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EQUIVARIANT POINT THEOREMS FOR FIBRE-PRESERVING MAPS

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1 Introduction

Let $p: X\to B$ and $p': X'\to B'$ be local trivial fibre spaces with fibre-preserving involutions $T: X\to X$ and $T': X'\to X'$ respectively, and let $f: X\to X'$ be a fibre-preserving map. Denote by A_f the set of equivariant points of f:

$$A_f=\{x\in X; fT(x)=T'f(x)\}\ ,$$

and by $\bar{A}_f$ its orbit space under T. In this paper we shall study $H^*(\bar{A}_f)$ in connection with $H^*(B)$, where H^* is the Čech cohomology with coefficients in Z_2. Two theorems will be proved by making use of the technique of establishing a transfer homomorphism, which was initiated by Becker and Gottlieb ([1], [2])

In case $p: X\to B$ is an m-sphere bundle with the antipodal involution and $p': X'\to B$ is an R^n-bundle with the trivial involution, Jaworowski gave in [4], [5] the following theorem which is a "continuous" version of the Borsuk-Ulam theorem: If $k=m-n\geqq 0$ and the all the Stiefel-Whitney classes of $p': X'\to B$ are zero then the composition

$$H^i(B)\xrightarrow{\bar{p}^*} H^i(\bar{A}_f)\xrightarrow{\smile\omega(A_f)^k} H^{i+k}(\bar{A}_f)$$

is injective for every i, where $\bar{p}: \bar{A}_f\to B$ is induced by $p|A_f$, and $\omega(A_f)$ is the characteristic class of the double covering $A_f\to\bar{A}_f$. It is seen in this paper that the assumption on the Stiefel-Whitney classes is superfluous in the theorem of Jaworowski.

Throughout this paper we use the Čech cohomology with coefficients in Z_2.

2 Equivariant fundamental cohomology class

Let $M\to X\xrightarrow{p}B$ be a local trivial fibre space such that both the fibre M and the base B are manifolds without boundary. Suppose that there is given a fibre-preserving involution $T: X\to X$, that is, an involution satisfying $pT=T$.

We take the fibre square $X \times_B X$ of the map p: $X \to B$, and define an involution on it by permutation of factors. Then there is an equivariant imbedding Δ: $X \to X \times_B X$ defined by $\Delta(x)=(x, Tx)$. Consider now the normal bundle of $\Delta X \subset X \times_B X$ in which the total space E is regarded as an invariant tubular neighborhood of ΔX in $X \times_B X$. Then we have an R^m-bundle π: $E \to \Delta X$ with involution, where $m = \dim M$. Let S^∞ be the infinite dimensional sphere with the antipodal involution, and consider the orbit spaces $S^\infty \times_{Z_2} E$ and $S^\infty \times_{Z_2} (\Delta X)$ under the diagonal action. Then we have an R^m-bundle $1 \times_{Z_2} \pi$: $S^\infty \times_{Z_2} E — S^\infty \times_{Z_2} (\Delta X)$, so that the Thom class $U(1 \times_{Z_2} \pi) \in H^m(S^\infty \times_{Z_2} E, S^\infty \times_{Z_2} E - S^\infty \times_{Z_2} (\Delta X)) = H^m_{Z_2}(E, E - \Delta X)$. We define $\hat{U}(p) \in H^m_{Z_2}(X \times_B X, X \times_B X - \Delta X)$ to be the element corresponding to $U(1 \times_{Z_2} \pi)$ under the excision isomorphism, and call it the *equivariant fundamental cohomology class of* p: $X \to B$. The restriction $\hat{U}(p) | X \times_B X \in H^m_{Z_2}(X \times_B X)$ is denoted by $\hat{U}'(p)$ and is called the *equivariant diagonal cohomology class* of p. If B is a single point, then $\hat{U}(p) \in H^m_{Z_2}(M \times M, M \times M - \Delta M)$ and $\Delta(p) \in H^m_{Z_2}(M \times M)$ are denoted by $\hat{U}(M)$ and $\hat{U}'(M)$ respectively. If M is a closed manifold, we have $\hat{U}'(M) = \Delta_!(1)$ for the Gysin homomorphism $\Delta_!$: $H^*_{Z_2}(M) \to H^*_{Z_2}(M \times M)$.

