.29)\qquad X_{s,t}(x) = \sum_{k=1}^n\int_s^t \tilde{F}_k(\tau, x)dB_\tau^k + \int_s^t \tilde{G}(\tau, x)d\tau\,, \qquad t>s\,.$$

(iii) *Brownian motion $\phi_{s,t}$ on G=Homeo (R^d) satisfies the following Stratonovich stochastic differential equation*

$$\phi_{s,t}(x) = x+\sum_{k=1}^{n}\int_s^t \tilde{F}_k(\tau, \phi_{s,\tau}(x))\circ dB_\tau^k+\int_s^t \tilde{G}(\tau, \phi_{s,\tau}(x))d\tau + \sum_{0\leq k\leq l\leq n}\frac{1}{2}(r_{kl}-r_{lk})\int_s^t [\tilde{F}_k, \tilde{F}_l](\tau, \phi_{s,\tau}(x))d\tau, \tag{1.30}$$

where $[\tilde{F}_k, \tilde{F}_l]$ *is the Lie bracket defined by*

$$[\tilde{F}_k, \tilde{F}_l] = \sum_i \tilde{F}_k^i \frac{\partial}{\partial x_i}\tilde{F}_l - \sum_i \tilde{F}_l^i \frac{\partial}{\partial x_i}\tilde{F}_k . \tag{1.31}$$

It is interesting to compare the above result with the approximating theorems of stochastic differential equation studied by Wong–Zakai [18] and Ikeda–Nakao–Yamato [5]. In these works, stochastic ordinary differential equation of the form

$$\frac{d}{dt}\phi_t^\varepsilon = \sum_{k=1}^{n}\tilde{F}_k(t, \phi_t^\varepsilon)\dot{B}_t^{k,\varepsilon}+\tilde{G}(t, \phi_t^\varepsilon), \quad \phi_s^\varepsilon = x$$

is considered, where $B_t^{k,\varepsilon}$, $\varepsilon>0$ are piecewise smooth approximations of B_t^k and $\dot{B}_t^{k,\varepsilon}=\frac{d}{dt}B_t^{k,\varepsilon}$. In [18], the polygonal approximation of B_t^k:

$$B_t^{k,\varepsilon} = B_{i\varepsilon}^k+\frac{1}{\varepsilon}(B_{(i+1)\varepsilon}^k-B_{i\varepsilon}^k)(t-i) \qquad \text{if } i\varepsilon<t<(i+1)\varepsilon$$

is taken. It is shown that ϕ_t^ε converges strongly (L^2-convergence) to the solution of (1.30) without the last correction term, i.e., it corresponds to the case that (r_{kl}) is symmetric. In [5], more general approximations such as Mcshane's and regularizations by mollifiers are considered. There, the limit ϕ_t satisfies (1.30) with the correction term. See also Ikeda–Watanabe [6]. The correction term is related to the stochastic area enclosed by the curve $(B_t^{k,\varepsilon}, B_t^{l,\varepsilon})$ and its chord:

$$\frac{1}{2}(r_{kl}-r_{lk}) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}E[\frac{1}{2}(\int_0^\varepsilon B_s^{k,\varepsilon}dB_s^{l,\varepsilon}-\int_0^\varepsilon B_s^{l,\varepsilon}dB_s^{k,\varepsilon})].$$

Finally we consider the case where coefficients $F(\tau, x, t, \omega)$ and $G(\tau, x, t, \omega)$ of equation (1.12) are smooth in x. We shall introduce the following.

(A.IV) $F(\tau, x, t, \omega)$ is $(r+2)$-times continuously differentiable in x and derivatives are all bounded in x. $G(\tau, x, t, \omega)$ is $(r+1)$-times continuously differentiable and their derivatives are all bounded. Furthermore, derivatives of $\sum_i F^i(\tau, x, t, \omega)D^kF^j(\tau', x, t', \omega)$, $|k|\leq r+1$ are all bounded.

Now a flow of homeomorphisms $\phi_{s,t}$, s, $t\in[0, T]$ is called a flow of C^r-diffeomorphisms if for each $s<t$, the maps $\phi_{s,t}$; $R^d\to R^d$ are C^r-diffeomorphisms and their derivatives in x up to r are continuous in (s, t, x). Two parameter family of C^r-vector fields $X_{s,t}(x)$ is defined similarly. We denote

by W_1^r the totality of flows of C^r-diffeomorphisms and by W_2^r the totality of two parameter families of C^r-vector fields with the additive property. For $\phi_{s,t}$ and $\psi_{s,t}$ of W_1^r, we define the metric d_r by

$$d_r(\phi, \psi) = \sum_{|k| \leq r} \sup_{s,t \in [0,T]} \rho(D^k \phi_{s,t}, D^k \psi_{s,t}),$$

where ρ is the compact uniform topology on $C(R^d; R^d)$, k is the multi-index of nonnegative integers $k=(k_1, \cdots, k_d)$, $|k|=k_1+\cdots+k_d$, $D^k=\left(\frac{\partial}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial}{\partial x_d}\right)^{k_d}$. Noting $\phi_{s,t}=\phi_{t,s}^{-1}$, we see that (W_1^r, d_r) is a complete separable metric space. To the space W_2^r, we associate the same metric d_r. The product space $W^r = W_1^r \otimes W_2^r$ is then a complete separable metric space. Denote by $\mathcal{B}_{W^r}$ the topologycal Borel field of W^r.

Theorem 3. *Assume* (A.I), (A.II), (A.III) *and* (A.IV). *Let* $\hat{P}^{(\varepsilon)}$ *be the law of* $(X_{s,t}^\varepsilon, \phi_{s,t}^\varepsilon)$ *on* $(W^r, \mathcal{B}_{W^r})$. *Then* $\hat{P}^{(\varepsilon)}$ *converges weakly to* $\hat{P}^{(0)}$ *on* $(W^r, \mathcal{B}_{W^r})$ *relative to* d_r*-topology. Furthermore,* $\hat{P}^{(0)}$ *coincides with* $P^{(0)}$ *of Theorem* 1, *i.e.,* $P^{(0)}$ *is supported by* W^r *and the restriction* $P^{(0)} | W^r$ *coincides with* $\hat{P}^{(0)}$.

For the proof of these theorems, we will discuss two problems. The first one is the tightness of the measures $\{P^{(\varepsilon)}, \varepsilon>0\}$, $\{\tilde{P}^{(\varepsilon)}, \varepsilon>0\}$ and $\{\hat{P}^{(\varepsilon)}, \varepsilon>0\}$. This will be inspected at the next section by means of Kolmogorov's criterion of the tightness of continuous random fields. The next problem is to show that any weak limit $P^{(0)}$ is a solution of a suitable martingale problem. We will see at Section 3 that the $(n+m)$-point motion $(\phi_{t_0,t}(x_1), \cdots, {}_{t_0,t}(x_n), X_{t_0,t}(y_1), \cdots, X_{t_0,t}(y_m))$ is a diffusion process with local characteristics a^{ij}, b^i and c^i for any t_0. Theorems 1–3 will then be proved at Section 4.

2. Tightness of measures

In this section, we are concerned with the tightness of the family of laws $P^{(\varepsilon)}$, $\varepsilon>0$ induced by the solutions $(\phi_{s,t}^\varepsilon(x), X_{s,t}^\varepsilon(x))$. We shall first quote Kolmogorov's tightness criterion for a sequence of continuous random fields.

Kolmogorov's theorem.[1] *Let* $X_n(\lambda)$, $\lambda \in \Lambda$ *be a sequence of continuous* R^d*-valued random fields with parameter* $\Lambda=[-N, N]^k$. *Suppose that there are positive constants* α, β, γ *and* K *such that*

$$E[|X_n(0)|^\alpha] \leq K,$$
$$E[|X_n(\lambda)-X_n(\mu)|^\beta] \leq K|\lambda-\mu|^{k+\gamma}$$

hold for any λ, μ *and* n. *Let* $P^{(n)}$ *be the law of* $X_n(\lambda)$ *defined on* $C(\Lambda; R^d)$. *Then the sequence of measures* $P^{(n)}$, $n=1, 2, \cdots$ *is tight.*

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1) The theorem is well known in case that X_n is a sequence of stochastic processes. See Theorem 12.3 in Billingsley [2]. The extension to the random field is not difficult.

In the following two propositions, we will check the above criterion to the random fields $X^{\varepsilon}_{s,t}(x)$ and $\phi^{\varepsilon}_{r,t}(x)$, separately.

Proposition 2.1. *Let p be an arbitrary positive integer. Then there is a positive constant K_p not depending on ε such that*

$$E[|X^{\varepsilon}_{s,t}(x)|^{2p}] \leqq K_p|t-s|^p(1+|x|)^{2p}, \tag{2.1}$$

$$E[|X^{\varepsilon}_{s,t}(x)-X^{\varepsilon}_{s',t'}(x')|^{2p}] \leqq K_p\{(1+|x|)^{2p}(|s-s'|^p+|t-t'|^p)+|x-x'|^{2p}\} \tag{2.2}$$

hold for any $s, s', t, t' \in [0, T]$ and $x, x' \in R^d$.

Proof. The above estimate is clearly satisfied for $X^{\varepsilon}_{s,t}(x) \equiv \int_s^t G(\tau, x, \frac{\tau}{\varepsilon^2})d\tau$. Therefore it is enough to consider the case $G\equiv 0$. For the convenience, we set $\mathcal{F}^{\varepsilon}_{s,t} = \mathcal{F}_{s/\varepsilon^2, t/\varepsilon^2}$, $\mathcal{F}^{\varepsilon}_t = \mathcal{F}^{\varepsilon}_{0,t}$ for $0 \leqq s \leqq t \leqq +\infty$. The i-th component of $X^{\varepsilon}_t(x)$ is denoted by $\tilde{X}^{\varepsilon}_t$, since s, x and i are fixed. $F^i_{\varepsilon}(t, x)$ is abbreviated as $F^i_{\varepsilon}(t)$. We will first prove (2.1) in case $s<t$. The case $t<s$ can be proved similarly. It holds

$$\begin{aligned} E[(\tilde{X}^{\varepsilon}_t)^{2p}] &= 2p(2p-1)E[\int_s^t d\tau \int_s^{\tau} d\sigma F^i_{\varepsilon}(\tau)F^i_{\varepsilon}(\sigma)(\tilde{X}^{\varepsilon}_{\sigma})^{2p-2}] \\ &= 2p(2p-1)E[\int_s^t d\sigma \int_{\sigma}^t d\tau E[F^i_{\varepsilon}(\tau)|\mathcal{F}^{\varepsilon}_{\sigma}]F^i_{\varepsilon}(\sigma)(\tilde{X}^{\varepsilon}_{\sigma})^{2p-2}]\,. \end{aligned} \tag{2.3}$$

Since $E[F^i_{\varepsilon}(\tau)]=0$ and $F^i_{\varepsilon}(\tau)$ is $\mathcal{F}^{\varepsilon}_{\tau,\tau}$-measurable, it holds by the mixing property (A.I),

$$\begin{aligned} |E[F^i_{\varepsilon}(\tau)|\mathcal{F}^{\varepsilon}_{\sigma}]| &\leqq 2\sup |F^i_{\varepsilon}(\tau)|\rho(\frac{\tau-\sigma}{\varepsilon^2}) \\ &\leqq \frac{2}{\varepsilon}C(1+|x|)\rho(\frac{\tau-\sigma}{\varepsilon^2})\,, \end{aligned}$$

where C is a constant in Assumption (A.II). (See Papanicolaou–Kohler [13], Lemma 1) Therefore

$$\begin{aligned} |\int_{\sigma}^t d\tau E[F^i_{\varepsilon}(\tau)|\mathcal{F}^{\varepsilon}_{\sigma}]F^i_{\varepsilon}(\sigma)| &\leqq 2C^2(1+|x|)^2\frac{1}{\varepsilon^2}\int_{\sigma}^t \rho(\frac{\tau-\sigma}{\varepsilon^2})d\tau \\ &\leqq 2C^2(1+|x|)^2\int_0^{\infty}\rho(s)ds\,. \end{aligned}$$

Then (2.3) is estimated as

$$E[(\tilde{X}^{\varepsilon}_t)^{2p}] \leqq 2p(2p-1)C_1(1+|x|)^2\int_s^t E[(\tilde{X}^{\varepsilon}_{\sigma})^{2p-2}]d\sigma$$

where $C_1 = 2C^2\int_0^{\infty}\rho(s)ds$. Then by iteration, we get

$$E[(\tilde{X}_t^\varepsilon)^{2p}] \leqq (2p)!\,C_1^p(1+|x|)^{2p}\int_s^t dt_1 \cdots \int_s^{t_{p-1}} dt_p$$
$$\leqq \frac{(2p)!}{p!} C_1^p(1+|x|)^{2p}|t-s|^p .$$

This proves (2.1).

