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ON THE K-THEORY OF SO(n)

Dedicated to Professor Nobuo Shimada on his 60th birthday

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The purpose of this paper is to compute $KO^*(SO(n))$ for $n\equiv -1, 0, 1$ mod 8 and $K^*(SO(n))$ as algebras, where $SO(n)$ is the rotation group of degree n.

The ring $K^*(SO(n))$ has been already determined in [6], [8], [10] and [15]. The calculations in these papers are based on the theorem of Hodgkin in [9] for the K-group of the spinor group Spin(n). Here using also this theorem and the Thom isomorphism theorem we shall show that there exists a short exact sequence in the equivariant K-theory associated with Z_2 involving the injection $K^*(SO(n))\to K^*(P^{n-1})\otimes K^*(\mathrm{Spin}(n))$ as in [8] where P^l is the real projective l-space, and making use of this exact sequence we shall give a proof of the result on $K^*(SO(n))$.

Using the theorem of Hodgkin, Seymour has proved in [14] a theorem on the additive structure of $KO^*(\mathrm{Spin}(n))$. From his result we have a similar short exact sequence in the equivariant KO-theory as in the complex case. So we shall next determine the algebra structure of $KO^*(SO(n))$ for n as above by arguments parallel to $K^*(SO(n))$. Then we shall use the result of Crabb in [4] on the squares of elements in $KO^{-1}(X)$.

Throughout this paper an A-module generated by x is denoted by $A\cdot x$ where A is a ring.

1. Preliminaries

a) By G we denote the multiplicative group $\{-1, 1\}$ throughout this paper. Let $R^{p,q}$ be the euclidean space R^{p+q} with a G-action such that -1 reverses the first p coordinates and fixes the last q. Let $S^{p,q}$ and $B^{p,q}$ be the unit sphere and unit ball in $R^{p,q}$ and $\Sigma^{p,q}=B^{p,q}/S^{p,q}$ with the collapsed $S^{p,q}$ as base point. Thus -1 acts on $S^{k,0}$ as antipodal involution and $P^{k-1}=S^{k,0}/G$. We consider that G acts on Spin(m) as the subgroup $\{-1, 1\}$ of Spin(m) and Spin$(m)/G=SO(m)$ (see [11]). We denote the natural projections $S^{k,0}\to P^{k-1}$ and Spin$(m)\to SO(m)$ by the same letter π.

For any $x=(x_1, \cdots, x_k)$ of R^k we write $x=x_1e_1+ \cdots +x_ke_k$ as a vector where

$e_1, \cdots, e_k$ are the unit vectors of R^k. Then for $k \leqslant m$ we have an equivariant embedding $\bar{\iota}: S^{k,0} \to \mathrm{Spin}(m)$ defined by $\bar{\iota}(x) = xe_1$ viewing e_i as an element of R^m canonically. This yields via π's the well-known embedding $\iota: P^{k-1} \to SO(m)$ given by $\iota\pi(x) = (\delta_{ij} - 2x_i x_j)((-1) \times I_{k-1}) \times I_{m-k}$ where δ_{ij} is the Kronecker index and I_i is the unit matrix of degree i. So these embeddings give a commutative diagram

$$\begin{array}{ccc} S^{k,0} & \xrightarrow{\bar{\iota}} & \mathrm{Spin}(m) \\ \pi\downarrow & & \downarrow\pi \\ P^{k-1} & \xrightarrow[\iota]{} & SO(m) \end{array} \tag{1.1}$$

for $k \leqslant m$ ([8], (1.12)). Since $\bar{\iota}$ is an equivariant map, the assignment

$$(x, g) \mapsto (\pi(x), \bar{\iota}(x)g) \quad \text{for } (x, g) \in S^{k,0} \times \mathrm{Spin}(m)$$

yields a homeomorphism

$$S^{k,0} \times_G \mathrm{Spin}(m) \overset{\approx}{\to} P^{k-1} \times \mathrm{Spin}(m) \tag{1.2}$$

for $k \leqslant m$ ([8], (1.14)). Here let G act diagonally on the product of G-spaces.

For a compact Lie group L, let $h_L = K_L$ or KO_L, the equivariant complex or real K-functor associated with L. If L is a trivial group then we omit the suffix L as usual. *Assume* that $p \equiv 0 \bmod 2$ or 8 according as $h_G = K_G$ or KO_G, then there exists a Thom isomorphism

$$\phi_p: \tilde{h}_G(X) \overset{\cong}{\to} \tilde{h}_G(\Sigma^{p,0} \wedge X)$$

for a compact based G-space X ([3], [13] and [12]).

As is well known, if X is a compact free G-space then $h_G(X) \cong h(X/G)$. So we identify $h_G(X)$ with $h(X/G)$ through this isomorphism in the following. Then for any compact G-space X

$$h_G(X \times \mathrm{Spin}(m)) = h(X \times_G \mathrm{Spin}(m))$$

since G acts on $X \times \mathrm{Spin}(m)$ freely.

Apply h_G to a cofiber sequence

$$S^{p+q,0} \times \mathrm{Spin}(m) \to B^{p+q,0} \times \mathrm{Spin}(m) \to \Sigma^{p+q,0} \wedge \mathrm{Spin}(m)_+$$

where $X_+ = X \cup \{+\}$, the disjoint sum of X and a point $+$, then the above identification, the homeomorphism of (1.2) and the Thom isomorphism give rise to an exact sequence

$$\begin{aligned} \cdots \leftarrow \tilde{h}_G^{i+1}(\Sigma^{q,0} \wedge \mathrm{Spin}(m)_+) &\overset{\delta}{\leftarrow} h^i(P^{p+q-1} \times \mathrm{Spin}(m)) \\ &\overset{I}{\leftarrow} h^i(SO(m)) \overset{J}{\leftarrow} \tilde{h}_G^i(\Sigma^{q,0} \wedge \mathrm{Spin}(m)_+) \leftarrow \cdots \end{aligned} \tag{1.3}$$

for $p+q\leqslant m$ and p as above, provided with the formula

$$\delta(I(x)y)=(-1)^i x\delta(y)$$

for $x\in h^i(SO(m))$, $y\in h^j(P^{p+q-1}\times \mathrm{Spin}(m))$.

b) We take here a vector bundle to be complex or real according as we confine ourself to the complex or real K-theory. Let $GL(m, F)$ be the general linear group of degree m over F where $F=R$ or C. We view a representation of a compact Lie group L as a continuous homomorphism $L\to GL(d, F)$.

Let Δ_{2n}^+ (resp. Δ_{2n}^-) be the even (resp. odd) half-spin representation of degree 2^{n-1} of Spin$(2n)$ and Δ_{2n+1} the spin representation of degree 2^n of Spin $(2n+1)$. According to [11], §13, 12.4 Δ_{8n}^+, Δ_{8n}^-, Δ_{8n-1} and Δ_{8n+1} are real, and the other simple spin representations are not real. Besides by [11], §13, Proposition 9.1 we have

$$\Delta_{2n}^+(-1)=\Delta_{2n}^-(-1)=-I_{2^{n-1}} \text{ and } \Delta_{2n+1}(-1)=-I_{2^n}\,. \tag{1.4}$$

Put $V=R^{1,0}$ or $R^{1,0}\otimes C$. Let ξ_m' denote the line bundle $\mathrm{Spin}(m)\times_G V\to SO(m)$ and $\xi_m=\xi_m'-1$ as an element of $h(SO(m))$ where 1 denotes the trivial line bundle. Clearly $(\xi_m')^2=1$. By (1.4) we see that $2^{n-1}\xi_{2n}'$ and $2^n\xi_{2n+1}'$ are isomorphic to the product bundles. Hence we have

$$\xi_m^2=-2\xi_m,\ 2^{n-1}\xi_{2n}=0 \text{ and } 2^n\xi_{2n+1}=0\,. \tag{1.5}$$

Because of (1.4) we can define base point preserving maps

$$\delta_{2n},\ \varepsilon_{2n}\colon SO(2n)\to GL(2^{n-1}, F) \text{ and } \varepsilon_{2n+1}\colon SO(2n+1)\to GL(2^n, F)$$

by $\delta_{2n}\pi(g)=\Delta_{2n}^-(g)^{-1}\Delta_{2n}^+(g)$, $\varepsilon_{2n}\pi(g)=\Delta_{2n}^+(g)^2$ for $g\in\mathrm{Spin}(2n)$ and $\varepsilon_{2n+1}\pi(g)=\Delta_{2n+1}(g)^2$ for $g\in\mathrm{Spin}(2n+1)$ respectively. (Here the unit element is specified as base point of a Lie group.)