Put $M_b = p^{-1}(b)$ for $b \in B$. Then the restriction of the normal bundle π: $E \to \Delta X$ on ΔM_b may be regarded as the normal bundle of $\Delta(M_b) \subset M_b \times M_b$. Therefore it follows that

$$\hat{U}(p) | (M_b \times M_b, M_b \times M_b - \Delta M_b) = \hat{U}(M_b),$$

so that

$$\hat{U}'(p) | (M_b \times M_b) = \hat{U}'(M_b).$$

In some cases, the equivariant diagonal cohomology class $\hat{U}'(M)$ of a closed manifold M with an involution T is expressed in terms of cohomology of M. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_s\}$ be a homogeneous basis of $H^*(M)$, and let $C = (c_{ij})$ be the inverse of the matrix $Y = (y_{ij})$ with $y_{ij} = \langle \alpha_i \smile T^* \alpha_j, [M] \rangle$. Let $\pi_!$: $H^*(M \times M) \to H^*_{Z_2}(M \times M)$ denote the transfer homomorphism for the covering $S^\infty \times (M \times M) \to S^\infty \times_{Z_2} (M \times M)$. We have

Proposition 1. (i) *If T is trivial, then*

$$\hat{U}'(M) = \sum_{i=0}^{[m/2]} q^* \omega^{m-2i} \smile P_0(V_i) + \sum_{i<j} (c_{ij} + c_{ii} c_{jj}) \pi_!(\alpha_i \times \alpha_j),$$

where q: $S^\infty \times_{Z_2} (M \times M) \to S^\infty / Z_2$ is the projection, $\omega \in H^1(S^\infty / Z_2)$ is the generator,

V_i is the i-th Wu class of M, and P_0 is the external Steenrod square (See [3], [7], [8]).

(ii) *If T is free, then*

$$\hat{U}'(M)=\sum_{i<j}c_{ij}\pi_1(\alpha_i\times\alpha_j)\quad\text{(See [7], [8]).}$$

3 Equivariant point theorem of Borsuk-Ulam type

For any space X with a free involution, we denote by $\bar{X}$ the orbit space of X under the involution, and by $\omega(X)\in H^1(\bar{X})$ the characteristic class of the double covering $X\to\bar{X}$.

Theorem 1. *Let $M\to X\xrightarrow{p}B$ and $M'\to X'\xrightarrow{p'}B'$ be local trivial fibre spaces over* ENR's (*=Euclidean neighborhood retracts*), *where the fibre M is a closed m-manifold, and M' is a compact n-manifold with or without boundary. Let $f\colon X\to X'$ be a fibre-preserving map covering a map $g\colon B\to B'$, and let $f_b\colon M_b=p^{-1}(b)\to M'_{g(b)}=p'^{-1}(g(b))$ $(b\in B)$ denote the restriction of f. Suppose that X provides a fibre-preserving free involution T, and put $A_f=\{x\in X; f(x)=f(Tx)\}$. If $k=m-n\geqq 0$ and, for some point b of each connected component of B,*

$$\omega(M_b)^m\neq 0\,,\qquad f_b^*=0\colon \tilde{H}^*(M'_{g(b)})\to\tilde{H}^*(M_b)\,,$$

then the composition

$$H^i(B)\xrightarrow{\bar{p}^*}H^i(\bar{A}_f)\xrightarrow{\smile\omega(A_f)^k}H^{i+k}(\bar{A}_f)$$

is injective for every $i\geqq 0$.

Proof. Case 1: B and B' are manifolds without boundary, and M' is a closed manifold.