We will next prove

$$(2.4) \qquad E[|\tilde{X}_t^\varepsilon(x)-\tilde{X}_t^\varepsilon(y)|^{2p}] \leqq C_2|x-y|^{2p}|t-s|^p$$

in case $t>s$. The case $t<s$ can be proved similarly. It holds

$$(2.5) \qquad E[|\tilde{X}_t^\varepsilon(x)-\tilde{X}_t^\varepsilon(y)|^{2p}]$$
$$= 2p(2p-1)E[\int_s^t d\sigma \int_\sigma^t d\tau E[F_\varepsilon^i(\tau, x)-F_\varepsilon^i(\tau, y)\,|\,\mathcal{F}_\sigma^\varepsilon]$$
$$\times (F_\varepsilon^i(\sigma, x)-F_\varepsilon^i(\sigma, y))(\tilde{X}_\sigma^\varepsilon(x)-\tilde{X}_\sigma^\varepsilon(y))^{2p-2}] .$$

Since

$$F_\varepsilon^i(\tau, x)-F_\varepsilon^i(\tau, y) = \sum_j \Big(\int_0^1 \partial_j F_\varepsilon^i(\tau, y+v(x-y))dv\Big)(x^j-y^j) ,$$

we have by the mixing property mentioned above,

$$\Big|\int_\sigma^t d\tau E[F_\varepsilon^i(\tau, x)-F_\varepsilon^i(\tau, y)\,|\,\mathcal{F}_\sigma^\varepsilon](F_\varepsilon^i(\sigma, x)-F_\varepsilon^i(\sigma, y))\Big|$$
$$\leqq 2C^2\Big(\int_0^\infty \rho(s)ds\Big)\Big(\sum_j |x^j-y^j|\Big)^2$$
$$\leqq 2C^2 d\Big(\int_0^\infty \rho(s)ds\Big)|x-y|^2 .$$

Therefore (2.5) implies

$$E[|\tilde{X}_t^\varepsilon(x)-\tilde{X}_t^\varepsilon(y)|^{2p}] \leqq C_2'|x-y|^2\int_s^t E[|\tilde{X}_\sigma^\varepsilon(x)-\tilde{X}_\sigma^\varepsilon(y)|^{2p-2}]d\sigma .$$

By iteration we get (2.4).

For the proof of (2.2), observe that $X_{s,t}^\varepsilon$ has the additive property $X_{s,u}^\varepsilon = X_{s,t}^\varepsilon + X_{t,u}^\varepsilon$. Then we have

$$E[|X_{s,t}^\varepsilon(x)-X_{s',t'}^\varepsilon(x')|^{2p}]$$
$$\leqq 3^{2p}\{E[|X_{s,s'}^\varepsilon(x)|^{2p}]+E[|X_{s',t}^\varepsilon(x)-X_{s',t}^\varepsilon(x')|^{2p}]+E[|X_{t,t'}^\varepsilon(x')|^{2p}]\} .$$

Apply (2.1) and (2.4) to the right hand side of the above. Then we get the desired inequality (2.2).

We will next estimate the solution $\phi_{s,t}^\varepsilon$.

Proposition 2.2. *Let p be an arbitrary positive integer. Then there is a*

Therefore, the first term of (2.11) is dominated by

$$4pC^2\left(\int_0^\infty \rho(s)ds\right)E\left[\int_s^t (1+|\phi_\sigma^\varepsilon|)|\tilde{\psi}_\sigma^\varepsilon|^{2p-1}d\sigma \,|\, \mathcal{F}_s^\varepsilon\right]. \tag{2.12}$$

Similarly, the second term of (2.11) is dominated by

$$2p(2p-1)2C^2\left(\int_0^\infty \rho(u)du\right)E\left[\int_s^t (1+|\phi_\sigma^\varepsilon|)^2(\tilde{\psi}_\sigma^\varepsilon)^{2p-2}d\sigma \,|\, \mathcal{F}_s^\varepsilon\right]. \tag{2.13}$$

Sum up (2.10), (2.12) and (2.13) and note the relation $|1+\phi_\sigma^\varepsilon|\leqq|\psi_\sigma^\varepsilon|+1+|x|$. Then we get

$$\begin{aligned} E[|\psi_t^\varepsilon|^{2p}|\mathcal{F}_s^\varepsilon] \leqq C_3\int_s^t & E[|\psi_\sigma^\varepsilon|^{2p}|\mathcal{F}_s^\varepsilon]d\sigma \\ &+C_4(1+|x|)\int_s^t E[|\psi_\sigma^\varepsilon|^{2p-1}|\mathcal{F}_s^\varepsilon]d\sigma \\ &+C_5(1+|x|)^2\int_s^t E[|\psi_\sigma^\varepsilon|^{2p-2}|\mathcal{F}_s^\varepsilon]d\sigma, \end{aligned}$$

where C_3, C_4 and C_5 are constants not depending on s, t, x and ε. By Gronwall's inequality, we have

$$\begin{aligned} E[|\psi_t^\varepsilon|^{2p}|\mathcal{F}_s^\varepsilon] \leqq C_6\Big\{(1+|x|)&\int_s^t E[|\psi_\sigma^\varepsilon|^{2p-1}|\mathcal{F}_s^\varepsilon]d\sigma \\ &+(1+|x|)^2\int_s^t E[|\psi_\sigma^\varepsilon|^{2p-2}|\mathcal{F}_s^\varepsilon]d\sigma\Big\}. \end{aligned}$$

By iteration, we get

$$E[|\psi_t^\varepsilon|^{2p}|\mathcal{F}_s^\varepsilon] \leqq C_7(1+|x|)^{2p}|t-s|^p.$$

Lemma 2.4. *There is a positive constant C_p not depending on ε such that*

$$E[|\phi_{s,t}^\varepsilon(x)-\phi_{s,t}^\varepsilon(y)-(x-y)|^{2p}|\mathcal{F}_s^\varepsilon] \leqq C_p|x-y|^{2p}|t-s|^p \quad a.s. \tag{2.14}$$

holds for any s, $t\in[0, T]$ and x, $y\in R^d$.

Proof. We prove the lemma in case $s<t$ only. Set $\psi_t^\varepsilon=\phi_{s,t}^\varepsilon(x)-\phi_{s,t}^\varepsilon(y)-(x-y)$ and denote the i-th component by $\tilde{\psi}_t^\varepsilon$. Then it holds

$$\begin{aligned} E[(\tilde{\psi}_t^\varepsilon)^{2p}|\mathcal{F}_s^\varepsilon] = 2pE\Big[\int_s^t & (\bar{F}_\varepsilon^i(\tau, \phi_\tau^\varepsilon(x))-\bar{F}_\varepsilon^i(\tau, \phi_\tau^\varepsilon(y)))(\tilde{\psi}_\tau^\varepsilon)^{2p-1}d\tau\,|\,\mathcal{F}_s^\varepsilon\Big] \\ &+2pE\Big[\int_s^t (\tilde{F}_\varepsilon^i(\tau, \phi_\tau^\varepsilon(x))-\tilde{F}_\varepsilon^i(\tau, \phi_\tau^\varepsilon(y)))(\tilde{\psi}_\tau^\varepsilon)^{2p-1}d\tau\,|\,\mathcal{F}_s^\varepsilon\Big]. \end{aligned}$$

Since $|\bar{F}_\varepsilon^i(\tau, x)-\bar{F}_\varepsilon^i(\tau, y)|\leqq C|x-y|$, the first term of the right hand side is dominated by

$$2pCE\left[\int_s^t |\phi_\tau^\varepsilon(x)-\phi_\tau^\varepsilon(y)|\,|\tilde{\psi}_\tau^\varepsilon|^{2p-1}d\tau\,|\,\mathcal{F}_s^\varepsilon\right]. \tag{2.15}$$

positive constant K_p *not depending on* ε *such that*

(2.6) $$E[|\phi^{\varepsilon}_{s,t}(x)|^{2p}] \leqq K_p(1+|x|)^{2p},$$

(2.7) $$E[|\phi^{\varepsilon}_{s,t}(x)-\phi^{\varepsilon}_{s',t'}(x')|^{2p}] \leqq K_p\{(1+|x|)^{2p}(|t-t'|^p+|s-s'|^p)+|x-x'|^{2p}\}$$

hold for any $s, t, s', t' \in [0, T]$ *and* $x, x' \in R^d$.

Before the proof, we prepare two related estimates.

Lemma 2.3. *There is a constant* C_p *not depending on* ε *such that*

(2.8) $$E[|\phi^{\varepsilon}_{s,t}(x)-x|^{2p}|\mathcal{F}^{\varepsilon}_s] \leqq C_p(1+|x|)^{2p}|t-s|^p \quad a.s.$$

holds for any $s, t \in [0, T]$ *and* $x \in R^d$.

Proof. We will prove the lemma in case $s<t$ only. The other case can be shown similarly. In the following discussion we write $\phi^{\varepsilon}_t=\phi^{\varepsilon}_{s,t}(x)$, $\psi^{\varepsilon}_t=\phi^{\varepsilon}_{s,t}(x)-x$ since s and x are fixed. Further, $\tilde{\psi}^{\varepsilon}_t$ denotes the i-th component of ψ^{ε}_t. It holds

(2.9) $$E[(\tilde{\psi}^{\varepsilon}_t)^{2p}|\mathcal{F}^{\varepsilon}_s] = 2pE[\int_s^t \bar{F}^i_{\varepsilon}(\tau, \phi^{\varepsilon}_{\tau})(\tilde{\psi}^{\varepsilon}_{\tau})^{2p-1}d\tau|\mathcal{F}^{\varepsilon}_s]$$
$$+2pE[\int_s^t \tilde{F}^i_{\varepsilon}(\tau, \phi^{\varepsilon}_{\tau})(\tilde{\psi}^{\varepsilon}_{\tau})^{2p-1}d\tau|\mathcal{F}^{\varepsilon}_s]$$

where $\bar{F}^i_{\varepsilon}(\tau, x)=E[F^i_{\varepsilon}(\tau, x)]$ and $\tilde{F}^i_{\varepsilon}(\tau, x)=F^i_{\varepsilon}(\tau, x)-\bar{F}^i_{\varepsilon}(\tau, x)$. Since $\bar{F}^i_{\varepsilon}(\tau, x)$ $=E[G(\tau, x, \frac{\tau}{\varepsilon^2})]$, it is dominated by $C(1+|x|)$ by (A.II). Therefore the first term is dominated by

(2.10) $$2pCE[\int_s^t(1+|\phi^{\varepsilon}_{\tau}|)|\tilde{\psi}^{\varepsilon}_{\tau}|^{2p-1}d\tau|\mathcal{F}^{\varepsilon}_s].$$