Let ρ_m be the standard faithful real representation of degree m of $SO(m)$. Then $\rho_m\pi$, denoted also by ρ_m, is a real representation of degree m of Spin(m).

Let us denote by $\beta(\alpha)$ the element of $\tilde{h}^{-1}(X)$ represented by a base point preserving map $\alpha\colon X\to GL(l, F)$ in a canonical way for a compact based space X. Then we have the following elements of $\tilde{h}^{-1}(\mathrm{Spin}(m))$ or $\tilde{h}^{-1}(SO(m))$

$$\beta(\lambda^i\rho_m)\ (1\leqslant i\leqslant m),\ \beta(\Delta_{2n}^+),\ \beta(\Delta_{2n}^-),\ \beta(\Delta_{2n+1}),\ \beta(\delta_{2n}),\ \beta(\varepsilon_{2n}) \text{ and } \beta(\varepsilon_{2n+1})$$

where $\lambda^i\sigma$ denotes the i-th exterior power of a representation σ.

As for the squares of elements of $\tilde{h}^{-1}(X)$, we have $x^2=0$ for $x\in\tilde{K}^{-1}(X)$, which is well known, and by the result on p. 67 of [4]

$$x^2=\eta_1\lambda^2 x \quad \text{for } x\in\widetilde{KO}^{-1}(X) \tag{1.6}$$

where $\lambda^2 x$ denotes the 2nd exterior power of x and η_1 the generator of $KO^{-1}(+)=Z_2$. If σ is a real representation of degree l of a compact Lie group L, then applying (1.6) to $\beta(\sigma)$ we have the relation

(1.7) $$\beta(\sigma)^2=\eta_1(\beta(\lambda^2\sigma)+l\beta(\sigma)) \quad \text{in } \widetilde{KO}^{-2}(L).$$

c) We consider generators of $\tilde{K}_G(\Sigma^{2n,0})$ and $\widetilde{KO}_G(\Sigma^{8n,0})$ over $R(G)$ and $RO(G)$ respectively. Here let $R(G)$ (resp. $RO(G)$) denote the complex (resp. real) representation ring of G.

Put $L=R^{1,0}\otimes C$. Let E_+ be the quotient of the disjoint sum $B^{2n,0}\times C^{2^{n-1}}\cup B^{2n,0}\times 2^{n-1}L$ by the equivalence relation which identifies (x, v) with $(x, \Delta_{2n}^+\bar{\iota}(x)v)$ for $x\in S^{2n,0}$, $v\in C^{2^{n-1}}$ where $\bar{\iota}$ is as in a) for $m=k=2n$. Identifying the corresponding quotient of $B^{2n,0}\cup B^{2n,0}$ with $\Sigma^{2n,0}$, we see easily that E_+ is a G-vector bundle over $\Sigma^{2n,0}$. Moreover when we regard the center of the latter ball $B^{2n,0}$ as the base point of $\Sigma^{2n,0}$, we see that $\tau_n^+=[E_+]-2^{n-1}[L]\in\tilde{K}_G(\Sigma^{2n,0})$. Here we consider L as the product bundle with fibre L and $[F]$ denotes the isomorphism class of a G-vector bundle F. Similarly by using Δ_{2n}^- for Δ_{2n}^+ we can define a G-vector bundle E_- over $\Sigma^{2n,0}$ and $\tau_n^-=[E_-]-2^{n-1}[L]\in\tilde{K}_G(\Sigma^{2n,0})$.

Lemma 1.8. *τ_n^+ and τ_n^- are generators of $\tilde{K}_G(\Sigma^{2n,0})$ as an $R(G)$-module and $\tau_n^+=-L\tau_n^-$.*

Proof. Let i denote the inclusion $\mathrm{Spin}(2n)\subset\mathrm{Spin}(2n+1)$. Because $i^*(\Delta_{2n+1})=\Delta_{2n}^+\oplus\Delta_{2n}^-$, $\Delta_{2n}^+\bar{\iota}\times\Delta_{2n}^-\bar{\iota}$ is extended on $B^{2n,0}$ as a non-equivariant map. This shows that $\psi(E_+\oplus E_-)$ is isomorphic to a product bundle where let ψ denote the forgetful functor. Therefore $\psi(\tau_n^+)=-\psi(\tau_n^-)$.

Let $\bar{\delta}$: $\mathrm{Spin}(2n)/\mathrm{Spin}(2n-1)\rightarrow GL(2^{n-1}, C)$ be a map defined by $\bar{\delta}([g])=\Delta_{2n}^+(g)\Delta_{2n}^-(g)^{-1}$ for $g\in\mathrm{Spin}(2n)$ where $[g]$ denotes the equivalence class of g. Then $\tilde{K}^{-1}(\mathrm{Spin}(2n)/\mathrm{Spin}(2n-1))=Z\cdot\beta(\bar{\delta})$ (e.g. see [7]). Let λ_{nL} be the Thom class of [13] such that $\lambda_{-1}(nL)=2^{n-1}(1-L)$, so that $\tilde{K}_G(\Sigma^{2n,0})=R(G)\cdot\lambda_{nL}$.

Obviously the composition of $\bar{\iota}$: $S^{2n-1}=\psi(S^{2n,0})\rightarrow\mathrm{Spin}(2n)$ and the natural projection p: $\mathrm{Spin}(2n)\rightarrow\mathrm{Spin}(2n)/\mathrm{Spin}(2n-1)$ factors as follows: $S^{2n-1}\xrightarrow{\pi}P^{2n-1}\xrightarrow{q}\mathrm{Spin}(2n)/\mathrm{Spin}(2n-1)$. By the commutativity of (1.1) and the inspection of the identifications $\mathrm{Spin}(2n)/\mathrm{Spin}(2n-1)=SO(2n)/SO(2n-1)=S^{2n-1}$ we see that $q\pi$ is of degree ±2. This shows that $(p\bar{\iota})^*(\beta(\bar{\delta}))=\pm2\psi(\lambda_{nL})$ and so $q^*(\beta(\bar{\delta}))$ is a generator of $K^{-1}(P^{2n-1})=Z$. By definition $\bar{\iota}^*(\beta(\Delta_{2n}^+))=\psi(\tau_n^+)$, $\bar{\iota}^*(\beta(\Delta_{2n}^-))=\psi(\tau_n^-)$ and $p^*(\beta(\bar{\delta}))=\beta(\Delta_{2n}^+)-\beta(\Delta_{2n}^-)$. Hence we have $\psi(\tau_n^+)=-\psi(\tau_n^-)=\pm\psi(\lambda_{nL})$.

Put $\tau_n^+=(a+bL)\lambda_{nL}$. Since $i^*(\tau_n^+)=i^*(\lambda_{nL})=2^{n-1}(1-L)$ where i is the inclusion $\Sigma^{0,0}\subset\Sigma^{2n,0}$, we have $a-b=1$. And since $\psi(\tau_n^+)=\pm\psi(\lambda_{nL})$ we have $a+b=\pm1$ so that $a=1$, $b=0$ or $a=0$, $b=-1$, which implies that $\tau_n^+=\lambda_{nL}$ or $-L\lambda_{nL}$. The similar fact holds for τ_n^- also. Therefore using $\psi(\tau_n^+)=-\psi(\tau_n^-)$ again we get $\tau_n^+=-L\tau_n^-$. q.e.d.

Let put $H=R^{1,0}$. Using Δ_{8n}^+ and Δ_{8n}^- viewed as real representations, by the parallel construction as τ_n^+ and τ_n^- we get elements ω_n^+ and ω_n^- of $\widetilde{KO}_G(\Sigma^{8n,0})$, which satisfy the relations $i^*(\omega_n^+)=i^*(\omega_n^-)=2^{4n-1}(1-H)$ where i is the inclusion $\Sigma^{0,0}\subset\Sigma^{8n,0}$. Since the complexification of ω_n^+ (resp. ω_n^-) is τ_{4n}^+ (resp. τ_{4n}^-) we obtain

Lemma 1.9. *ω_n^+ and ω_n^- are generators of $\widetilde{KO}_G(\Sigma^{8n,0})$ as an $RO(G)$-module and $\omega_n^+=-H\omega_n^-$.*

In the sections 3 and 4 we *assume* that the Thom isomorphism in a) is given by multiplication by $\tau_{p/2}^+$ or $\omega_{p/8}^+$.

2. $h^*(P^n)$ and $h^*(\mathrm{Spin}(n))$

We consider the K-groups which we need in the following sections.