By the continuity property of Čech cohomology, it suffices to prove that, for any invariant neighborhood V of A_f, the composition

$$\bar{p}_k\colon H^i(B)\xrightarrow{\bar{p}^*}H^i(\bar{V})\xrightarrow{\smile\omega(V)^k}H^{i+k}(\bar{V})$$

is injective for every $i\geqq 0$. To do this we shall establish a transfer homomorphism

$$\tau_k\colon H^{i+k}(\bar{V})\to H^i(B)$$

such that $\tau_k\circ\bar{p}_k=id$.

Regard X' as a space with involution by the trivial action. Then we have the equivariant fundamental cohomology class $\hat{U}(p')\in H^n_{Z_2}(X'\times_{B'}X', X'\times_{B'}X'-dX')$ of $p'\colon X'\to B'$, where dX' is the diagonal. There is an equivariant map $\hat{f}\colon (X, X-A_f)\to(X'\times_{B'}X', X'\times_{B'}X'-dX')$ defined by $\hat{f}(x)=(f(x),$

$fT(x))$. Consider $\hat{f}^*(\hat{U}(p')) \in H^n_{Z_2}(X, X-A_f)=H^n(\bar{X}, \bar{X}-\bar{A}_f)$, and define τ_k to be the composition

$$H^{i+k}(\bar{V}) \xrightarrow{\smile l^*\hat{f}^*(\hat{U}(p'))} H^{i+m}(\bar{V}, \bar{V}-\bar{A}_f) \underset{\cong}{\overset{l^*}{\leftarrow}} H^{i+m}(\bar{X}, \bar{X}-\bar{A}_f) \xrightarrow{j^*} H^{i+m}(\bar{X}) \xrightarrow{\bar{p}_!} H^i(B),$$

where l: $(\bar{V}, \bar{V}-\bar{A}_f) \subset (\bar{X}, \bar{X}-\bar{A}_f)$, j: $\bar{X} \subset (\bar{X}, \bar{X}-\bar{A}_f)$, and $\bar{p}_!$ is the integration along the fibre ([2]) for the fibre space $\bar{p}$: $\bar{X} \to B$. We shall show $\tau_k \circ \bar{p}_k = id$.

For any $\beta \in H^i(B)$ we have

$$\begin{aligned} &\tau_k \bar{p}_k(\beta) \\ &= \bar{p}_! j^* l^{*^{-1}}(\bar{p}^*(\beta) \smile \omega(V)^k \smile l^* \hat{f}^*(\hat{U}(p'))) \\ &= \bar{p}_! j^*(\bar{p}^*(\beta) \smile \omega(X)^k \smile \hat{f}^*(\hat{U}(p'))) \\ &= \bar{p}_!(\bar{p}^*(\beta) \smile \omega(X)^k \smile \hat{f}^*(\hat{U}'(p'))) \\ &= \beta \smile \bar{p}_!(\omega(X)^k \smile \hat{f}^*(\hat{U}'(p'))) . \end{aligned}$$

Therefore it remains to prove

$$\bar{p}_!(\omega(X)^k \smile \hat{f}^*(\hat{U}'(p'))) = 1 .$$

We have a commutative diagram:

$$\begin{array}{ccccc} H^*_{Z_2}(X' \times_{B'} X') & \xrightarrow{\hat{f}^*} & H^*_{Z_2}(X) & \xrightarrow{\bar{p}_!} & H^*(B) \\ \downarrow i^* & & \downarrow i^* & & \downarrow i^* \\ H^*_{Z_2}(M'_{g(b)} \times M'_{g(b)}) & \xrightarrow{\hat{f}^*_b} & H^*_{Z_2}(M_b) & \xrightarrow{\bar{p}_!} & H^*(b) \end{array}$$

where i are inclusions. Therefore we see

$$\begin{aligned} &i^* \bar{p}_!(\omega(X)^k \smile \hat{f}^*(\hat{U}'(p'))) \\ &= \bar{p}_!(\omega(M_b)^k \smile \hat{f}^*_b(\hat{U}'(M'_b))) . \end{aligned}$$

From (i) of Proposition 1 and our assumption $f^*_b=0$, it follows that

$$\hat{f}^*_b(\hat{U}'(M'_b)) = \omega(M_b)^n .$$

Since $\omega(M_b)^m \neq 0$ we have $p_!(\omega M_b)^m)=1$. Thus it holds that

$$i^* \bar{p}_!(\omega(X)^k \smile \hat{f}^*(\hat{U}'(p'))) = 1$$

which shows the desired result.