The second term is written as

(2.11) $$2pE[\int_s^t d\tau\int_s^{\tau} d\sigma(\sum_j \partial_j\tilde{F}^i_{\varepsilon}(\tau, \phi^{\varepsilon}_{\sigma})F^j_{\varepsilon}(\sigma, \phi^{\varepsilon}_{\sigma}))(\tilde{\psi}^{\varepsilon}_{\sigma})^{2p-1}|\mathcal{F}^{\varepsilon}_s]$$
$$+2p(2p-1)E[\int_s^t d\tau\int_s^{\tau} d\sigma\tilde{F}^i_{\varepsilon}(\tau, \phi^{\varepsilon}_{\sigma})F^i_{\varepsilon}(\sigma, \phi^{\varepsilon}_{\sigma})(\tilde{\psi}^{\varepsilon}_{\sigma})^{2p-2}|\mathcal{F}^{\varepsilon}_s]$$
$$=2p\int_s^t d\sigma\int_{\sigma}^t d\tau E[\sum_j E[\partial_j\tilde{F}^i_{\varepsilon}(\tau, \phi^{\varepsilon}_{\sigma})|\mathcal{F}^{\varepsilon}_{\sigma}]F^j_{\varepsilon}(\sigma, \phi^{\varepsilon}_{\sigma})(\tilde{\psi}^{\varepsilon}_{\sigma})^{2p-1}|\mathcal{F}^{\varepsilon}_s]$$
$$+2p(2p-1)\int_s^t d\sigma\int_{\sigma}^t d\tau E[E[\tilde{F}^i_{\varepsilon}(\tau, \phi^{\varepsilon}_{\sigma})|\mathcal{F}^{\varepsilon}_{\sigma}]F^i_{\varepsilon}(\sigma, \phi^{\varepsilon}_{\sigma})(\tilde{\psi}^{\varepsilon}_{\sigma})^{2p-2}|\mathcal{F}^{\varepsilon}_s].$$

Since $E[\partial_j\tilde{F}^i_{\varepsilon}(\tau, x)]=0$ and $\partial_j\tilde{F}^i_{\varepsilon}$ is $\mathcal{F}^{\varepsilon}_{\tau,\tau}$-measurable, we have by the mixing property,

$$|E[\partial_j\tilde{F}^i_{\varepsilon}(\tau, \phi^{\varepsilon}_{\sigma})|\mathcal{F}^{\varepsilon}_{\sigma}]| \leqq \frac{2}{\varepsilon}C\rho(\frac{\tau-\sigma}{\varepsilon^2}).$$

The second term equals

$$2pE[\int_s^t d\sigma\int_\sigma^t d\tau E[\{\sum_j \partial_j\tilde{F}_\varepsilon^i(\tau,\phi_\sigma^\varepsilon(x))F_\varepsilon^j(\sigma,\phi_\sigma^\varepsilon(x)) \\ -\sum_j \partial_j\tilde{F}_\varepsilon^i(\tau,\phi_\sigma^\varepsilon(y))F_\varepsilon^j(\sigma,\phi_\sigma^\varepsilon(y))\}\,|\,\mathcal{F}_\sigma^\varepsilon](\tilde{\psi}_\sigma^\varepsilon)^{2p-1}\,|\,\mathcal{F}_s^\varepsilon] \\ +2pE[\int_s^t d\sigma\int_\sigma^t d\tau E[\tilde{F}_\varepsilon^i(\tau,\phi_\sigma^\varepsilon(x))-\tilde{F}_\varepsilon^i(\tau,\phi_\sigma^\varepsilon(y))\,|\,\tilde{\mathcal{F}}_\sigma^\varepsilon] \\ \times(F_\varepsilon^i(\sigma,\phi_\sigma^\varepsilon(x))-F_\varepsilon^i(\sigma,\phi_\sigma^\varepsilon(y)))(\tilde{\psi}_\sigma^\varepsilon)^{2p-2}\,|\,\mathcal{F}_s^\varepsilon] \\ =I_1+I_2\,.$$

We will consider I_1. By assumptions (A.I) and (A.II), the absolute value of the conditional expectation $E[\{\cdots\}\,|\,\mathcal{F}_\sigma^\varepsilon]$ is dominated by

$$2\rho(\frac{\tau-\sigma}{\varepsilon^2})\sup_{\omega'}|H_\varepsilon^i(\tau,\sigma,\phi_\sigma^\varepsilon(x),\omega')-H_\varepsilon^i(\tau,\sigma,\phi_\sigma^\varepsilon(y),\omega')|^{1)} \\ \leqq 2C^2\frac{1}{\varepsilon^2}\rho(\frac{\tau-\sigma}{\varepsilon^2})\,|\phi_\sigma^\varepsilon(x)-\phi_\sigma^\varepsilon(y)|\,,$$

where $H_\varepsilon^i(\tau,\sigma,x,\omega')=\sum\partial_j\tilde{F}_\varepsilon^i(\tau,x,\omega')F_\varepsilon^j(\sigma,x)$. Therefore $|I_1|$ is dominated by the same quantity as (2.15). We can estimate $|I_2|$ similarly. We have in fact

$$|I_2|\leqq C_8E[\int_s^t|\phi_\sigma^\varepsilon(x)-\phi_\sigma^\varepsilon(y)|^2(\tilde{\psi}_\sigma^\varepsilon)^{2p-2}d\sigma\,|\,\mathcal{F}_s^\varepsilon]\,.$$

Summing up these estimations and noting the relation $|\phi_\sigma^\varepsilon(x)-\phi_\sigma^\varepsilon(y)|\leqq|x-y|+|\psi_\sigma^\varepsilon|$, we obtain

$$E[|\psi_t^\varepsilon|^{2p}\,|\,\mathcal{F}_s^\varepsilon]\leqq C_9E[\int_s^t|\psi_\sigma^\varepsilon|^{2p}d\sigma\,|\,\mathcal{F}_s^\varepsilon] \\ +C_{10}|x-y|E[\int_s^t|\psi_\sigma^\varepsilon|^{2p-1}d\sigma\,|\,\mathcal{F}_s^\varepsilon] \\ +C_{11}|x-y|^2E[\int_s^t|\psi_\sigma^\varepsilon|^{2p-2}d\sigma\,|\,\mathcal{F}_s^\varepsilon]$$

where C_9, C_{10} and C_{11} are constants not depending on s, t, x and ε. By Gronwall's lemma,

$$E[|\psi_t^\varepsilon|^{2p}\,|\,\mathcal{F}_s^\varepsilon]\leqq C_{12}|x-y|E[\int_s^t|\psi_\sigma^\varepsilon|^{2p-1}d\sigma\,|\,\mathcal{F}_s^\varepsilon] \\ +C_{13}|x-y|^2E[\int_s^t|\psi_\sigma^\varepsilon|^{2p-2}d\sigma\,|\,\mathcal{F}_s^\varepsilon]\,.$$

This implies by iteration the estimate of the lemma.

Proof of Proposition 2.2. The estimate (2.6) is immediate from (2.8). We will prove (2.7) in case $s\leqq s'\leqq t\leqq t'$ only: Other cases can be proved similarly. Since

1) See Lemma 1 in [13]

$$\phi^{\varepsilon}_{s,t}(x)-\phi^{\varepsilon}_{s',t'}(x')=x-x'+\int_s^{s'}F_\varepsilon(\tau,\phi^{\varepsilon}_{s,\tau}(x))d\tau-\int_t^{t'}F_\varepsilon(\tau,\phi^{\varepsilon}_{s',\tau}(x'))d\tau$$
$$+\int_{s'}^{t}\{F_\varepsilon(\tau,\phi^{\varepsilon}_{s',\tau}\circ\phi^{\varepsilon}_{s,s'}(x))-F_\varepsilon(\tau,\phi^{\varepsilon}_{s',\tau}(x'))\}d\tau\,,$$

we shall estimate each of the right side. Similarly as in Lemma 2.3,

$$E[|\int_s^{s'}F_\varepsilon(\tau,\phi^{\varepsilon}_{s,\tau}(x))d\tau|^{2p}]\leqq C_p(1+|x|)^{2p}|s'-s|^p\,,$$

and

$$E[|\int_t^{t'}F_\varepsilon(\tau,\phi^{\varepsilon}_{s',\tau}(x))d\tau|^{2p}]=E[E[|\int_t^{t'}F_\varepsilon(\tau,\phi^{\varepsilon}_{t,\tau}(y))d\tau|^{2p}|\mathcal{F}^{\varepsilon}_t]_{y=\phi^{\varepsilon}_{s',t}(x)}]$$
$$\leqq C_p|t'-t|^pE[(1+|\phi^{\varepsilon}_{s',t}(x)|)^{2p}]$$
$$\leqq C'_p|t'-t|^p(1+|x|)^{2p}\,.$$

Similarly as in Lemma 2.4, we have

$$E[|\int_{s'}^{t}F_\varepsilon(\tau,\phi^{\varepsilon}_{s',\tau}\circ\phi^{\varepsilon}_{s,s'}(x'))-F_\varepsilon(\tau,\phi^{\varepsilon}_{s',\tau}(x))d\tau|^{2p}]$$
$$=E[\{E[|\int_{s'}^{t}F_\varepsilon(\tau,\phi^{\varepsilon}_{s',\tau}(z))-F_\varepsilon(\tau,\phi^{\varepsilon}_{s',\tau}(x))d\tau|^{2p}|\mathcal{F}^{\varepsilon}_{s'}]_{z=\phi^{\varepsilon}_{s,s'}(x')}\}]$$
$$\leqq C_p|t-s'|^pE[|\phi^{\varepsilon}_{s,s'}(x')-x|^{2p}]\,.$$

By Lemma 2.3,

$$E[|\phi^{\varepsilon}_{s,s'}(x')-x|^{2p}]\leqq 2^{2p}\{|x-x'|^{2p}+E[|\phi^{\varepsilon}_{s,s'}(x')-x'|^{2p}]\}$$
$$\leqq 2^{2p}\{|x-x'|^{2p}+C_p(1+|x'|)^{2p}|s-s'|^p\}\,.$$

Summing up all these estimates, we get (2.7).

We next discuss the tightness of the family of laws $\hat{P}^{(\varepsilon)}$, $\varepsilon>0$ on $(W^r, \mathcal{B}_{W_r})$ assuming the additional assumption (A.IV).

Proposition 2.5. *Assume* (A.I)–(A.IV). *Let p be an arbitrary positive integer. Then there is a positive constant K_p not depending on ε such that*

$$\sum_{|k|\leqq r}E[|D^kX^{\varepsilon}_{s,t}(x)|^{2p}]\leqq K_p|t-s|^p(1+|x|)^{2p}\,,\tag{2.16}$$

$$\sum_{|k|\leqq r}E[|D^kX^{\varepsilon}_{s,t}(x)-D^kX^{\varepsilon}_{s',t'}(x')|^{2p}]\,,\tag{2.17}$$
$$\leqq K_p\{(1+|x|)^{2p}(|s-s'|^p+|t-t'|^p)+|x-x'|^{2p}\}\,,$$

$$\sum_{|k|\leqq r}E[|D^k\phi^{\varepsilon}_{s,t}(x)|^{2p}]\leqq K_p(1+|x|)^{2p}\,,\tag{2.18}$$

$$\sum_{|k|\leqq r}E[|D^k\phi^{\varepsilon}_{s,t}(x)-D^k\phi^{\varepsilon}_{s',t'}(x')|^{2p}]\tag{2.19}$$
$$\leqq K_p\{(1+|x|)^{2p}(|t-t'|^p+|s-s'|^p)+|x-x'|^{2p}\}$$

hold for any x, $x'\in R^d$ and s, s', t, $t'\in[0,T]$.

We only give the proof of (2.19), which is most complicated among the four inequalities. The case $r=1$ is only considered, since the case $r\geqq 2$ can be shown similarly.

Lemma 2.6. *Let* $\partial_j=\dfrac{\partial}{\partial x_j}$. *There is a positive constant* C_p *not depending on* ε *such that*

$$E[|\partial_j(\phi^\varepsilon_{s,t}(x)-x)|^{2p}|\mathcal{F}^\varepsilon_s]\leqq C_p|t-s|^p, \tag{2.20}$$

$$E[|\partial_j\phi^\varepsilon_{s,t}(x)-\partial_j\phi^\varepsilon_{s,t}(y)-\partial_jx+\partial_jy|^{2p}|\mathcal{F}^\varepsilon_s]\leqq C_p|x-y|^{2p}|t-s|^p \tag{2.21}$$

hold for any s, $t\in[0, T]$ *and* $x, y\in R^d$.