Let $\gamma_{m-1}=\iota^*\xi_m$ where ι is as in a) for $k=m$ and let $\nu_{2n-1}=\beta(\delta)$ where δ is a map $P^{2n-1}\to GL(2^{n-1}, F)$ defined by $\delta\pi(x)=\Delta_{2n}^+(\bar{\iota}(x))\Delta_{2n}^-(\bar{\iota}(x))^{-1}$ for $x\in S^{2n,0}$, If $n\equiv 0 \bmod 4$, then we may view ν_{2n-1} as an element of $\widetilde{KO}^{-1}(P^{2n-1})$, because Δ_{2n}^+ and Δ_{2n}^- are real. In the case when $F=C$, as noted in the proof of Lemma 1.8, $\nu_{2n-1}=q^*(\beta(\bar{\delta}))$ which is a generator of $\tilde{K}^{-1}(P^{2n-1})$.

Proposition 2.1. [2]. 1) $\tilde{K}^0(P^{2n-1})=Z_{2^{n-1}}\cdot\gamma_{2n-1}$, $\tilde{K}^{-1}(P^{2n-1})=Z\cdot\nu_{2n-1}$ *with relations* $\gamma_{2n-1}^2=-2\gamma_{2n-1}$, $\nu_{2n-1}^2=\gamma_{2n-1}\nu_{2n-1}=0$.

2) $\tilde{K}^0(P^{2n})=Z_{2^n}\cdot\gamma_{2n}$, $\tilde{K}^{-1}(P^{2n})=0$ *with relations* $\gamma_{2n}^2=-2\gamma_{2n}$.

We next recall that as a Z_8-graded algebra

$$KO^*(+) = Z[\eta_1, \eta_4]/(2\eta_1, \eta_1^3, \eta_1\eta_4, \eta_4^2-4)$$

where $\eta_i\in KO^{-i}(+)$. Then by [1], [5] and [16] we have

Proposition 2.2. 1) $\widetilde{KO}^0(P^{8n-1})=Z_{2^{4n-1}}\cdot\gamma_{8n-1}$,

$$\widetilde{KO}^{-1}(P^{8n-1}) = Z\cdot\nu_{8n-1}\oplus Z_2\cdot\eta_1\gamma_{8n-1}\,,$$
$$\widetilde{KO}^{-2}(P^{8n-1}) = Z_2\cdot\mu_{8n-1}\oplus Z_2\cdot\eta_1\nu_{8n-1}\oplus Z_2\cdot\eta_1^2\gamma_{8n-1}\,,$$
$$\widetilde{KO}^{-3}(P^{8n-1}) = Z_2\cdot\eta_1\mu_{8n-1}\oplus Z_2\cdot\eta_1^2\nu_{8n-1}\,,$$
$$\widetilde{KO}^{-4}(P^{8n-1}) = Z_{2^{4n-1}}\cdot\eta_4\gamma_{8n-1}\,,$$
$$\widetilde{KO}^{-5}(P^{8n-1}) = Z\cdot\eta_4\nu_{8n-1}\,,$$
$$\widetilde{KO}^{-6}(P^{8n-1}) = \widetilde{KO}^{-7}(P^{8n-1}) = 0$$

with relations

$$\gamma_{8n-1}^2 = -2\gamma_{8n-1},\ \gamma_{8n-1}\nu_{8n-1} = 0,\ \gamma_{8n-1}\mu_{8n-1} = \nu_{8n-1}^2 = 0\,,$$
$$\nu_{8n-1}\mu_{8n-1} = 0,\ \mu_{8n-1}^2 = 0,\ \eta_1^2\mu_{8n-1} = 2^{4n-2}\eta_4\gamma_{8n-1}\,,\quad \eta_4\mu_{8n-1} = 0\,.$$

2) $\widetilde{KO}^0(P^{8n-2}) = Z_{2^{4n-1}}\cdot\gamma_{8n-2}$,

$\widetilde{KO}^{-1}(P^{8n-2}) = Z_2\cdot\eta_1\gamma_{8n-2}$,

$\widetilde{KO}^{-2}(P^{8n-2}) = Z_2\cdot\mu_{8n-2}\oplus Z_2\cdot\eta_1^2\gamma_{8n-2}$,

$\widetilde{KO}^{-3}(P^{8n-2}) = Z_2\cdot\eta_1\mu_{8n-2}$,

$\widetilde{KO}^{-4}(P^{8n-2}) = Z_{2^{4n-1}}\cdot\eta_4\gamma_{8n-2}$,

$\widetilde{KO}^{-5}(P^{8n-2}) = \widetilde{KO}^{-6}(P^{8n-2}) = \widetilde{KO}^{-7}(P^{8n-2}) = 0$

with relations

$$\gamma_{8n-2}^2 = -2\gamma_{8n-2},\ \gamma_{8n-2}\mu_{8n-2} = 0,\ \mu_{8n-2}^2 = 0\,,$$
$$\eta_1^2\mu_{8n-2} = 2^{4n-2}\eta_4\gamma_{8n-2},\ \eta_4\mu_{8n-2} = 0\,.$$

3) $\widetilde{KO}^0(P^{8n}) = Z_{2^{4n}}\cdot\gamma_{8n}$,

$\widetilde{KO}^{-1}(P^{8n}) = Z_2\cdot\eta_1\gamma_{8n}$,

$\widetilde{KO}^{-2}(P^{8n}) = Z_2\cdot\bar{\nu}_{8n}\oplus Z_2\cdot\eta_1^2\gamma_{8n}$,

$\widetilde{KO}^{-3}(P^{8n}) = Z_2\cdot\eta_1\bar{\nu}_{8n}$,

$\widetilde{KO}^{-4}(P^{8n}) = Z_{2^{4n}}\cdot\eta_4\gamma_{8n}$,

$\widetilde{KO}^{-5}(P^{8n}) = \widetilde{KO}^{-6}(P^{8n}) = \widetilde{KO}^{-7}(P^{8n}) = 0$

with relations

$$\gamma_{8n}^2 = -2\gamma_{8n},\ \gamma_{8n}\bar{\nu}_{8n} = 0,\ \bar{\nu}_{8n}^2 = 0,\ \eta_1^2\bar{\nu}_{8n} = 2^{4n-1}\eta_4\gamma_{8n}\,,\quad \eta_4\bar{\nu}_{8n} = 0\,.$$

Proof. We refer the reader to the table in [5] for the additive structure of $KO^*(P^l)$. The generators of the 0- and (-4)-terms are given in [1] and [5]. To determine the generators of the other terms it suffices to consider for 1) the exact sequence of $(B^{8n,0}, S^{8n,0})$ in the equivariant KO-theory and for 2) (resp. 3)) the exact sequence of (P^{8n-1}, P^{8n-2}) (resp. (P^{8n}, P^{8n-1})) in the KO-theory. But we omit all details.

We want here to determine the ring structure of $KO^*(P^l)$ for l as above. The relation $\gamma_{8n+i}^2=-2\gamma_{8n+i}$ for $i=-2, -1, 0$ follows from (1.5) and the relation $\nu_{8n-1}^2=0$ does from the definition of ν_{8n-1} because it comes from $\widetilde{KO}^{-1}(S^{8n-1})$.

1) Let i: $S^1=P^1\to P^{8n-1}$ be the inclusion. Then $i^*(\gamma_{8n-1})=\eta_1$ and $i^*(\nu_{8n-1})=0$ by definition. If $\gamma_{8n-1}\nu_{8n-1}\neq 0$ then $\gamma_{8n-1}\nu_{8n-1}=\eta_1\gamma_{8n-1}$ since it is a torsion element of $\widetilde{KO}^{-1}(P^{8n-1})$. Apply i^* to this equality then we have $\eta_1^2=0$ which is a contradiction. Hence we get $\gamma_{8n-1}\nu_{8n-1}=0$.

Since $P^{8n-1}/P^{8n-3}\simeq S^{8n-2}\vee S^{8n-1}$, $\widetilde{KO}^*(P^{8n-1}/P^{8n-3})=\widetilde{KO}^*(S^{8n-2})\oplus\widetilde{KO}^*(S^{8n-1})$.

When we consider the exact sequence of (P^{8n-1}, P^{8n-3}) in the KO-theory, we see that μ_{8n-1} and ν_{8n-1} come from $\widetilde{KO}^{-2}(S^{8n-2})$ and $\widetilde{KO}^{-1}(S^{8n-1})$ respectively. From this it follows readily that $\nu_{8n-1}\mu_{8n-1}=\mu_{8n-1}^2=\gamma_{8n-1}\mu_{8n-1}=0$.