Case 2: B and B' are manifolds without boundary, and M' is a compact manifold with boundary.

In this case X' is a manifold with boundary, and a local trivial fibre space

$$DM' \to DX' \overset{\tilde{p}'}{\to} B'$$

is defined naturally, where DX' and DM' are the doubles of X' and M' respectively. Put $\tilde{f}=i\circ f$: $X\to DX'$ where i: $X'\subset DX'$. Obviously $A_{\tilde{f}}=A_f$. Therefore, by applying Case 1 to p, $\tilde{p}'$ and $\tilde{f}$, we have the result.

Case 3: B and B' are ENR's, and M' is a compact manifold.
There are continuous maps

$$B \overset{i}{\to} W \overset{r}{\to} B\,, \qquad B' \overset{i'}{\to} W' \overset{r'}{\to} B'$$

such that $r\circ i=id$, $r'\circ i'=id$, where W and W' are open sets in Euclidean spaces. Let q: $Z\to W$ and q': $Z'\to W'$ be the induced fibre spaces of p: $X\to B$ and p': $X'\to B'$ under r and r' respectively. Define $\tilde{r}$: $Z\to X$, $\tilde{i}'$: $X'\to Z'$ and S: $Z\to Z$ by

$$\tilde{r}(w, x) = x, \quad \tilde{i}'(x') = (i'p'(x'), x')\,,$$
$$S(w, x) = (w, T(x)), \quad (x\in X, x'\in X', w\in W)\,.$$

Then $h=\tilde{i}'\circ f\circ\tilde{r}$: $Z\to Z'$ is a fibre-preserving map, and S is a fibre-preserving free involution. We see $\tilde{r}(A_{\tilde{f}})\subset A_f$. In a commutative diagram

$$\begin{array}{ccccc} H^i(B) & \overset{\bar{p}^*}{\to} & H^i(\bar{A}_f) & \xrightarrow{\smile\omega(A_f)^k} & H^{i+k}(\bar{A}_f) \\ \downarrow r^* & & \downarrow \tilde{r}^* & & \downarrow \tilde{r}^* \\ H^i(W) & \overset{\bar{q}^*}{\to} & H^i(\bar{A}_h) & \xrightarrow{\smile\omega(A_h)^k} & H^{i+k}(\bar{A}_h)\,, \end{array}$$

r^* is injective and the lower composition is injective by Cases 1 and 2. Therefore the upper composition is injective.

Corollary 1. *Let f: $X\to X'$ be a fibre-preserving map of an m-sphere bundle p: $X\to B$ with the antipodal involution into an R^n-bundle p': $X'\to B'$, where B and B' are ENR's. Then if $k=m-n\geqq 0$ the composition*

$$H^i(B) \overset{\bar{p}^*}{\to} H^i(\bar{A}_f) \xrightarrow{\smile\omega(A_f)^k} H^{i+k}(\bar{A}_f)$$

is injective for every i.

Proof. Taking one point compactification of each fibre, p': $X'\to B'$ may be regarded as a subbundle of an n-sphere bundle. Regard f as a fibre-preserving map between the sphere bundles, and apply Theorem 1. Then we get the corollary.

Corollary 2. *Let $M\to X\overset{p}{\to} B$ be a local trivial fibre space with a fibre-preserving free involution, where B is a connected ENR, and M is a closed m-mani-*

fold. Then, if $\omega(M_b)^m \neq 0$ *for some* $b \in B$, *the composition*

$$H^i(B) \xrightarrow{\bar{p}^*} H^i(\bar{X}) \xrightarrow{\smile \omega(X)^k} H^{i+k}(\bar{X})$$

is injective for every $i \geqq 0$ *and* $k=0, 1, \cdots, m$.