Proof. We prove (2.21) in case $s<t$ only. Set $\psi^\varepsilon_t=\partial_j\phi^\varepsilon_{s,t}(x)-\partial_j\phi^\varepsilon_{s,t}(y)-\partial_jx+\partial_jy$ and denote the i-th component by $\tilde{\psi}^\varepsilon_t$. Then it holds

$$\tilde{\psi}^\varepsilon_t=\sum_k\int_s^t\{\partial_kF^i_\varepsilon(r, \phi^\varepsilon_r(x))\partial_j\phi^{\varepsilon,k}_r(x)-\partial_kF^i_\varepsilon(r, \phi^\varepsilon_r(y))\partial_j\phi^{\varepsilon,k}_r(y)\}dr\,.$$

Therefore $(\tilde{\psi}^\varepsilon_t)^{2p}$ equals

$$\begin{aligned}
&2p\int_s^t\sum_k\{\partial_k\bar{F}^i_\varepsilon(r, \phi^\varepsilon_r(x))-\partial_k\bar{F}^i_\varepsilon(r, \phi^\varepsilon_r(y))\}\partial_j\phi^{\varepsilon,\kappa}_r(x)(\tilde{\psi}^\varepsilon_r)^{2p-1}dr\\
&+2p\int_s^t\sum_k\partial_k\bar{F}^i_\varepsilon(r, \phi^\varepsilon_r(y))(\partial_j\phi^{\varepsilon,k}_r(x)-\partial_j\phi^{\varepsilon,k}_r(y))(\tilde{\psi}^\varepsilon_r)^{2p-1}dr\\
&+2p\int_s^t\sum_k\{\partial_k\tilde{F}^i_\varepsilon(r, \phi^\varepsilon_r(x))-\partial_k\tilde{F}^i_\varepsilon(r, \phi^\varepsilon_r(y))\}\partial_j\phi^{\varepsilon,k}_r(x)(\tilde{\psi}^\varepsilon_r)^{2p-1}dr\\
&+2p\int_s^t\sum_k\partial_k\tilde{F}^i_\varepsilon(r, \phi^\varepsilon_r(y))(\partial_j\phi^{\varepsilon,k}_r(x)-\partial_j\phi^{\varepsilon,k}_r(y))(\tilde{\psi}^\varepsilon_r)^{2p-1}dr\\
&=I_1+I_2+I_3+I_4\,,
\end{aligned}$$

where $\bar{F}_\varepsilon=E[F_\varepsilon]$ and $\tilde{F}_\varepsilon=F_\varepsilon-\bar{F}_\varepsilon$.

In the following argument, constants C_i are chosen to be independent of ε. Since $\partial_j\bar{F}_\varepsilon=\partial_j\bar{G}$ is Lipschitz continuous by assamption (A.II), $|E[I_1|\mathcal{F}^\varepsilon_s]|$ is dominated by

$$\begin{aligned}
&C_1E[\int_s^t|\phi^\varepsilon_r(x)-\phi^\varepsilon_r(y)|\,|\partial_j\phi^\varepsilon_r(x)|\,|\tilde{\psi}^\varepsilon_r|^{2p-1}dr|\mathcal{F}^\varepsilon_s]\\
&\leqq C_1E[\int_s^t\{|x-y|+|\phi^\varepsilon_r(x)-\phi^\varepsilon_r(y)-x+y|\}\{1+|\partial_j\phi^\varepsilon_r(x)-\partial_jx|\}|\tilde{\psi}^\varepsilon_r|^{2p-1}dr|\mathcal{F}^\varepsilon_s]\\
&\leqq C_1|x-y|E[\int_s^t|\psi^\varepsilon_r|^{2p-1}dr|\mathcal{F}^\varepsilon_s]\\
&\quad+\frac{C_1}{2p}|x-y|^{2p}E[\int_s^t|\partial_j(\phi^\varepsilon_r(x)-x)|^{2p}dr|\mathcal{F}^\varepsilon_s]\\
&\quad+\frac{C_1}{2p}E[\int_s^t|\phi^\varepsilon_r(x)-x-\phi^\varepsilon_r(y)+y|^{2p}dr|\mathcal{F}^\varepsilon_s]
\end{aligned}$$

$$+\frac{C_1}{2p}E[\int_s^t|\phi_r^\varepsilon(x)-x-\phi_r^\varepsilon(y)+y|^{2p}|\partial_j(\phi_r^\varepsilon(x)-x)|^{2p}dr|\mathcal{F}_s^\varepsilon]$$
$$+\frac{3(2p-1)}{2p}C_1E[\int_s^t|\psi_r^\varepsilon|^{2p}dr|\mathcal{F}_s^\varepsilon].$$

Apply Lemma 2.4 and (2.20). Then we see that $|E[I_1|\mathcal{F}_s^\varepsilon]|$ is dominated by

$$\text{(2.22)}\qquad C_2|x-y|E[\int_s^t|\psi_r^\varepsilon|^{2p-1}dr|\mathcal{F}_s^\varepsilon]$$
$$+C_3|x-y|^{2p}|t-s|^{p+1}+C_4E[\int_s^t|\psi_r^\varepsilon|^{2p}dr|\mathcal{F}_s^\varepsilon]$$

if $s, t\in[0, T]$ and $x, y\in R^d$. By a similar calculation, we can show that $|E[I_2|\mathcal{F}_s^\varepsilon]|$ is also dominated by (2.22). On the other hand, $E[I_3|\mathcal{F}_s^\varepsilon]$ is rewritten by

$$2pE[\int_s^t d\sigma\int_\sigma^t d\tau E[\sum_{l,k}\{\partial_l\partial_k\tilde{F}_\varepsilon(\tau, \phi_\sigma^\varepsilon(x))F_\varepsilon^l(\sigma, \phi_\sigma^\varepsilon(x))$$
$$-\partial_l\partial_k\tilde{F}_\varepsilon(\tau, \phi_\sigma^\varepsilon(y))F_\varepsilon^l(\sigma, \phi_\sigma^\varepsilon(y))\}|\mathcal{F}_\sigma^\varepsilon]\partial_j\phi_\sigma^{\varepsilon,k}(x)(\tilde{\psi}_\sigma^\varepsilon)^{2p-1}|\mathcal{F}_s^\varepsilon]$$
$$+2pE[\int_s^t d\sigma\int_\sigma^t d\tau\sum_k E[\partial_k\tilde{F}_\varepsilon^i(\tau, \phi_\sigma^\varepsilon(x))-\partial_k\tilde{F}_\varepsilon^i(\tau, \phi_\sigma^\varepsilon(y))|\mathcal{F}_\sigma^\varepsilon]$$
$$\times\{\sum_l\partial_l F_\varepsilon^k(\sigma, \phi_\sigma^\varepsilon(x))\partial_j\phi_\sigma^{\varepsilon,l}(x)(\tilde{\psi}_\sigma^\varepsilon)^{2p-1}+(2p-1)\partial_j\phi_\sigma^{\varepsilon,k}(x)(\tilde{\psi}_\sigma^\varepsilon)^{2p-2}$$
$$\times(\sum_m\partial_m F_\varepsilon^i(\sigma, \phi_\sigma^\varepsilon(x))\partial_j\phi_\sigma^{\varepsilon,m}(x)-\partial_m F_\varepsilon^i(\sigma, \phi_\sigma^\varepsilon(y))\partial_j\phi_\sigma^{\varepsilon,m}(y))\}|\mathcal{F}_s^\varepsilon].$$

Then, we can show as in the proof of Lemma 2.4 that $|E[I_3|\mathcal{F}_s^\varepsilon]|$ is dominated by

$$C_5E[\int_s^t d\sigma|\phi_\sigma^\varepsilon(x)-\phi_\sigma^\varepsilon(y)||\partial_j\phi_\sigma^\varepsilon(x)||\psi_\sigma^\varepsilon|^{2p-1}|\mathcal{F}_s^\varepsilon]$$
$$+C_6E[\int_s^t d\sigma|\phi_\sigma^\varepsilon(x)-\phi_\sigma^\varepsilon(y)|^2|\partial_j\phi_\sigma^\varepsilon(x)|^2|\psi_\sigma^\varepsilon|^{2p-2}|\mathcal{F}_s^\varepsilon]$$
$$+C_7E[\int_s^t d\sigma|\phi_\sigma^\varepsilon(x)-\phi_\sigma^\varepsilon(y)||\partial_j\phi_\sigma^\varepsilon(x)||\partial_j\phi_\sigma^\varepsilon(x)-\partial_j\phi_\sigma^\varepsilon(y)||\psi_\sigma^\varepsilon|^{2p-2}|\mathcal{F}_s^\varepsilon].$$

Like the case of $|E[I_1|\mathcal{F}_s^\varepsilon]|$, we can prove that the above is dominated by

$$C_8|x-y|E[\int_s^t|\psi_r^\varepsilon|^{2p-1}dr|\mathcal{F}_s^\varepsilon]$$
$$+C_9|x-y|^2E[\int_s^t|\psi_r^\varepsilon|^{2p-2}dr|\mathcal{F}_s^\varepsilon]$$
$$+C_{10}|x-y|^{2p}|t-s|^{p+1}$$
$$+C_{11}E[\int_s^t|\psi_r^\varepsilon|^{2p}dr|\mathcal{F}_s^\varepsilon].$$

Also, $|E[I_4|\mathcal{F}_s^\varepsilon]|$ is dominated by the above. Summing up all these estimations for $|E[I_i|\mathcal{F}_s^\varepsilon]|$, $i=1, \cdots, 4$, we arrive at

$$E[|\psi_t^\varepsilon|^{2p}]\leqq C_{12}|x-y|^{2p}|t-s|^{p+1}$$

$$+C_{13}|x-y|^2E[\int_s^t|\psi_r^\varepsilon|^{2p-2}dr|\mathcal{F}_s^\varepsilon]$$
$$+C_{14}|x-y|E[\int_s^t|\psi_r^\varepsilon|^{2p-1}dr|\mathcal{F}_s^\varepsilon]$$
$$+C_{15}E[\int_s^t|\psi_r^\varepsilon|^{2p}dr|\mathcal{F}_s^\varepsilon]\,.$$

By Gronwall's lemma,

$$E[|\psi_t^\varepsilon|^{2p}]\leqq C'_{12}|x-y|^{2p}|t-s|^{p+1}$$
$$+C'_{13}|x-y|^2E[\int_s^t|\psi_r^\varepsilon|^{2p-2}dr|\mathcal{F}_s^\varepsilon]$$
$$+C'_{14}|x-y|E[\int_s^t|\psi_r^\varepsilon|^{2p-1}dr|\mathcal{F}_s^\varepsilon]\,.$$

By iteration, this implies the estimate of (2.21).

Now, Proposition 2.5 can be proved using Lemma 2.6 just as in the proof of Proposition 2.2.

We now summarize the tightness of the family of laws of solutions $(\phi_{s,t}^\varepsilon, X_{s,t}^\varepsilon)$.