Because of $\widetilde{KO}^{-4}(P^{8n-3})=Z_{2^{4n-2}}\cdot\eta_4\gamma_{8n-3}$ by [5] we have $\eta_1^2\mu_{8n-1}=2^{4n-2}\eta_4\gamma_{8n-1}$ similarly. The last relation is obvious because $\widetilde{KO}^{-6}(P^{8n-1})=0$.

2) Immediate from the relations of 1).

3) We see by using the exact sequence of (P^{8n}, P^{8n-1}) in the KO-theory and 1) that the restriction $\widetilde{KO}^{-2}(P^{8n})\to\widetilde{KO}^{-2}(P^{8n-1})$ is injective and by the definition of $\bar{\nu}_{8n}$ it sends $\gamma_{8n}\bar{\nu}_{8n}$ to $\eta_1\gamma_{8n-1}\nu_{8n-1}$. From this and the relation $\gamma_{8n-1}\nu_{8n-1}=0$ it follows that $\gamma_{8n}\bar{\nu}_{8n}=0$.

Let π: $P^{8n}\to P^{8n}/P^{8n-1}$ be the natural projection. Then we see that $\bar{\nu}_{8n}$ belongs to $\mathrm{Im}\,\pi^*$. But P^{8n}/P^{8n-2} is a suspension of a space, so we have $\bar{\nu}_{8n}^2=0$.

We consider the following exact sequence of Atiyah [2], (3.4)

$$\cdots\to\widetilde{KO}^{-3}(P^{8n})\overset{\chi}{\to}\widetilde{KO}^{-4}(P^{8n})\overset{c}{\to}\tilde{K}^{-4}(P^{8n})\to\widetilde{KO}^{-2}(P^{8n})\to\cdots$$

where χ is multiplication by η_1 and c is the complexification homomorphism. Since $\widetilde{KO}^{-3}(P^{8n})=Z_2\cdot\eta_1\bar{\nu}_{8n}$, $\widetilde{KO}^{-4}(P^{8n})=Z_{2^{4n}}\cdot\eta_4\gamma_{8n}$, $\tilde{K}^{-4}(P^{8n})=Z_{2^{4n}}\cdot\mu^2\gamma_{8n}$ by Proposition 2.1 and $c(\eta_4\gamma_{8n})=2\mu^2\gamma_{8n}$, then we obtain $\eta_1^2\bar{\nu}_{8n}=2^{4n-1}\eta_4\gamma_{8n}$ where $\mu\in K^{-2}(+)$ is the Bott class. The last relation is also clear because $\widetilde{KO}^{-6}(P^{8n})=0$. q.e.d.

By [9], [14] and (1.7) we have the following propositions.

Proposition 2.3 [9]. *As rings*

$$K^*(\mathrm{Spin}(2n))=\Lambda(\beta(\lambda^1\rho_{2n}),\cdots,\beta(\lambda^{n-2}\rho_{2n}),\beta(\Delta_{2n}^+),\beta(\Delta_{2n}^-)),$$
$$K^*(\mathrm{Spin}(2n-1))=\Lambda(\beta(\lambda^1\rho_{2n-1}),\cdots,\beta(\lambda^{n-2}\rho_{2n-1}),\beta(\Delta_{2n-1})).$$

Proposition 2.4 [14]. *As $KO^*(+)$-modules*

$$KO^*(\mathrm{Spin}(8n))=\Lambda_{KO^*(+)}(\beta(\lambda^1\rho_{8n}),\cdots,\beta(\lambda^{4n-2}\rho_{8n}),\beta(\Delta_{8n}^+),\beta(\Delta_{8n}^-)),$$
$$KO^*(\mathrm{Spin}(8n-1))=\Lambda_{KO^*(+)}(\beta(\lambda^1\rho_{8n-1}),\cdots,\beta(\lambda^{4n-2}\rho_{8n-1}),\beta(\Delta_{8n-1})),$$
$$KO^*(\mathrm{Spin}(8n+1))=\Lambda_{KO^*(+)}(\beta(\lambda^1\rho_{8n+1}),\cdots,\beta(\lambda^{4n-1}\rho_{8n+1}),\beta(\Delta_{8n+1}))$$

in which there hold the relations

$$\beta(\lambda^i\rho_m)^2=\eta_1(\beta(\lambda^2(\lambda^i\rho_m))+\binom{m}{i}\beta(\lambda^i\rho_m))$$

for $m=8n$, $8n-1$, $8n+1$ and $1\leqslant i\leqslant(m-3)/2-(1+(-1)^m)/4$,

$$\beta(\Delta_m)^2=\eta_1\beta(\lambda^2\Delta_m)$$

for $m=8n-1$, $8n+1$ and

$$\beta(\Delta_{8n}^{+})^{2}=\eta_{1}\beta(\lambda^{2}\Delta_{8n}^{+}),\ \beta(\Delta_{8n}^{-})^{2}=\eta_{1}\beta(\lambda^{2}\Delta_{8n}^{-})\,.$$

Since $h^*(\mathrm{Spin}(n))$, observed above, is free over $h^*(+)$, there exists a canonical isomorphism

$$h^*(X)\otimes_{h^*(+)}h^*(\mathrm{Spin}(n))\cong h^*(X\times\mathrm{Spin}(n))$$

for a finite CW-complex X, under whose isomorphism $h^*(X)\otimes_{h^*(+)}h^*(\mathrm{Spin}(n))$ is identified with $h^*(X\times\mathrm{Spin}(n))$ in the following.

3. $K^*(SO(n))$

Take $h_G=K_G$, $p=0$ and $q=m$ in (1.3), then we have the following exact sequence

$$(3.1)\quad \cdots\leftarrow\tilde{K}_G^*(\Sigma^{m,0}\wedge\mathrm{Spin}(m)_+)\overset{\delta}{\leftarrow}K_G^*(P^{m-1}\times\mathrm{Spin}(m))\overset{I}{\leftarrow}$$
$$K^*(SO(m))\overset{J}{\leftarrow}\tilde{K}_G^*(\Sigma^{m,0}\wedge\mathrm{Spin}(m)_+)\leftarrow\cdots\,.$$

First we discuss $K^*(SO(2n))$. Let j be the inclusion $\Sigma^{0,0}\wedge\mathrm{Spin}(2n)_+\subset\Sigma^{2n,0}\wedge\mathrm{Spin}(2n)_+$. Then by definition $j^*(\tau_n^+\wedge 1)=2^{n-1}(1-L)$ and hence $J(\tau_n^+\wedge 1)=-2^{n-1}\xi_{2n}$ for J in (3.1) when $m=2n$, which implies by (1.5) that J is zero. Therefore by (3.1) we obtain

Lemma 3.2.

$$0\leftarrow K^*(SO(2n))\overset{\delta}{\leftarrow}K^*(P^{2n-1})\otimes K^*(\mathrm{Spin}(2n))\overset{I}{\leftarrow}K^*(SO(2n))\leftarrow 0$$

is exact where $\delta=\phi_{2n}^{-1}\delta$, *and* $\delta(I(x)-)=(-1)^i x\delta(-)$ *for* $x\in K^i(SO(2n))$.

Lemma 3.3. i) $I(\xi_{2n})=\gamma_{2n-1}\otimes 1$,

ii) $I(\beta(\lambda^k\rho_{2n}))=1\otimes\beta(\lambda^k\rho_{2n})\quad(1\leqslant k\leqslant 2n)$,

iii) $I(\beta(\delta_{2n}))=1\otimes(\beta(\Delta_{2n}^{+})-\beta(\Delta_{2n}^{-}))-\nu_{2n-1}\otimes 1$,

iv) $I(\beta(\varepsilon_{2n}))=(\gamma_{2n-1}+2)\otimes\beta(\Delta_{2n}^{+})-\nu_{2n-1}\otimes 1$.