Proof. Take the disc D^{m-k}, and regard a constant map $f: X \to D^{m-k}$ as a fibre-preserving map of $p: X \to B$ to $p': D^{m-k} \to pt$. Then $A_f = X$, and we get the result by Theorem 1.

4 Equivariant point theorem of Lefschetz type

We shall first recall from [7], [8] the definition of *equivariant Lefschetz number* $\hat{L}(f)$ for a continuous map $f: M \to N$, where M and N are closed n-manifolds with free involutions S and T respectively. There exists a homogeneous basis $\{\alpha_1, \cdots, \alpha_r, \alpha'_1, \cdots, \alpha'_r\}$ of $H^*(N)$ such that

$$\langle \alpha_i \smile T^*\alpha_i, [N] \rangle = 0, \langle \alpha'_i \smile T^*\alpha'_i, [N] \rangle = 0$$
$$\langle \alpha_i \smile T^*\alpha'_j, [N] \rangle = \delta_{ij},$$

where $[N]$ is the fundamental homology class of N. Then the number

$$\sum_{i=1}^{r} \langle f^*\alpha_i \smile S^* f^* \alpha'_i, [M] \rangle \in Z_2$$

is independent of the choice of $\{\alpha_1, \cdots, \alpha_r, \alpha'_1, \cdots, \alpha'_r\}$. This number is $\hat{L}(f)$ by definition. If $M=N$, $S=T$ and $f^*=id$, $\hat{L}(f)$ coincides with the mod 2 semi-characteristic $\hat{\chi}(M)$ of M.

Theorem 2. *Let* $M \to X \xrightarrow{p} B$ *and* $M' \to X' \xrightarrow{p'} B'$ *be local trivial fibre spaces over ENR's such that the fibres are closed n-manifolds, and let* $f: X \to X'$ *be a fibre-preserving map. Suppose there are given fibre-preserving free involutions* T: $X \to X$ *and* T': $X' \to X'$, *and put* $A_f = \{x \in X \mid fT(x) = T'f(x)\}$. *If the equivariant Lefschetz number* $\hat{L}(f_b)$ *is not zero for some point b of each connected component of* B, *then*

$$\bar{p}^*: H^*(B) \to H^*(\bar{A}_f)$$

is injective.

Proof. If we use (ii) of Proposition 1, Theorem 2 can be proved similarly to the proof of Theorem 1.

Corollary. *Let* $M \to X \to B$ *be a local trivial fibre space with a fibre preserving free involution, where B is an ENR and M is a closed manifold. If the mod* 2 *semi-characteristic* $\hat{\chi}(M) \neq 0$ *then*

$$\bar{p}^*: H^*(B) \to H^*(\bar{X})$$

is injective.

Proof. Take $f=id$ in Theorem 2.
This corollary can be applied to prove

Theorem 3 ([7], [8], [9]). *If a closed manifold M admits a free action of $Z_2\times X_2$, then $\hat{\chi}(M)=0$.*

Proof. Let T_1 and T_2 generate $G=Z_2\times Z_2$. Take an n-sphere S^n for sufficiently large n, and consider the orbit space $X=S^n\times_{Z_2} M$ of $S^n\times M$ under the diagonal action of the antipodal involution on S^n and the involution T_1 on M. A fibre space $M\to X\xrightarrow{p} S^n/Z_2$ and a fibre-preserving free involution T on X are given by $p(z, x)=x$ and $T(z, x)=(z, T_2(x))$, where $z\in S^n$ and $x\in M$. We have also a fibre bundle $S^n\to X\to M/Z_2$, so that $H^i(X)=0$ if $m<i<n$, where $m=\dim M$. Therefore $\bar{p}$: $H^i(S^n/Z_2)\to H^i(\bar{X})$ is not injective if $m<i<n$. Thus $\hat{\chi}(M)=0$ by the above corollary.

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