Theorem 2.7. *Assume* (A.I), (A.II) *and* (A.III). *Then the family of laws* $\{P^{(\varepsilon)}, \varepsilon>0\}$ *of* $(\phi_{s,t}^\varepsilon, X_{s,t}^\varepsilon)$ *defined on* $(W, \mathcal{B}_W)$ *is tight. Assume further* (A.IV). *Then the family of laws* $\{\hat{P}^{(\varepsilon)}, \varepsilon>0\}$ *defined on* $(W^m, \mathcal{B}_{W_m})$ *is tight.*

Proof. We will show that for any $\eta>0$ there is a compact subset M of W such that $P^{(\varepsilon)}(M)>1-\eta$ holds for any $\varepsilon>0$. Let N be a positive integer. Given a positive number δ, we define the modulus of continuity of $\phi_{s,t}(x)$, $s, t\in[0, T]$, $x\in[-N, N]^d$ by

$$w_\phi^N(\delta)=\sup_{|x-x'|+|s-s'|+|t-t'|\leqq\delta}|\phi_{s,t}(x)-\phi_{s',t'}(x')|\,.$$

Then, Kolmogorov's theorem tells us that for any $\eta>0$ and $\zeta>0$ there is a positive number $\delta=\delta(\eta, \zeta, N)$ independent of ε such that

$$P^{(\varepsilon)}\{\phi;\, w_\phi^N(\delta)>\zeta\}>\frac{\eta}{4}\,,\quad P^{(\varepsilon)}\{X;\, w_X^N(\delta)>\zeta\}>\frac{\eta}{4}$$

hold for any $\varepsilon>0$ in view of (2.7) and (2.2). See Billingsley [2], Theorem 12.3 and its proof. Further, there is a positive number $a=a(\eta)$ independent of ε such that

$$P^{(\varepsilon)}\{\phi;\, |\phi_{0,0}(0)|>a\}<\frac{\eta}{4}\,,\quad P^{(\varepsilon)}\{X;\, |X_{0,0}(0)|>a\}<\frac{\eta}{4}$$

hold for any $\varepsilon>0$ in view of (2.6) and (2.1). Set

$$A(\eta, \zeta, N)=\{(\phi, X)\in W;\, |\phi_{0,0}(0)|\leqq a,\, w_\phi^N(\delta)\leqq\zeta \text{ and } |X_{0,0}(0)|\leq a,\, |w_X^N(\delta)|\leqq\zeta\}\,.$$

Then we have $P^{(\varepsilon)}(A(\eta, \zeta, N)) \geqq 1-\eta$ for any $\varepsilon > 0$. Define now $A_{n,N} = A(\frac{\eta}{2^{n+1}}, \frac{1}{n}, N)$ and M=closure of $\bigcap_{N\geqq 1} \bigcap_{n>N} A_{n,N}$. Then it holds $P^{(\varepsilon)}(M) \geqq 1-\eta$ for any $\varepsilon > 0$. Further, the set M is compact, In fact, let (ϕ^n, X^n) be any sequence in M. Then by Ascoli–Arzela's theorem, there is a subsequence (ϕ^{n_i}, X^{n_i}) converging uniformly in $[0, T]^2 \times [-N, N]^d$ for any N. This means that $d(\phi^{n_i}, \phi^{n_j}) + d(X^{n_i}, X^{n_j})$ converges to 0 as $n_i, n_j \to \infty$. Therefore the set M is compact. The tightness of $\{P^{(\varepsilon)}, \varepsilon > 0\}$ is established.

We next consider the second assertion. Let k be a multi-index such that $|k| \leqq r$. Then, given $\eta > 0$ and $\zeta > 0$ there is a positive number $\delta_k = \delta_k(\eta, \zeta, N)$ such that

$$\hat{P}^{(\varepsilon)}\{\phi;\, w^N_{D^k\phi}(\delta_k) > \zeta\} \leqq \frac{\eta}{4}, \quad \hat{P}^{(\varepsilon)}\{X;\, w^N_{D^k X}(\delta_k) > \zeta\} \leqq \frac{\eta}{4}$$

in view of (2.19) and (2.17). Also, there is a positive number $a_k = a_k(\eta)$ such that

$$\hat{P}^{(\varepsilon)}\{\phi;\, |D^k\phi_{0,0}(0)| > a_k\} > \frac{\eta}{4}, \quad \hat{P}^{(\varepsilon)}\{X;\, |D^k X_{0,0}(0)| > a_k\} < \frac{\eta}{4}.$$

Set $a = \max_k a_k$, $\delta = \min_k \delta_k$ and

$$\begin{aligned} A(\eta, \zeta, N) = \{(\phi, X) \in W^r;\ & w^N_{D^k\phi}(\delta) \leqq \zeta,\ |D^k\phi_{0,0}(0)| \leqq \zeta \text{ and} \\ & w^N_{D^k X}(\delta) \leqq \zeta,\ |D^k X_{0,0}(0)| \leqq \zeta \ \text{ for any } k \text{ with } |k| \leqq r\}. \end{aligned}$$

Then we have $P(A(\eta, \zeta, N)) \geqq 1-2(r+1)^d\eta$. Set now $A_{n,N} = A(\frac{\eta}{2^{n+1}}, \frac{1}{n}, N)$ and M=closure of $\bigcap_{N\geqq 1} \bigcap_{n>N} A_{n,N}$. Then it holds $\hat{P}^{(\varepsilon)}(M) \geqq 1-4(r+1)^d\eta$. We can prove similarly as the above that M is a compact subset of W^r. Therefore $\{\hat{P}^{(\varepsilon)}, \varepsilon > 0\}$ is tight. The proof is complete.

3. Characterization of limiting measures by martingale problem

Let $P^{(\varepsilon)}$ be the law of the random field $(\phi^\varepsilon_{s,t}, X^\varepsilon_{s,t})$ defined on $(W, \mathcal{B}_W)$. We have seen in the previous section that the family of laws $\{P^{(\varepsilon)}, \varepsilon > 0\}$ on $(W, \mathcal{B}_W)$ is tight. Hence there is a sequence ε_k converging to 0 such that $\{P^{(\varepsilon_k)}, k = 1, 2, \cdots\}$ converges weakly to a law $P^{(0)}$ on $(W, \mathcal{F}_W)$. In this section, we shall prove that $P^{(0)}$ is a solution of a suitable martingale problem. At the next section, the result will be applied to proving the uniqueness of the limiting law $P^{(0)}$.

Let n and m be arbitrarily fixed nonnegative integers. We shall define an elliptic differential operator on $R^{nd} \times R^{md}$ with time parameter s and state parameters $y^0_1, \cdots, y^0_m \in R^d$ as follow:

$$\begin{aligned}(3.1)\qquad & L^{(n,m)}_{s,y_1^0,\cdots,y_m^0} f(x_1, \cdots, x_n, y_1, \cdots, y_m) \\ &= \frac{1}{2}\sum_{i,j}\sum_{k,l} a^{kl}(s, x_i, x_j)\frac{\partial^2 f}{\partial x_i^k \partial x_j^l} + \sum_{i,k}\{b^k(s, x_i)+c^k(s, x_i)\}\frac{\partial f}{\partial x_i^k} \\ &\quad + \sum_{i,j}\sum_{k,l} a^{kl}(s, x_i, y_j^0)\frac{\partial^2 f}{\partial x_i^k \partial y_j^l} \\ &\quad + \frac{1}{2}\sum_{i,j}\sum_{k,l} a^{kl}(s, y_i^0, y_j^0)\frac{\partial^2 f}{\partial y_i^k \partial y_j^l} + \sum_{i,k} b^k(s, y_i^0)\frac{\partial f}{\partial y_i^k},\end{aligned}$$

where $x_i=(x_i^1, \cdots, x_i^d)$, $y_i=(y_i^1, \cdots, y_i^d)$ are points in R^d.

Theorem 3.1. *For any C^∞-function $f(x_1, \cdots, x_n, y_1, \cdots, y_m)$ with compact support, the following is a martingale relative to $(\mathcal{B}_{t_0,t}$,[1] $P^{(0)})$ for any fixed t_0:*

$$\begin{aligned}(3.2)\qquad & f(\phi_{t_0,t}(x_1^0), \cdots, \phi_{t_0,t}(x_n^0), X_{t_0,t}(y_1^0), \cdots, X_{t_0,t}(y_m^0)) \\ &- \int_{t_0}^t L^{(n,m)}_{\tau,y_1^0,\cdots,y_m^0} f(\phi_{t_0,\tau}(x_1^0), \cdots, \phi_{t_0,\tau}(x_n^0), X_{t_0,\tau}(y_1^0), \cdots, X_{t_0,\tau}(y_m^0))d\tau\,.\end{aligned}$$

Before we proceed to the proof of the theorem, we will mention some consequences of the theorem. For simplicity, we write $\phi_{t_0,t}$, $X_{t_0,t}$ etc. as ϕ_t, X_t etc.

The operator $L^{(n,m)}_{\tau,y_1^0,\cdots,y_m^0}$ is degenerate, obviously. However, if coefficients $a(\tau, x, y)$, $b(\tau, x)$ and $c(\tau, x)$ are smooth with bounded derivatives, the martingale problem of the above proposition has a unique solution. (See Stroock-Varadhan [15]). This means that the law of $(n+m)$-point motion $(\phi_t(x_1^0), \cdots, \phi_t(x_n^0), X_t(y_1^0), \cdots, X_t(y_m^0))$ where t_0 and $x_1^0, \cdots, x_n^0, y_1^0, \cdots, y_m^0$ are fixed, is unique. Then the law of the random field $(\phi_t(x), X_t(x))$ is unique. As a consequence, we see that the law $P^{(\varepsilon)}$ of $(\phi_t^\varepsilon, X_t^\varepsilon)$ converges weakly to $P^{(0)}$ as $\varepsilon\to 0$. We will prove the uniqueness of the limiting law under assumptions (A.I)–(A.III) at the next section.

Suppose further that the function f of the theorem depends only on $x_1, \cdots, x_n$. Then

$$(3.3)\qquad L^{(n,m)}_{\tau,y_1^0,\cdots,y_m^0} f = \sum_{i=1}^n L_\tau^i f + \frac{1}{2}\sum_{i\neq j}\sum_{k,l} a^{kl}(\tau, x_i, x_j)\frac{\partial^2 f}{\partial x_i^k \partial x_j^l},$$

where

$$(3.4)\qquad L_\tau^i f = \frac{1}{2}\sum_{k,l} a^{kl}(\tau, x_i, x_i)\frac{\partial^2 f}{\partial x_i^k \partial x_i^l} + \sum_k \{b^k(\tau, x_i)+c^k(\tau, x_i)\}\frac{\partial f}{\partial x_i^k}\,.$$

Hence the n-point motion $(\phi_t(x_1^0), \cdots, \phi_t(x_n^0))$ is a diffusion process, and each component $\phi_t(x_i^0)$ is also a diffusion process with the generator L_τ^i. The operator $\frac{1}{2}\sum_{k,l} a^{kl}(\tau, x_i, x_j)\frac{\partial^2 f}{\partial x_i^k \partial x_j^l}$ indicates the interaction between $\phi_t(x_i^0)$ and

[1] The least σ-field of W for which $(\phi_{u,v}, X_{u,v})$, $t_0\leq u, v\leq t$ are measurable.

$\phi_t(x_j)$. Note that n-point motion is determined by the two point motion. See Baxendale [1].

Suppose next that f is a function of $y_1, \cdots, y_m$. Then

$$L^{(n,m)}_{\tau,y_1^0,\cdots,y_m^0}f=\sum_{i=1}^m L_{\tau,y_i^0}f+\frac{1}{2}\sum_{i\neq j}\sum_{k,l}a^{kl}(\tau, y_i^0, y_j^0)\frac{\partial^2 f}{\partial y_i^k\partial y_j^l}$$

where

$$L_{\tau,y_i^0}f=\frac{1}{2}\sum_{k,l}a^{kl}(\tau, y_i^0, y_i^0)\frac{\partial^2 f}{\partial y_i^k\partial y_i^l}+\sum_k b^k(\tau, y_i^0)\frac{\partial f}{\partial y_i^k}, \tag{3.5}$$

which is a second order operator with constant coefficients, depending on the parameters $y_1^0, \cdots, y_m^0$. Then the corresponding m point motion $(X_t(y_1^0), \cdots, X_t(y_m^0))$ is a Brownian motion, or continuous Gaussian process with independent increments.

The remaining part of the operator $L^{(n,m)}_{\tau,y_1^0,\cdots,y_m^0}$ is the cross term:

$$\sum_{i,j}\sum_{k,l}a^{kl}(\tau, x_i, y_j^0)\frac{\partial^2 f}{\partial x_i^k\partial y_j^l},$$

which control the interaction between ϕ_t and X_t. The interaction is described by the stochastic differential equation of Theorem 1. Thus ϕ_t is a functional of X_s, $t_0\leqq s\leqq t$.

We shall prove Theorem 3.1 in case $n=m=1$ only. The following argument is close to Kesten–Papanicolaou [7]. It is enough to prove

$$\begin{aligned} &E^{(0)}[\{f(\phi_t(x^0), X_t(y^0))-f(\phi_s(x^0), X_s(y^0))\}\Phi] \\ &= E^{(0)}[\{\int_s^t L^{(1,1)}_{\tau,y^0}f(\phi_\tau(x^0), X_\tau(y^0))d\tau\}\Phi] \end{aligned} \tag{3.6}$$

where Φ is a bounded continuous $\mathcal{B}_{t_0,s}$-adapted function of the form

$$\Phi=\Phi(\phi_{s_1}(x_1), \cdots, \phi_{s_k}(x_k), X_{s_1}(y_1), \cdots, X_{s_k}(y_k))$$

where $t_0\leqq s_i\leqq s$.