Proof. i) Obvious.

ii) It follows from definition that $I(\beta(\lambda^k\rho_{2n}))$ is represented by a map $f\colon P^{2n-1}\times\mathrm{Spin}(2n)\to GL(\binom{2n}{k}, C)$ given by $f(\pi(x),g)=\lambda^k\rho_{2n}(\bar{\iota}(x))^{-1}\lambda^k\rho_{2n}(g)$ for $x\in S^{2n,0}$, $g\in\mathrm{Spin}(2n)$. Therefore we see that $I(\beta(\lambda^k\rho_{2n}))=1\otimes\beta(\lambda^k\rho_{2n})-\beta((\lambda^k\rho_{2n})\iota)\otimes 1$. But $\lambda^k\rho_{2n}$ comes from representations of $SO(2n+1)$, hence $\beta((\lambda^k\rho_{2n})\iota)$ is contained in the image of the restriction $K^{-1}(P^{2n})\to K^{-1}(P^{2n-1})$. This shows that $\beta((\lambda^k\rho_{2n})\iota)=0$, because of $K^{-1}(P^{2n})=0$.

iii) Observe that $I(\beta(\delta_{2n}))$ is represented by a map $f\colon P^{2n-1}\times\mathrm{Spin}(2n)\to GL(2^{n-1},C)$ given by $f(\pi(x),g)=\Delta_{2n}^{-}(g)^{-1}\Delta_{2n}^{-}(\bar{\iota}(x))\Delta_{2n}^{+}(\bar{\iota}(x))^{-1}\Delta_{2n}^{+}(g)$ for $x\in S^{2n,0}$, $g\in\mathrm{Spin}(2n)$. From this the claim follows immediately.

iv) Define $f: P^{2n-1}\to GL(2^{n-1}, C)$ and $h: P^{2n-1}\times \mathrm{Spin}(2n)\to GL(2^{n-1}, C)$ by $f\pi(x)=\Delta_{2n}^{+}(\bar{\iota}(x))^{-2}$ and $h(\pi(x), g)=\Delta_{2n}^{+}(\bar{\iota}(x))\Delta_{2n}^{+}(g)\Delta_{2n}^{+}(\bar{\iota}(x))^{-1}$ for $x\in S^{2n,0}$, $g\in \mathrm{Spin}(2n)$ respectively. Then we see similarly that $I(\beta(\varepsilon_{2n}))=\beta(f)\otimes 1+\beta(h)+1\otimes\beta(\Delta_{2n}^{+})$.

Considering the injection $\pi^*: K^{-1}(P^{2n-1})\to K^{-1}(S^{2n-1})$ we get $\pi^*(\beta(f))=-2\psi(\tau_n^+)$. On the other hand by the definition of ν_{2n-1} and Lemma 1.8 we have $\pi^*(\nu_{2n-1})=\psi(\tau_n^+-\tau_n^-)=2\psi(\tau_n^+)$. Therefore $\beta(f)=-\nu_{2n-1}$.

Let $C\,\mathrm{Spin}(2n)=[0, 1]\times \mathrm{Spin}(2n)/\{1\}\times \mathrm{Spin}(2n)$. For $(x, \lambda)\in S^{2n,0}\times L$, $(t, g)\in[0, 1]\times \mathrm{Spin}(2n)$, $v\in C^{2n-1}$ the assignment $[x, \lambda]\otimes([t, g], v)\mapsto(\pi(x), [t, g], \Delta_{2n}^{+}(\bar{\iota}(x))\lambda v)$ gives an isomorphism of vector bundles

$$a: (S^{2n,0}\times_G L)\otimes(C\,\mathrm{Spin}(2n)\times C^{2^{n-1}})\overset{\cong}{\to} P^{2n-1}\times C\,\mathrm{Spin}(2n)\times C^{2^{n-1}}$$

where [ ]'s denote the equivalence classes. Let E and F be the quotients of the two copies of $C\,\mathrm{Spin}(2n)\times C^{2^{n-1}}$ and $P^{2n-1}\times C\,\mathrm{Spin}(2n)\times C^{2^{n-1}}$ by the equivalence relations which identify $([0, g], v)$ with $([0, g], \Delta_{2n}^{+}(g)v)$ and $(\pi(x), [0, g], v)$ with $(\pi(x), [0, g], \Delta_{2n}^{+}(\bar{\iota}(x))\Delta_{2n}^{+}(g)\Delta_{2n}^{+}(\bar{\iota}(x))^{-1}v)$ for $x\in S^{2n,0}$, $g\in\mathrm{Spin}(2n)$, $v\in C^{2^{n-1}}$ respectively. Then we may view E and F as vector bundles over $S^1\wedge \mathrm{Spin}(2n)_+$ and $P^{2n-1}\times S^1\wedge\mathrm{Spin}(2n)_+$ respectively and we have $\gamma'_{2n-1}\otimes E\cong F$ through a where $\gamma'_{2n-1}=i^*\xi_{2n}$. Therefore we obtain by definition

$$\begin{aligned}\beta(h) &= [F]-2^{n-1}\\ &= \gamma'_{2n-1}\otimes[E]-2^{n-1}\\ &= \gamma'_{2n-1}\otimes([E]-2^{n-1}) \quad \text{by (1.5)}\\ &= (\gamma_{2n-1}+1)\otimes\beta(\Delta_{2n}^{+})\end{aligned}$$

from which we obtain the claim. q.e.d.

Lemma 3.4. i) $\delta(1\otimes\beta(\Delta_{2n}^{+}))=\xi_{2n}+1$, $\delta(1\otimes\beta(\Delta_{2n}^{-}))=-1$,

ii) $\delta(\nu_{2n-1}\otimes 1)=\xi_{2n}+2_n$,

iii) $\delta(\nu_{2n-1}\otimes\beta(\Delta_{2n}^{+}))=(\xi_{2n}+1)\beta(\varepsilon_{2n})$,

$\delta(\nu_{2n-1}\otimes\beta(\Delta_{2n}^{-}))=(\xi_{2n}+1)\beta(\varepsilon_{2n})-(\xi_{2n}+2)\beta(\delta_{2n})$,

iv) $\delta(1\otimes\beta(\Delta_{2n}^{+})\beta(\Delta_{2n}^{-}))=(\xi_{2n}+1)\,(\beta(\delta_{2n})-\beta(\varepsilon_{2n}))$,

v) $\delta(\nu_{2n-1}\otimes\beta(\Delta_{2n}^{+})\beta(\Delta_{2n}^{-}))=(\xi_{2n}+1)\beta(\delta_{2n})\beta(\varepsilon_{2n})$.

Proof. i) Here we consider $\tilde{K}_G^*(\Sigma^{2n,0}\wedge\mathrm{Spin}(2n)_+)$ as a subgroup of $K_G^*(\Sigma^{2n,0}\times\mathrm{Spin}(2n))$ in a canonical manner. Let E and F be the quotients of the disjoint sums $B_1^{2n,0}\times\mathrm{Spin}(2n)\times C^{2^{n-1}}\cup B_2^{2n,0}\times\mathrm{Spin}(2n)\times C^{2^{n-1}}$ and $B_2^{2n,0}\times\mathrm{Spin}(2n)\times C^{2^{n-1}}\cup B_2^{2n,0}\times\mathrm{Spin}(2n)\times 2^{n-1}L$ by the equivalence relations which identify (x, g, v) with $(x, g, \Delta_{2n}^{+}(\bar{\iota}(x)g)v)$ and (x, g, v) with $(x, g, \Delta_{2n}^{+}\bar{\iota}(x)v)$ for $x\in S^{2n,0}$, $g\in\mathrm{Spin}(2n)$, $v\in C^{2^{n-1}}$ respectively where $B_1^{2n,0}=B_2^{2n,0}=B^{2n,0}$. Then by definition $\bar{\delta}(1\otimes\beta(\Delta_{2n}^{+}))=[E]-2^{n-1}$ and $\tau_n^+\wedge 1=[F]-2^{n-1}[L]$. Through the isomorphism $B_1^{2n,0}\times\mathrm{Spin}(2n)\times C^{2^{n-1}}\cong B_1^{2n,0}\times\mathrm{Spin}(2n)\times 2^{n-1}L$ given by the assignment $(tx, g, v)\mapsto(tx, g, \Delta_{2n}^+(g)v)$, $0\leqslant t\leqslant 1$, we get $L\otimes F\cong E$. Hence $L\tau_n^+\wedge 1=[E]-2^{n-1}$ and so $\bar{\delta}(1\otimes\beta(\Delta_{2n}^+))=L\tau_n^+\wedge 1$ which implies $\delta(1\otimes\beta(\Delta_{2n}^+))=[\xi'_{2n}]=\xi_{2n}+1$.

By the same argument as above we obtain $\bar{\delta}(1\otimes\beta(\Delta_{2n}^-))=L\tau_n^-\wedge 1$. Hence it follows from Lemma 1.8 that $\delta(1\otimes\beta(\Delta_{2n}^-))=-1$.

ii) Immediate from i) and Lemma 3.3, iii), because $\delta I=0$.

iii), iv) By Lemma 3.2 we have $\delta(I(\beta(\varepsilon_{2n}))(1\otimes\beta(\Delta_{2n}^+)))=-\beta(\varepsilon_{2n})\delta(1\otimes\beta(\Delta_{2n}^+))$. Because of this, the first formula of iii) follows from i) and Lemma 3.3, iv). From this, i) and Lemma 3.3, iii), we get iv) by using the equality $\delta(I(\beta(\delta_{2n})(1\otimes\beta(\Delta_{2n}^+)))=-\beta(\delta_{2n})\delta(1\otimes\beta(\Delta_{2n}^+))$. The rest follows from the equality $\delta(I(\beta(\delta_{2n}))(1\otimes\beta(\Delta_{2n}^-)))=-\beta(\delta_{2n})\delta(1\otimes\beta(\Delta_{2n}^-))$, the above results and Lemma 3.3, iii).

v) Similarly this follows from the equality $\delta(I(\beta(\delta_{2n}))(\nu_{2n-1}\otimes\beta(\Delta_{2n}^-))=-\beta(\delta_{2n})\delta(\nu_{2n-1}\otimes\beta(\Delta_{2n}^-))$. q.e.d.

Theorem 3.5 (cf. [6, 8, 10, 15]).

$$K^*(SO(2n))=\Lambda(\beta(\lambda^1\rho_{2n}),\cdots,\beta(\lambda^{n-2}\rho_{2n}),\beta(\delta_{2n}),\beta(\varepsilon_{2n}))\otimes(Z\cdot 1\oplus Z_{2^{n-1}}\cdot\xi_{2n})$$

as a ring with relations

$$\xi_{2n}^2=-2\xi_{2n},\quad \beta(\varepsilon_{2n})\otimes\xi_{2n}=0\,.$$

Proof. Since I is injective by Lemma 3.2 and multiplicative, we see by (1.5) and Lemma 3.3, iv) that the relations hold. Using Lemma 3.3 we also see by Propositions 2.1 and 2.3 that the right-hand side R of the desired equality is a subring of $KO^*(SO(2n))$. Let m be a monomial of ξ_{2n} and $\beta(\lambda^k\rho_{2n})$ $(1\leqslant k\leqslant n-2)$. Then by Lemma 3.3, i) and ii) we have $\delta(I(m)y)=m\delta(y)$, $y\in K^*(P^{2n-1})\otimes K^*(\mathrm{Spin}(2n))$, up to sign. Using this, it follows from Lemma 3.4 that $K^*(SO(2n))=R$, because δ is surjective by Lemma 3.2. q.e.d.

Next we consider $K^*(SO(2n-1))$. Observe the exact sequence of the pair $(\Sigma^{2,0}\wedge\mathrm{Spin}(2n-1)_+,\ \Sigma^{1,0}\wedge\mathrm{Spin}(2n-1)_+)$ in the equivariant K-theory, then using an equivariant homeomorphism of [12], Lemma 4.1 we have the following exact sequence

$$(3.6)\qquad \cdots\to\tilde{K}^*(S^2\wedge\mathrm{Spin}(2n-1)_+)\xrightarrow{\delta^*}\tilde{K}_G^*(\Sigma^{2,0}\wedge\mathrm{Spin}(2n-1)_+)$$
$$\xrightarrow{\chi}\tilde{K}_G^*(\Sigma^{1,0}\wedge\mathrm{Spin}(2n-1)_+)\xrightarrow{\psi}K^*(S^1\wedge\mathrm{Spin}(2n-1)_+)\to\cdots.$$

Here we can check easily that δ^* agrees with the transfer. Since $\psi\delta^*=2$ and $K^*(\mathrm{Spin}(2n-1))$ is torsion free by Proposition 2.3, we see that δ^* is injective and hence χ is surjective.

From this, for any $x\in\tilde{K}_G^*(\Sigma^{2n-1,0}\wedge\mathrm{Spin}(2n-1)_+)$ we may write $x=\tau_{n-1}^+$

$\wedge\chi(\tau_1^+\wedge y)$ for some $y\in K^*(SO(2n-1))$. Let J: $\tilde{K}_G^*(\Sigma^{2n-1,0}\wedge \text{Spin}(2n-1)_+)\to\tilde{K}^*(SO(2n-1))$ be as in (3.1). Then by definition we obtain $J(x)=-2^{n-1}\xi_{2n-1}y$, which shows by (1.5) that J is zero. Thus by (3.1) we obtain

Lemma 3.7.

$$0\leftarrow \tilde{K}_G^*(\Sigma^{1,0}\wedge \text{Spin}(2n-1)_+)\overset{\delta}{\leftarrow} K^*(P^{2n-2})\otimes K^*(\text{Spin}(2n-1)) \overset{I}{\leftarrow} K^*(SO(2n-1))\leftarrow 0$$

is exact where $\delta=\phi_{2n-2}^{-1}\delta$, *and* $\delta(I(x)-)=(-1)^i x\delta(-)$ *for* $x\in K^i(SO(2n-1))$.

Consider the following commutative diagram with the exact rows as in Lemmas 3.2 and 3.7

$$\begin{array}{ccccccc}
0\leftarrow K^*(SO(2n)) & \overset{\delta}{\leftarrow} & K^*(P^{2n-1})\otimes K^*(\text{Spin}(2n)) & \overset{I}{\leftarrow} & K^*(SO(2n)) & \leftarrow 0 \\
\downarrow r\phi_2 & & \downarrow r & & \downarrow r & \\
0\leftarrow \tilde{K}_G^*(\Sigma^{1,0}\wedge \text{Spin}(2n-1)_+) & \overset{\delta}{\leftarrow} & K^*(P^{2n-2})\otimes K^*(\text{Spin}(2n-1)) & \overset{I}{\leftarrow} & K^*(SO(2n-1)) & \leftarrow 0
\end{array}$$

where r's denote the homomorphisms induced by the natural inclusions. Apply r (resp. $r\phi_2$) to the formulas of Lemma 3.3 (resp. Lemma 3.4) then we obtain

Lemma 3.8. i) $I(\xi_{2n-1})=\gamma_{2n-2}\otimes 1$,

ii) $I(\beta(\lambda^k\rho_{2n-1}))=1\otimes\beta(\lambda^k\rho_{2n-1})\quad(1\leqslant k\leqslant 2n-1)$,

iii) $I(\beta(\varepsilon_{2n-1}))=(2+\gamma_{2n-2})\otimes\beta(\Delta_{2n-1})$.

Lemma 3.9. $\delta(1\otimes\beta(\Delta_{2n-1}))=-\chi(\tau_1^+\wedge 1)=L\chi(\tau_1^+\wedge 1)$ *and* $\chi(\tau_1^+\wedge\beta(\varepsilon_{2n-1}))=0$ *where* χ *is as in* (3.6).

Theorem 3.10 (cf. [6, 8, 10, 15]).

$$K^*(SO(2n-1))=\Lambda(\beta(\lambda^1\rho_{2n-1}),\cdots,\beta(\lambda^{n-2}\rho_{2n-1}),\beta(\varepsilon_{2n-1}))\otimes(Z\cdot 1\oplus Z_{2^{n-1}}\cdot\xi_{2n-1})$$

as a ring with relations

$$\xi_{2n-1}^2=-2\xi_{2n-1},\quad \beta(\varepsilon_{2n-1})\otimes\xi_{2n-1}=0\,.$$

Proof. Similar to the proof of Theorem 3.5 except the fact that $K^*(SO(2n-1))$ is generated by 1, $\beta(\lambda^k\rho_{2n-1})$ $(1\leqslant k\leqslant n-2)$, $\beta(\varepsilon_{2n-1})$ and ξ_{2n-1}.

Let $m=m(b_1,\cdots,b_{2n-1})=\beta(\lambda^1\rho_{2n-1})^{b_1}\cdots\beta(\lambda^{n-2}\rho_{2n-1})^{b_{n-2}}(b_1,\cdots,b_{n-2}=0,1)$. Since δ is surjective by Lemma 3.7 it follows from Lemmas 3.8, 3.9 and the equality of Lemma 3.7 that $\tilde{K}_G^*(\Sigma^{1,0}\wedge\text{Spin}(2n-1)_+)$ is generated by $\chi(\tau_1^+\wedge$

$m(b_1, \cdots, b_{n-2}))$ $(b_1, \cdots, b_{n-2}=0, 1)$ as a module. To prove the rest, by (3.6) it therefore suffices to prove that the image of $\phi_2^{-1}\delta^*$: $\tilde{K}^*(S^2\wedge \mathrm{Spin}(2n-1)_+)$ $\to K_G^*(\mathrm{Spin}(2n-1))=K^*(SO(2n-1))$ is contained in the right-hand side of the required equality. Now a short computation shows that $\phi_2^{-1}\delta^*(\psi(\tau_1^+\wedge m))$ $=(2+\xi_{2n-1})m$ and $\phi_2^{-1}\delta^*(\psi(\tau_1^+)\wedge m\beta(\Delta_{2n-1}))=m\beta(\varepsilon_{2n-1})$. This completes the proof.