We shall evaluate the quantity for $(\phi_t^\varepsilon, X_t^\varepsilon)$ corresponding to (3.6). It holds

$$\begin{aligned} &f(\phi_t^\varepsilon, X_t^\varepsilon)-f(\phi_s^\varepsilon, X_s^\varepsilon) \\ &=\sum_i\int_s^t\frac{\partial f}{\partial x^i}(\phi_\tau^\varepsilon, X_\tau^\varepsilon)F_\varepsilon^i(\tau, \phi_\tau^\varepsilon)d\tau+\sum_i\int_s^t\frac{\partial f}{\partial y^i}(\phi_\tau^\varepsilon, X_\tau^\varepsilon)F_\varepsilon^i(\tau, y_0)d\tau\,. \end{aligned} \tag{3.7}$$

The first member of the right hand side is the sum of the following for $i=1, \cdots, d$.

$$\{\int_s^t\frac{\partial f}{\partial x^i}(\phi_\tau^\varepsilon, X_\tau^\varepsilon)\bar{F}_\varepsilon^i(\tau, \phi_\tau^\varepsilon)d\tau+\int_s^t\frac{\partial f}{\partial x^i}(\varphi_s^\varepsilon, X_s^\varepsilon)\tilde{F}_\varepsilon^i(\tau, \varphi_s^\varepsilon)d\tau\}$$

$$+\int_s^t d\tau \int_s^\tau d\sigma \{\sum_j \frac{\partial^2 f}{\partial x^j \partial x^i}(\phi_\sigma^\varepsilon, X_\sigma^\varepsilon) F_\varepsilon^j(\sigma, \phi_\sigma^\varepsilon) \tilde{F}_\varepsilon^i(\tau, \phi_\sigma^\varepsilon)\}$$

$$+\int_s^t d\tau \int_s^\tau d\sigma \{\sum_j \frac{\partial^2 f}{\partial y^j \partial x^i}(\phi_\sigma^\varepsilon, X_\sigma^\varepsilon) F_\varepsilon^j(\sigma, y_0) \tilde{F}_\varepsilon^i(\tau, \phi_\sigma^\varepsilon)\}$$

$$+\int_s^t d\tau \int_t^\tau d\sigma \{\frac{\partial f}{\partial x^i}(\phi_\sigma^\varepsilon, X_\sigma^\varepsilon) H_\varepsilon^i(\tau, \sigma, \phi_\sigma^\varepsilon)\}$$

$$= I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon + I_4^\varepsilon,$$

where $H_\varepsilon^i(\tau, \sigma, x) = \sum_j \{\partial_j \tilde{F}_\varepsilon^i(\tau, x)\} F_\varepsilon^j(\sigma, x)$. Set $\Phi^\varepsilon = \Phi(\phi_{s_1}^\varepsilon(x_1), \cdots, \phi_{s_k}^\varepsilon(x_k), X_{s_1}^\varepsilon(y_1), \cdots X_{s_k}^\varepsilon(y_k))$. We want to prove

$$\lim_{\varepsilon\to 0} E[I_1^\varepsilon \phi^\varepsilon] = E^{(0)}[\{\int_s^t \frac{\partial f}{\partial x^i}(\phi_\tau, X_\tau) b^i(\tau, \phi_\tau) d\tau\} \Phi], \tag{3.8}$$

$$\lim_{\varepsilon\to 0} E[I_2^\varepsilon \Phi^\varepsilon] = \sum_j E^{(0)}[\{\int_s^t \frac{\partial^2 f}{\partial x^j \partial x^i}(\phi_\tau, X_\tau) A^{ji}(\tau, \phi_\tau, \phi_\tau) d\tau\} \Phi], \tag{3.9}$$

$$\lim_{\varepsilon\to 0} E[I_3^\varepsilon \Phi^\varepsilon] = \sum_j E^{(0)}[\{\int_s^t \frac{\partial^2 f}{\partial y^j \partial x^i}(\phi_\tau, X_\tau) A^{ji}(\tau, y_0, \phi_\tau) d\tau\} \Phi], \tag{3.10}$$

$$\lim_{\varepsilon\to 0} E[I_4^\varepsilon \Phi^\varepsilon] = E^{(0)}[\{\int_s^t \frac{\partial f}{\partial x^i}(\phi_\tau, X_\tau) c^i(\tau, \phi_\tau) d\tau\} \Phi]. \tag{3.11}$$

Once these four formulas are proved, then we have

$$\lim_{\varepsilon\to 0} \sum_i E[\{\int_s^t \frac{\partial f}{\partial x^i}(\phi_\tau^\varepsilon, X_\tau^\varepsilon) F_\varepsilon^i(\tau, \phi_\tau^\varepsilon) d\tau\} \Phi^\varepsilon] \tag{3.12}$$
$$= E^{(0)}[\{\int_s^t L_\tau f(\phi_\tau, X_\tau) d\tau\} \Phi] + E^{(0)}[\{\sum_{j,i} \int_s^t \frac{\partial^2 f}{\partial y^j \partial x^i}(\phi_\tau, X_\tau) A^{ji}(\tau, y_0, \phi_\tau) d\tau\} \Phi],$$

where L_τ is the operator of (3.4). By the similar argument, the second term of (3.7) can be calculated as

$$\lim_{\varepsilon\to 0} \sum_i E[\{\int_s^t \frac{\partial f}{\partial y^i}(\phi_\tau^\varepsilon, X_\tau^\varepsilon) F_\varepsilon^i(\tau, y_0) d\tau\} \Phi] \tag{3.13}$$
$$= E^{(0)}[\{\int_s^t L_{\tau, y_0} f(\phi_\tau, X_\tau) d\tau\} \Phi]$$
$$+ \sum_{j,i} E^{(0)}[\{\int_s^t \frac{\partial^2 f}{\partial x^j \partial y^i}(\phi_\tau, X_\tau) A^{ji}(\tau, \phi_\tau, y_0) d\tau\} \Phi].$$

Then (3.12) and (3.13) imply (3.2) and the assertion of the theorem follows in case $n=m=1$.

In the following, we prove (3.8) and (3.9). Proofs of (3.10) and (3.11) can be done similarly and are omitted.

Proof of (3.8). Let $\delta = \{s = s_0 < s_1 < \cdots\}$ be a partition such that $s_{k+1} - s_k$

$=\varepsilon$. Let $\delta(t)$ be the function such that $\delta(t)=s_k$ if $s_k\leqq t<s_{k+1}$. Then we have by assumption (A.III)

$$|\int_s^t F_\varepsilon^i(\tau, \phi_{\delta(\tau)}^\varepsilon)\frac{\partial f}{\partial x^i}(\phi_{\delta(\tau)}^\varepsilon, X_{\delta(\tau)}^\varepsilon)d\tau - \int_s^t b^i(\delta(\tau), \phi_{\delta(\tau)}^\varepsilon)\frac{\partial f}{\partial x^i}(\phi_{\delta(\tau)}^\varepsilon, X_{\delta(\tau)}^\varepsilon)d\tau| \leqq C\varepsilon\int_s^t(1+|\phi_{\delta(\tau)}^\varepsilon|)d\tau .$$

We have also

$$E[|\int_s^t\{F_\varepsilon(\tau, \phi_{\delta(\tau)}^\varepsilon)\frac{\partial f}{\partial x^i}(\phi_{\delta(\tau)}^\varepsilon, X_{\delta(\tau)}^\varepsilon)-F_\varepsilon(\tau, \phi_\tau^\varepsilon)\frac{\partial f}{\partial x^i}(\phi_\tau^\varepsilon, X_\tau^\varepsilon)\}d\tau|^{2p}]$$
$$\leqq \text{const } E[\int_s^t\{|\phi_{\delta(\tau)}^\varepsilon-\phi_\tau^\varepsilon|^{2p}+|X_{\delta(\tau)}^\varepsilon-X_\tau^\varepsilon|^{2p}\}d\tau] .$$

The above is $O(\varepsilon^p)$ because of Proposition 2.1 and Lemma 2.3. On the other hand, we have

$$\int_s^t b^i(\delta(\tau), \phi_{\delta(\tau)}^\varepsilon)\frac{\partial f}{\partial x^i}(\phi_{\delta(\tau)}^\varepsilon, X_{\delta(\tau)}^\varepsilon)d\tau \xrightarrow[\varepsilon\to 0]{} \int_s^t b^i(\tau, \phi_\tau)\frac{\partial f}{\partial x^i}(\phi_\tau, X_\tau)d\tau$$

in the weak convergence. Next, the property

$$E[\{\int_s^t\frac{\partial f}{\partial x^i}(\varphi_s^\varepsilon, X_s^\varepsilon)\tilde{F}_\varepsilon(\tau, \varphi_s^\varepsilon)d\tau\}\Phi^\varepsilon] \to 0, \ (\varepsilon \to 0)$$

is easily verified. See Kesten-Papanicolaou [7], p. 115 Hence we have (3.8).

Proof of (3.9). Set $K_\varepsilon^{ji}(\sigma, \tau, x)=F_\varepsilon^j(\sigma, x)\tilde{F}_\varepsilon^i(\tau, x)$. Then I_2^ε is written as

$$(3.14)\quad \sum_j\int_s^t d\tau\int_s^\tau d\sigma\frac{\partial^2 f}{\partial x^i\partial x^j}(\phi_\sigma^\varepsilon, X_\sigma^\varepsilon)\bar{K}_\varepsilon^{ji}(\sigma, \tau, \phi_\sigma^\varepsilon) + \sum_j\int_s^t d\tau\int_s^\tau d\sigma\frac{\partial^2 f}{\partial x^i\partial x^j}(\phi_\sigma^\varepsilon, X_\sigma^\varepsilon)\tilde{K}_\varepsilon^{ji}(\sigma, \tau, \phi_\sigma^\varepsilon) .$$

where $\bar{K}_\varepsilon^{ji}=E[K_\varepsilon^{ji}]$ and $\tilde{K}_\varepsilon^{ji}=K_\varepsilon^{ji}-\bar{K}_\varepsilon^{ji}$. By assumption (A.III), we have

$$|\int_{s_k}^{s_{k+1}}\int_{s_k}^\tau \bar{K}_\varepsilon^{ji}(\sigma, \tau, x)d\sigma d\tau-(s_{k+1}-s_k)A^{ji}(s_k, x, x)| = O(\varepsilon^2) .$$

We have further

$$|\int_{s_k}^{s_{k+1}}\int_s^{s_k}\bar{K}_\varepsilon^{ji}(\sigma, \tau, x)d\sigma d\tau| \leqq \varepsilon^2C^2(1+|x|)^2\int_{s_k/\varepsilon^2}^{s_{k+1}/\varepsilon^2}d\tau(\int_{s/\varepsilon^2}^{s_k/\varepsilon^2}\rho(\tau-\sigma)d\sigma) = O(\varepsilon^2) .$$

(See Papanicolaou–Varadhan [13], p. 504). Therefore we have

$$(3.15)\qquad \int_s^t d\tau\int_s^\tau K_\varepsilon^{ji}(\sigma,\tau,\phi^\varepsilon_{\delta(\sigma)})\frac{\partial^2 f}{\partial x^i\partial x^j}(\phi^\varepsilon_{\delta(\sigma)},X^\varepsilon_{\delta(\sigma)})d\sigma$$
$$=\int_s^t A^{ji}(\delta(\sigma),\phi^\varepsilon_{\delta(\sigma)},\phi^\varepsilon_{\delta(\sigma)})\frac{\partial^2 f}{\partial x^i\partial x^j}(\phi^\varepsilon_{\delta(\sigma)},X^\varepsilon_{\delta(\sigma)})d\sigma+o_\varepsilon(1)\,.$$

Also, the L^{2p}-metric between the above and

$$\int_s^t d\tau\int_s^\tau d\sigma K_\varepsilon^{ji}(\sigma,\tau,\phi^\varepsilon_\sigma)\frac{\partial^2 f}{\partial x^i\partial x^j}(\phi^\varepsilon_\sigma,X^\varepsilon_\sigma)$$

is estimated as $O(\varepsilon^p)$ as before. Since the last member of (3.15) converges to

$$\int_s^t A^{ji}(\sigma,\phi_\sigma,\phi_\sigma)\frac{\partial^2 f}{\partial x^i\partial x^j}(\phi_\sigma,X_\sigma)d\sigma\,,$$

we see that

$$\lim_{\varepsilon\to 0}E[(\int_s^t d\tau\int_s^\tau d\sigma\frac{\partial^2 f}{\partial x^i\partial x^j}(\phi^\varepsilon_\sigma,X^\varepsilon_\sigma)K_\varepsilon^{ji}(\sigma,\tau,\phi^\varepsilon_\sigma))\Phi^\varepsilon]$$
$$=E^{(0)}[(\int_s^t\frac{\partial^2 f}{\partial x^i\partial x^j}(\phi_\sigma,X_\sigma)A^{ji}(\sigma,\phi_\sigma,\phi_\sigma)d\sigma)\Phi]\,.$$

We can prove similarly as Kesten–Papanicolaou [7], p. 116–117 that the second term of (3.14) converges weakly to 0 as $\varepsilon\to 0$. There, we apply the following lemma instead of Lemma 2 in [7].

Lemma 3.2 (c.f. H. Watanabe [17]). *Let $X(s,\varepsilon)$ be $\mathcal{F}^\varepsilon_{0,s}$-measurable and let $U(t,x,\varepsilon)$ (resp. $V(u,x,\varepsilon)$) $\mathcal{F}^\varepsilon_{t,t}$ (resp. $\mathcal{F}^\varepsilon_{u,u}$)-measurable such that $E[V(u,x,\varepsilon)]=0$ and*

$$|X(s,\varepsilon)|\leqq c_1\,,\quad |U(t,x,\varepsilon)|\leqq c_2\,,\quad |V(u,x,\varepsilon)|\leqq c_3\,.$$

Set $W(t,u,x,\varepsilon)=E[U(t,x,\varepsilon)V(u,x,\varepsilon)]$. Then for $s\leqq t\leqq u$, we have

$$|E[X(s,\varepsilon)\{U(t,\phi^\varepsilon_s,\varepsilon)V(u,\phi^\varepsilon_s,\varepsilon)-W(t,u,\phi^\varepsilon_s,\varepsilon)\}]|$$
$$\leqq 8c_1c_2c_3\rho\Big(\frac{t-s}{\varepsilon^2}\Big)^{1/2}\rho\Big(\frac{u-t}{\varepsilon^2}\Big)^{1/2}\,.$$

It is convenient to extend Theorem 3.1 to a broader class of functions. For this purpose, we require a proposition.

Proposition 3.3. *$A^{ij}(\tau,x,y)$, $b^j(\tau,x)$, $c^j(\tau,x)$ are uniformly Lipschitz continuous and of linear growth in the following sense. There is a positive constant L such that*

$$|A^{ij}(\tau,x,y)|\leqq L(1+|x|)(1+|y|)\,,$$
$$|b^i(\tau,x)|+|c^i(\tau,x)|\leqq L(1+|x|)$$

hold for any τ, x, y.

Proof. The uniform Lipschitz continuity of $b^i(\tau, x)$ is obvious from the same property of $G(\tau, x, \frac{t}{\varepsilon^2})$ and (1.7). We shall consider $A^{ij}(\tau, x, y)$. Set

$$A_\varepsilon^{ij}(\tau, x, y) = \frac{1}{\varepsilon^3}\int_\tau^{\tau+\varepsilon}\int_\tau^\sigma E[F^i(s, x, \frac{s}{\varepsilon^2})F^j(\sigma, y, \frac{\sigma}{\varepsilon^2})]dsd\sigma .$$

Then

$$\begin{aligned}
&|A_\varepsilon^{ij}(\tau, x, x)-A_\varepsilon^{ij}(\tau, x, y)-A_\varepsilon^{ij}(\tau, y, x)+A_\varepsilon^{ij}(\tau, y, y)| \\
&= \frac{1}{\varepsilon^3}|\int_\tau^{\tau+\varepsilon}\int_\tau^\sigma E[(F^i(s, x, \frac{s}{\varepsilon^2})-F^i(s, y, \frac{s}{\varepsilon^2})) \\
&\qquad \times(F^j(\sigma, x, \frac{\sigma}{\varepsilon^2})-F^j(\sigma, y, \frac{\sigma}{\varepsilon^2}))]dsd\sigma| \\
&\leqq \frac{1}{\varepsilon^3}|\int_\tau^{\tau+\varepsilon}\int_\tau^\sigma\int_0^1\int_0^1\sum_{k,l} E[\partial_k F^i(s, y+v(x-y), \frac{s}{\varepsilon^2})\,(x^k-y^k) \\
&\qquad \times(\partial_l F^j(\sigma, y+u(x-y), \frac{\sigma}{\varepsilon^2})\,(x^l-y^l)]dudvdsd\sigma| \\
&\leqq C^2d^2|x-y|^2\frac{1}{\varepsilon^3}\int_\tau^{\tau+\varepsilon}\int_\tau^\sigma \rho(\frac{\sigma-s}{\varepsilon^2})dsd\sigma \\
&\leqq C^2d^2(\int_0^\infty \rho(s)ds)|x-y|^2 .
\end{aligned}$$

Now let ε tend to 0. Then we see that A^{ij} is uniformly Lipschitz continuous. The proof for $c^i(\tau, x)$ is similar. The linear growth property is clear from the uniformly Lipschitz continuity and the boundedness of $A^{ij}(\tau, 0, 0)$ etc.

Corollary to Theorem 3.1. *Let $f(x_1, \cdots, x_n, y_1, \cdots, y_m)$ be a C^∞-function such that f together with their derivatives up to the second order are polynomial growth*:

$$|D^k f|\leqq C_k(1+|x_1|+\cdots+|x_n|)^{p_k}(1+|y_1|+\cdots+|y_m|)^{p_k}, \quad |k|\leqq 2$$

etc. hold for some C_k, p_k. Then (3.2) *is a* $(\mathcal{B}_{t_0,t}, P^{(0)})$ *martingale.*

Proof. It holds from Proposition 2.1 and 2.2,

$$E^{(0)}[|X_{t_0,t}(y)|^{2p}]\leqq K_p(1+|y|)^{2p}, E^{(0)}[|\phi_{t_0,t}(x)|^{2p}]\leqq K_p(1+|x|)^{2p} .$$

Therefore $f(\phi_{t_0,t}(x_1^0), \cdots, \phi_{t_0,t}(x_n^0), X_{t_0,t}(y_1^0), \cdots, X_{t_0,t}(y_m^0))$ and $\int_{t_0}^t L_{\tau,y_1^0,\cdots,y_m^0}f(\phi_{t_0,\tau}(x_1^0)\cdots)d\tau$ are square integrable. Then we see easily that (3.2) is a square integrable martingale, approximating f by a sequence of functions with compact supports.

4. Proof of Theorems

This section is devoted to the proofs of Theorems 1–3.

Proof of Theorem 1. We fix any weak limit $P^{(0)}$ of $\{P^{(\varepsilon)}, \varepsilon>0\}$. Apply Corollary to Theorem 3.1 to the functions $f(y)=y^i$ and $f(y_1, y_2)=y_1^i y_2^j$, where $y=(y^1, \cdots, y^d)$ and $y_i=(y_i^1, \cdots, y_i^d)$ $(i=1, 2)$. Then we see that for any s, both of

$$Y_{s,t}^i(y) \equiv X_{s,t}^i(y)-\int_s^t b^i(\tau, y)d\tau\,, \tag{4.1}$$

$$X_{s,t}^i(y_1)X_{s,t}^j(y_2)-\int_s^t b^i(\tau, y_1)X_{s,\tau}^j(y_2)d\tau-\int_s^t b^j(\tau, y_2)X_{s,\tau}^i(y_1)d\tau -\int_s^t a^{ij}(\tau, y_1, y_2)d\tau \tag{4.2}$$

are continuous $(\mathcal{F}_{s,t}, P^{(0)})$-martingales. By Itô's formula, it holds

$$X_{s,t}^i(y_1)X_{s,t}^j(y_2) = \int_s^t X_{s,\tau}^i(y_1)dX_\tau^j(y_2)+\int_s^t X_{s,\tau}^j(y_2)dX_\tau^i(y_1) +\langle X_{s,t}^i(y_1), X_{s,t}^j(y_2)\rangle\,,$$

where the last term is the joint quadratic variation of the process $X_{s,t}^i(y_1)$ and $X_{s,t}^j(y_2)$. Therefore, we see that (4.2) is written as

$$\int_s^t X_{s,\tau}^i(y_1)dY_\tau^j(y_2)+\int_s^t X_{s,\tau}^j(y_2)dY_\tau^i(y_1)+\langle X_{s,t}^i(y_1), X_{s,t}^j(y_2)\rangle -\int_s^t a^{ij}(\tau, y_1, y_2)d\tau\,.$$

The first and the second term of the above are martingales. Thus the remaining part is 0 since it is a martingale of bounded variation. This proves that

$$\langle Y_{s,t}^i(y_1), Y_{s,t}^j(y_2)\rangle = \langle X_{s,t}^i(y_1), X_{s,t}^j(y_2)\rangle = \int_s^t a^{ij}(\tau, y_1, y_2)d\tau\,. \tag{4.3}$$

Since the right hand side of the above does not depend on ω, we can conclude that $(Y_{s,t}^i(y_1), Y_{s,t}^j(y_2))$ is a Brownian motion. (See Kunita–Watanabe [10]). By the same argument, linear sums of $Y_{s,t}^i(y_k)$, $i=1, \cdots, d$, $k=1, \cdots, n$ are also Brownian motions. We have thus proved that $X_{s,t}(y)$ is a Gaussian random field with independent increments. The mean of $X_{s,t}(y)$ is $\int_s^t b(\tau, y)d\tau$ because (4.1) is a martingale with zero-mean. The covariance of $Y_{s,t}^i(y_1)$ and $Y_{s,t}^j(y_2)$ is $\int_s^t a^{ij}(\tau, y_1, y_2)d\tau$ because of (4.3).

We next consider $\phi_{s,t}(x)$. By the mixing property (A.I), it is obvious that $\phi_{s,t}$ has independent increments. Now apply Corollary to Theorem 3.1 to $f(x)=x^i$ and $f(x)=x_1^i x_2^j$. Then we see that both of

$$M_{s,t}^i(x) \equiv \phi_{s,t}^i(x)-x^i-\int_s^t (b^i+c^i)\,(\tau, \phi_{s,\tau}(x))d\tau\,, \tag{4.5}$$

$$\phi_{s,t}^i(x_1)\phi_{s,t}^j(x_2)-\int_s^t (b^i+c^i)\,(\tau, \phi_{s,\tau}(x_1))\phi_{s,\tau}^j(x_2)d\tau \tag{4.6}$$

$$-\int_s^t (b^j+c^j)(\tau, \phi_{s,\tau}(x_2))\phi^i_{s,\tau}(x_1)d\tau$$
$$-\int_s^t a^{ij}(\tau, \phi_{s,\tau}(x_1), \phi_{s,\tau}(x_2))d\tau$$

are martingales. Then by the argument similar to the preceding paragraph, we find that the joint quadratic variation is given by

$$(4.7)\qquad \langle M^i_{s,t}(x_1), M^j_{s,t}(x_2)\rangle = \int_s^t a^{ij}(\tau, \phi_{s,\tau}(x_1), \phi_{s,\tau}(x_2))d\tau\,.$$

Now the property (1.18) follows immediately from the fact that $M^i_{s,t}(x)$ of (4.5) is a martingale with zero-mean. Also, (4.7) implies

$$(4.8)\qquad \lim_{h\to 0+} \frac{1}{h} E^{(0)}[(\phi^i_{s,s+h}(x)-x^i-\int_s^{s+h}(b^i+c^i)(\tau, \phi_{s,\tau}(x))d\tau)$$
$$\times(\phi^j_{s,s+h}(y)-y^j-\int_s^t(b^j+c^j)(\tau, \phi_{s,\tau}(y))d\tau)]$$
$$= a^{ij}(\tau, x, y)\,.$$

Using the estimate $E^{(0)}[|\phi_{s,t}(x)-x|^{2p}]\leqq C_p(1+|x|)^{2p}|t-s|^p$, which follows from Lemma 2.3, it is immediate to see that the above coincides with

$$(4.9)\qquad \lim_{h\to 0} \frac{1}{h} E^{(0)}[(\phi^i_{s,s+h}(x)-x^i)(\phi^j_{s,s+h}(y)-y^j)]\,.$$

Hence property (1.19) is proved.