4. $KO^*(SO(8n))$, $KO^*(SO(8n-1))$ and $KO^*(SO(8n+1))$

The calculation of $KO^*(SO(n))$ proceeds in a manner parallel to that of $K^*(SO(n))$. As before we have by (1.3) the following short exact sequences.

Lemma 4.1.

i) $0 \leftarrow KO^*(SO(8n)) \overset{\delta}{\leftarrow} KO^*(P^{8n-1})\otimes_{KO^*(+)}KO^*(\mathrm{Spin}(8n)) \overset{I}{\leftarrow} KO^*(SO(8n)) \leftarrow 0\,,$

ii) $0 \leftarrow \widetilde{KO}_G^*(\Sigma^{7,0}\wedge \mathrm{Spin}(8n-1)_+) \overset{\delta}{\leftarrow} KO^*(P^{8n-2})\otimes_{KO^*(+)}KO^*(\mathrm{Spin}(8n-1)) \overset{I}{\leftarrow} KO^*(SO(8n-1)) \leftarrow 0\,,$

iii) $0 \leftarrow \widetilde{KO}_G^*(\Sigma^{1,0}\wedge \mathrm{Spin}(8n+1)_+) \overset{\delta}{\leftarrow} KO^*(P^{8n})\otimes_{KO^*(+)}KO^*(\mathrm{Spin}(8n+1)) \overset{I}{\leftarrow} KO^*(SO(8n+1)) \leftarrow 0$

where $\delta=\phi_{8n}^{-1}\delta$, $\delta=\phi_{8n-8}^{-1}\delta$, $\delta=\phi_{8n}^{-1}\delta$ *respectively.*

Proof. It is enough to show that J: $\widetilde{KO}_G^*(\Sigma^{8n+i,0}\wedge \mathrm{Spin}(8n+i)_+)\to KO^*(SO(8n+i))$ is zero for $i=0, -1, 1$. Because of $J(\omega_n^+\wedge 1)=-2^{4n-1}\xi_{8n}$, i) follows from (1.5) immediately. But we postpone to prove ii), iii), whose proofs will follow after Lemmas 4.10, 4.14 respectively. q.e.d.

We note here that also there holds

(4.2) $\delta(I(x)-) = (-1)^i x\delta(-)$ for $x\in KO^i(SO(8n+j)), j=0, -1, 1\,.$

First we consider $KO^*(SO(8n))$. We have

Lemma 4.3. i) $I(\xi_{8n}) = \gamma_{8n-1}\otimes 1\,,$

ii) $I(\beta(\lambda^k\rho_{8n})) = 1\otimes\beta(\lambda^k\rho_{8n})+\varepsilon\eta_1\gamma_{8n-1}\otimes 1$ $(\varepsilon = 0, 1, 0\leqslant k\leqslant 8n)\,,$

iii) $I(\beta(\delta_{8n})) = 1\otimes(\beta(\Delta_{8n}^+)-\beta(\Delta_{8n}^-))-\nu_{8n-1}\otimes 1\,,$

iv) $I(\beta(\varepsilon_{8n})) = (\gamma_{8n-1}+2)\otimes\beta(\Delta_{8n}^+)-\nu_{8n-1}\otimes 1\,.$

Proof. Similar to the proof of Lemma 3.3. But in the proof of ii) it does not necessarily follow that $\beta((\lambda^k\rho_{8n})\iota)=0$ because $\widetilde{KO}^{-1}(P^{8n})=Z_2\cdot\eta_1\gamma_{8n}$. q.e.d.

Let $p=0$, $m=q=8n$ and $i=-2$ in (1.3) when $h=KO$. Then as it follows from the calculation of Proposition 2.2, 1) that μ_{8n-1} satisfies $\delta(\mu_{8n-1}\otimes 1)=\eta_1\omega_n^+\wedge 1$ we get

$$\delta(\mu_{8n-1}\otimes 1)=\eta_1 \tag{4.4}$$

Furthermore by the same proof as that of Lemma 3.4, i) we obtain

$$\delta(1\otimes\beta(\Delta_{8n}^-))=-1\,. \tag{4.5}$$

By (4.4) and (4.5) $\delta(\eta_1\otimes\beta(\Delta_{8n}^-)+\mu_{8n-1}\otimes 1)=0$, which shows that there exist an element $\zeta_{8n}\in\widetilde{KO}^{-2}(SO(8n))$ such that

$$I(\zeta_{8n})=\eta_1\otimes\beta(\Delta_{8n}^-)+\mu_{8n-1}\otimes 1\,. \tag{4.6}$$

For $\lambda^2\Delta_{8n}^+$ viewed as a real representation of $SO(8n)$ we have $I(\beta(\lambda^2\Delta_{8n}^+))=1\otimes\beta(\lambda^2\Delta_{8n}^+)+\varepsilon\eta_1\gamma_{8n-1}\otimes 1$ ($\varepsilon=0$, 1) by the same argument as Lemma 4.3, ii). Since $RO(\mathrm{Spin}(8n))\cong R(\mathrm{Spin}(8n))$, using Theorem 10.3 of [11], §13 and the real version of (2) of [9], I §4 we see that $\eta_1\beta(\lambda^2\Delta_{8n}^+)$, as an element of $KO^*(\mathrm{Spin}(8n))$, is a linear combination of $\eta_1\beta(\lambda^1\rho_{8n})^{b_1}\cdots\beta(\lambda^{4n-2}\rho_{8n})^{b_{4n-2}}$ ($b_1,\cdots,b_{4n-2}=0, 1$). Hence using i), ii) of Lemma 4.3 we see that $\eta_1\beta(\lambda^2\Delta_{8n}^+)$ is a linear combination of $\eta_1^i\xi_{8n}^a\beta(\lambda^1\rho_{8n})^{b_1}\cdots\beta(\lambda^{4n-2}\rho_{8n})^{b_{4n-2}}$ ($a, b_1,\cdots,b_{4n-2}=0, 1, i=1, 2$), since I is injective. For $\lambda^2\Delta_{8n}^-$ we also have the same result. Considering the restriction of $\eta_1\beta(\lambda^2\Delta_{8n}^+)$ to $KO^*(SO(8n-1))$ we see that a similar result holds for $\eta_1\beta(\lambda^2\Delta_{8n-1})$.

Then since I is injective and multiplicative we obtain by Lemma 4.3 the following

Proposition 4.7. *In $KO^*(SO(8n))$ the following relations hold*:

i) $\xi_{8n}^2=-2\xi_{8n}$

ii) $\beta(\delta_{8n})^2=\beta(\varepsilon_{8n})^2=\zeta_{8n}^2=0$

iii) $\eta_1^2\zeta_{8n}=2^{4n-2}\eta_4\xi_{8n}$, $\eta_4\zeta_{8n}=0$

iv) $\xi_{8n}\beta(\varepsilon_{8n})=0$

Proof. i) Immediate from (1.5).

ii) When we replace n by $4n$ in the proof of Lemma 1.8 we see that $\beta(\delta)$ defines an element of $\widetilde{KO}^{-1}(\mathrm{Spin}(8n)/\mathrm{Spin}(8n-1))$. Let p: $\mathrm{Spin}(8n)\to\mathrm{Spin}(8n)/\mathrm{Spin}(8n-1)$ be the natural projection. Then $p^*(\beta(\delta))=\beta(\Delta_{8n}^+)-\beta(\Delta_{8n}^-)$ by definition. Since $\mathrm{Spin}(8n)/\mathrm{Spin}(8n-1)\approx S^{8n-1}$, we have $\beta(\delta)^2=0$ and hence $\beta(\Delta_{8n}^+)^2+\beta(\Delta_{8n}^-)^2=0$.

From Lemma 4.3, iii) it follows that $I(\beta(\delta_{8n})^2)=1\otimes(\beta(\Delta_{8n}^+)^2+\beta(\Delta_{8n}^-)^2)+\nu_{8n-1}^2\otimes 1$. This shows $\beta(\varepsilon_{8n})^2=0$ because $\nu_{8n-1}^2=0$ by Proposition 2.2, 1).

Similarly we have $I(\beta(\varepsilon_{8n})^2)=(\gamma_{8n-1}+2)\otimes 2\beta(\Delta_{8n}^{+})^2+\nu_{8n-1}^2\otimes 1$. By Proposition 2.4 $\beta(\Delta_{8n}^{+})^2$ is divisible by η_1. Therefore we get $\beta(\varepsilon_{8n})^2=0$. The last formula follows analogously as in the second because $\mu_{8n-1}^2=0$ by Proposition 2.2, 1).

iii, iv) The proofs are parallel to the case ii), so we omit them. q.e.d.