For the proof of (iii) in Theorem 1, apply Corollary to Theorem 3.1 to $f(x, y)=x^iy^j$. Then we see that

$$\phi^i_{s,t}(x)X^j_{s,t}(y)-\int_s^t(b^i+c^i)(\tau, \phi_{s,\tau}(x))X^j_{s,\tau}(y)d\tau$$
$$-\int_s^t b^j(\tau, y)\phi^i_{s,\tau}(x)d\tau-\int_s^t a^{ij}(\tau, \phi_{s,\tau}(x), y))d\tau$$

is a martingale. Then we get as above that

$$(4.10)\qquad \langle M^i_{s,t}(x), Y^j_{s,t}(y)\rangle = \int_s^t a^{ij}(\tau, \phi_{s,\tau}(x), y)d\tau\,.$$

Define now $\tilde{M}^i_t(x)=\int_s^t dY^i_\tau(\phi_{s,\tau}(x))$. It is a martingale. From the relation (1.22), we have

$$(4.11)\qquad \langle \tilde{M}^i_{s,t}(x), \tilde{M}^j_{s,t}(x)\rangle = \int_s^t a^{ij}(\tau, \phi_{s,\tau}(x), \phi_{s,\tau}(x))d\tau\,.$$

On the other hand, we have from (4.10),

$$(4.12)\qquad \langle M^i_{s,t}(x), \tilde{M}^j_{s,t}(x)\rangle = \int_s^t a^{ij}(\tau, \phi_{s,\tau}(x), \phi_{s,\tau}(x))d\tau\,.$$

See [10]. Consequently, by (4.7), (4.11) and (4.12)

$$\langle M^i_{s,t}(x)-\tilde{M}^i_{s,t}(x)\rangle = \langle M^i_{s,t}(x), M^i_{s,t}(x)\rangle - 2\langle M^i_{s,t}(x), \tilde{M}^i_{s,t}(x)\rangle + \langle \tilde{M}^i_{s,t}(x), \tilde{M}^i_{s,t}(x)\rangle = 0\,.$$

This proves that $M^i_{s,t}(x)=\tilde{M}^i_{s,t}(x)$ for any $s<t$ and x. Then the formula (1.20) follows immediately.

Finally we will prove the uniqueness of the limiting law $P^{(0)}$. Consider *SDE* (1.20). Let $\tilde{\phi}_{s,t}$ be any solution of the following equation

$$\tilde{\phi}_{s,t}(x) = x+\int_s^t dX_\tau(\tilde{\phi}_{s,\tau})d\tau+\int_s^t c(\tau, \tilde{\phi}_{s,\tau}(x))d\tau\,.$$

Then, since a^{ij}, b^i and c^i are Lipschitz continuous, we can prove that it has a unique pathwise solution i.e. $\phi_{s,t}(x)=\tilde{\phi}_{s,t}(x)$ a.s. for any x by the standard argument of Itô's *SDE* (La Jan [11]). Now let $P^{(0)}_1$ be another limiting law. Then $(\phi_{s,t}, X_{s,t}, P^{(0)}_1)$ also satisfies (i)–(iii) of the theorem. Therefore the laws of $(X_{s,t}, P^{(0)}_1)$ and $(X_{s,t}, P^{(0)})$ coincide each other, since both are Gaussian random fields with the same means and covariances. Then the pathwise uniqueness of solutions implies the uniqueness of the law, i.e., $(\phi_{s,t}, X_{s,t}, P^{(0)}) =(\phi_{s,t}, X_{s,t}, P^{(0)}_1)$ (c.f. Yamada–Watanabe [19]). The proof of the theorem is now complete.

Proof of Theorem 2. For each $p\geqq 2$, there is a positive constant K_p such that $E[|B^{k,\varepsilon}_t-B^{k,\varepsilon}_{t'}|^{2p}]\leqq K_p|t-t'|^p$ holds for any $\varepsilon>0$ and $k=1, \cdots, n$. Then we see that the family of laws $\tilde{P}^{(\varepsilon)}$, $\varepsilon>0$ is tight as in the proof of Theorem 2.7. Let $\tilde{P}^{(0)}$ be a limiting measure. Then $(\phi_{s,t}, X_{s,t}, \tilde{P}^{(0)})$ has the same property as Theorem 1. On the other hand, $(B^1_t, \cdots, B^n_t, \tilde{P}^{(0)})$ is a Brownian motion with zero-mean and covariance $(r_{kl})t$ by the central limit theorem. (See Ibragimov–Linnik [4]).

We shall prove that $X_{s,t}(x)$ is represented by (1.29). Similarly as the proof of Theorem 1, we can prove that both of

$$Y_{s,t}(x)\equiv X_{s,t}(x)-\int_s^t \tilde{G}(\tau, x)d\tau\,,$$

$$X_{s,t}(x)(B^k_t-B^k_s)-\int_s^t \tilde{G}(\tau, x)(B^k_\tau-B^k_s)d\tau-\sum_l \bar{r}_{kl}\int_s^t \tilde{F}_l(\tau, x)d\tau$$

are martingales where $\bar{r}_{kl}=r_{kl}+r_{lk}$. Then we see as the proof of Theorem 1,

$$\langle Y_{s,t}(x), B^k_t-B^k_s\rangle = \sum_l \bar{r}_{kl}\int_s^t \tilde{F}_l(\tau, x)d\tau\,. \tag{4.13}$$

Define now

$$\tilde{Y}_{s,t}(x)\equiv \sum_k \int_s^t \tilde{F}_k(\tau, x)dB^k_\tau\,. \tag{4.14}$$

From (4.13), we get

$$(4.15)\qquad \langle Y^i_{s,t}(x),\ \tilde{Y}^j_{s,t}(x)\rangle = \sum_{k,l}\left(\int_s^t \tilde{F}^i_k(\tau, x)\tilde{F}^j_l(\tau, x)d\tau\right)\tilde{r}_{kl}\,.$$

We have also from (4.14)

$$(4.16)\qquad \langle \tilde{Y}^i_{s,t}(x),\ \tilde{Y}^j_{s,t}(x)\rangle = \sum_{k,l}\tilde{r}_{kl}\int_s^t \tilde{F}^i_k(\tau, x)\tilde{F}^j_l(\tau, x)d\tau\,.$$

On the other hand, we have from (4.3) and (1.25)

$$(4.17)\qquad \langle Y^i_{s,t}(x),\ Y^j_{s,t}(x)\rangle = \int_s^t a^{ij}(\tau, x, x)d\tau = \sum_{k,l}\tilde{r}_{kl}\int_s^t \tilde{F}^i_k(\tau, x)\tilde{F}^j_l(\tau, x)d\tau\,.$$

Then (4.15), (4.16) and (4.17) imply $\langle Y^i-\tilde{Y}^i\rangle\equiv 0$, proving $Y_{s,t}=\tilde{Y}_{s,t}$ and (1.29).

Now Itô *SDE* (1.20) is written as

$$\phi_{s,t}(x) = x+\sum_{k=1}^n\int_s^t \tilde{F}_k(\tau, \phi_{s,\tau}(x))dB^k_\tau+\int_s^t \tilde{G}(\tau, \phi_{s,\tau}(x))d\tau + \int_s^t c(\tau, \phi_{s,\tau}(x))d\tau\,,$$

where $c(\tau, x)$ is given by (1.27). On the other hand, Stratonovich integral and Itô integral are related by

$$\int_s^t \tilde{F}_k(\tau, \phi_{s,\tau}(x))\circ dB^k_\tau = \int_s^t \tilde{F}_k(\tau, \phi_{s,\tau}(x))dB^k_\tau + \frac{1}{2}\sum_{l,i}\tilde{r}_{kl}\int_s^t \frac{\partial}{\partial x^i}\tilde{F}_k(\tau, \phi_{s,\tau}(x))\tilde{F}^i_l(\tau, \phi_{s,\tau}(x))d\tau\,.$$

It holds

$$c(\tau, x)-\frac{1}{2}\sum_{k,l,i}\tilde{r}_{kl}\frac{\partial}{\partial x^i}\tilde{F}_k(\tau, x)F^i_l(\tau, x) = \frac{1}{2}\sum_{1\le k\le l\le n}(r_{kl}-r_{lk})\,[\tilde{F}_k, \tilde{F}_l]\,.$$

Therefore we get the expression (1.30). The proof is complete.

Proof of Theorem 3. The family of measures $\{\hat{P}^{(\varepsilon)}, \varepsilon>0\}$ on W^r is tight by Theorem 2.7. Let $\hat{P}^{(0)}$ by any weak limit. Obviously it coincides with the limiting measure of Theorem 1. Therefore the assertion of the theorem follows.

References

[1] P. Baxendale: *Brownian motions in the diffeomorphism group* I, Univ. Aberdeen.

[2] P. Billingsley: Convergence of probability measures, John Wiley & Sons, New

York, London, Sydney, Toronto, 1968.

[3] A.N. Borodin: *A limit theorem for solutions of differential equations with random right hand side*, Theory Probab. Appl. **22**, (1977) 482–497.

[4] I.A. Ibragimov-Yu. V. Linnik: Independent and stationary sequences of random variables, Groningen: Wolters-Noordhoff, 1971.

[5] N. Ikeda, S. Nakao and Y. Yamato: *A class of approximations of Brownian motion*, Publ. RIMS Kyoto Univ. **13** (1977), 285–300.

[6] N. Ikeda and S. Watanabe: Stochastic differential equations and diffusion processes, North-Holland/Kodansha, 1981.

[7] H. Kesten and G.C. Papanicolaou: *A limit theorem for turbulent diffusion*, Commun. Math. Phys. **65** (1979), 97–128.

[8] R.Z. Khasminskii: *A limit theorem for the solutions of differential equations with random right hand sides*, Theory. Probab. Appl. **11** (1966), 390–406.

[9] H. Kunita: *On the decomposition of solutions of stochastic differential equations*, Proc. Durham Conf. Stoch. Integrals, Lecture Notes in Math. 851 (1981), 213–255.

[10] H. Kunita and S. Watanabe: *On square integrable martingales*, Nagoya Math. J. **30** (1967), 209–245.

[11] Y. Le Jan: *Flots de deffusions dans R^d*, C.R. Acad. Sci. Paris **294**, Série I (1982), 697–699.

[12] Y. Le Jan and S. Watanabe: *Stochastic flows of diffeomorphisms*, Stochastic Analysis, Taniguchi Symp. SA Katata 1982, 192–224, North-Holland/Kinokuniya, 1984.

[13] G.C. Papanicolaou and W. Kohler: *Asymptotic theory of mixing stochastic ordinary differential equations*, Comm. Pure Appl. Math. **27** (1974), 641–668.

[14] G.C. Papanicolaou and S.R.S. Varadhan: *A limit theorem with strong mixing in Banach space and two applications to stochastic differential equations*, Comm. Pure Appl. Math. **26** (1973), 497–524.

[15] D.W. Stroock and S.R.S. Varadhan: Multidimensional diffusion processes, Springer-Verlag, Berlin, Heidelberg, New York, 1979.

[16] H. Watanabe: *A note on the weak convergence of solutions of certain stochastic ordinary differential equations*, Proc. fourth Japan-USSR Symp. Probab. Theory, Lecture Notes in Math. 1021 (1983), 690–698.

[17] S. Watanabe: *Stochastic flow of diffeomorphisms*, Proc. fourth Japan-USSR Symp. Probab. Theory, Lecture Notes in Nath. 1021 (1983), 699–708.

[18] E. Wong and M. Zakai: *On the relation between ordinary and stochastic differential equations*, Intern. J. Engng. Sci. **3** (1965), 213–229.

[19] T. Yamada and S. Watanabe: *On the uniqueness of solutions of stochastic differential equations*, J. Math. Kyoto Univ. **11** (1971), 155–167.

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