By the arguments applied to Lemma 3.4 and by (4.2), (4.4) and the above observation of $\beta(\lambda^2\Delta_{8n}^{+})$ and $\beta(\lambda^2\Delta_{8n}^{-})$ we obtain

Lemma 4.8. i) $\delta(1\otimes\beta(\Delta_{8n}^{+}))=\xi_{8n}+1,\ \delta(1\otimes\beta(\Delta_{8n}^{-}))=-1$,

ii) $\delta(1\otimes\beta(\Delta_{8n}^{+})\beta(\Delta_{8n}^{-}))=(\xi_{8n}+1)(\beta(\varepsilon_{8n})-\beta(\delta_{8n}))$,

iii) $\delta(\nu_{8n-1}\otimes 1)=\xi_{8n}+2$,

iv) $\delta(\nu_{8n-1}\otimes\beta(\Delta_{8n}^{+}))=(\xi_{8n}+1)\beta(\varepsilon_{8n})$,
$\delta(\nu_{8n-1}\otimes\beta(\Delta_{8n}^{-}))=(\xi_{8n}+1)\beta(\varepsilon_{8n})-(\xi_{8n}+2)\beta(\delta_{8n})$,

v) $\delta(\nu_{8n-1}\otimes\beta(\Delta_{8n}^{+})\beta(\Delta_{8n}^{-}))=(\xi_{8n}+1)\beta(\delta_{8n})\beta(\varepsilon_{8n})+\eta_1\xi_{8n}\beta(\lambda^2\Delta_{8n}^{+})$,

vi) $\delta(\mu_{8n-1}\otimes 1)=\eta_1$,

vii) $\delta(\mu_{8n-1}\otimes\beta(\Delta_{8n}^{+}))=\eta_1(\xi_{8n}+1)(\beta(\varepsilon_{8n})-\beta(\delta_{8n}))+(\xi_{8n}+1)\zeta_{8n}$,
$\delta(\mu_{8n-1}\otimes\beta(\Delta_{8n}^{-}))=\zeta_{8n}$,

viii) $\delta(\mu_{8n-1}\otimes\beta(\Delta_{8n}^{+})\beta(\Delta_{8n}^{-}))=(\xi_{8n}+1)\zeta_{8n}(\beta(\varepsilon_{8n})-\beta(\delta_{8n}))$
$+\eta_1^2(\xi_{8n}+1)\beta(\lambda^2\Delta_{8n}^{-})$,

ix) $\delta(\gamma_{8n-1}\otimes 1)=\delta(1\otimes\beta(\lambda^{i_1}\rho_{8n})\cdots\beta(\lambda^{i_s}\rho_{8n}))=0$
$(1\leqslant i_1<\cdots<i_s\leqslant 8n)$.

Theorem 4.9.

$$KO^*(SO(8n))=\Lambda_{KO^*(+)}(\beta(\lambda^1\rho_{8n}),\cdots,\beta(\lambda^{4n-2}\rho_{8n}),\beta(\delta_{8n}),\beta(\varepsilon_{8n}))\otimes_Z(Z\cdot 1\oplus Z_{2^{4n-1}}\cdot\xi_{8n}\oplus Z_2\cdot\zeta_{8n})$$

as a $KO^*(+)$-module with the ring structure given by

$$\beta(\lambda^k\rho_{8n})^2=\eta_1(\beta(\lambda^2(\lambda^k\rho_{8n}))+\binom{8n}{k}\beta(\lambda^k\rho_{8n}))\quad(1\leqslant k\leqslant 4n-2)$$

and the relations of Proposition 4.7 *in which* $\otimes_Z$ *is left out.*

Proof. The first relation follows from (1.7). The rest is quite similar to the proof of Theorem 3.5. q.e.d.

We next consider $KO^*(SO(8n-1))$. Define $\zeta_{8n-1}\in\widetilde{KO}^{-2}(SO(8n-1))$ by $i^*(\zeta_{8n})=\zeta_{8n-1}$ where i: $SO(8n-1)\to SO(8n)$ is the inclusion. As in the complex case by Lemma 4.3 and (4.6) we have

Lemma 4.10. i) $I(\xi_{8n-1})=\gamma_{8n-2}\otimes 1$,

ii) $I(\beta(\lambda^k\rho_{8n-1}))=1\otimes\beta(\lambda^k\rho_{8n-1})+\varepsilon\eta_1\gamma_{8n-2}\otimes 1\quad(\varepsilon=0,1,\ 0\leqslant k\leqslant 8n-1)$,

iii) $I(\beta(\varepsilon_{8n-1}))=(\gamma_{8n-2}+2)\otimes\beta(\Delta_{8n-1})$,

iv) $I(\zeta_{8n-1})=\eta_1\otimes\beta(\Delta_{8n-1})+\mu_{8n-2}\otimes 1$.

Proof of Lemma 4.1, ii). As we got (3.6) we have an exact sequence

$$\cdots\to\widetilde{KO}^*(S^8\wedge \mathrm{Spin}(8n-1)_+)\xrightarrow{\delta^*}\widetilde{KO}_G^*(\Sigma^{8,0}\wedge\mathrm{Spin}(8n-1)_+)$$
$$\xrightarrow{\chi}\widetilde{KO}_G^*(\Sigma^{7,0}\wedge\mathrm{Spin}(8n-1)_+)\xrightarrow{\psi}\widetilde{KO}^*(S^7\wedge\mathrm{Spin}(8n-1)_+)\to\cdots$$

where δ^* also agrees with the transfer and ψ denotes the forgetful homomorphism. Observe the composition $I\phi_8^{-1}\delta^*$. (Here $KO_G^*(\mathrm{Spin}(8n-1))$ is of course identified with $KO^*(SO(8n-1))$ as noted in §1, a).) As before we have by definition $\delta^*(\psi(\omega_1^+\wedge m))=\omega_1^+\wedge(\xi_{8n-1}+2)m$ and $\delta^*(\psi(\omega_1^+\wedge 1)(1\wedge m\beta(\Delta_{8n-1}))=\omega_1^+\wedge m\beta(\varepsilon_{8n-1})$ where m is a monomial of $\beta(\lambda^k\rho_{8n-1})$ $(1\leqslant k\leqslant 4n-2)$. Using Lemma 4.10 we therefore see that $I\phi_8^{-1}\delta^*$ is injective. This implies that δ^* is injective and so χ is surjective. Hence for any $x\in\widetilde{KO}_G^*(\Sigma^{8n-1,0}\wedge\mathrm{Spin}(8n-1)_+)$ we may write $x=\omega_{n-1}^+\wedge\chi(\omega_1^+\wedge y)$ for some $y\in KO^*(SO(8n-1))$. Thus by (1.5) $J(x)=-2^{4n-1}\xi_{8n-1}y=0$, which completes the proof.

From the injectivity of I and Lemma 4.10 we obtain

Proposition 4.11. *In $KO^*(SO(8n-1))$ the following relations hold*:

i) $\xi_{8n-1}^2=-2\xi_{8n-1}$

ii) $\beta(\varepsilon_{8n-1})^2=\zeta_{8n-1}^2=0$

iii) $\eta_1^2\zeta_{8n-1}=2^{4n-2}\eta_4\xi_{8n-1},\ \eta_4\zeta_{8n-1}=0$

iv) $\xi_{8n-1}\zeta_{8n-1}=\eta_1\beta(\varepsilon_{8n-1})$

v) $\beta(\varepsilon_{8n-1})\zeta_{8n-1}=\eta_1^2\xi_{8n-1}\beta(\lambda^2\Delta_{8n-1})$

vi) $\xi_{8n-1}\beta(\varepsilon_{8n-1})=0$

Let $\chi\colon\widetilde{KO}_G^*(\Sigma^{8,0}\wedge\mathrm{Spin}(8n-1)_+)\to\widetilde{KO}_G^*(\Sigma^{7,0}\wedge\mathrm{Spin}(8n-1)_+)$ be as in the proof of Lemma 4.1, ii). Under the identification $KO_G^*(\mathrm{Spin}(8n-1))=KO^*(SO(8n-1))$ we obtain by Lemma 4.8 the following

Lemma 4.12. i) $\delta(1\otimes\beta(\Delta_{8n-1}))=-\chi(\omega_1^+\wedge 1)=H\chi(\omega_1^+\wedge 1)$,

ii) $\delta(\mu_{8n-2}\otimes 1)=\eta_1\chi(\omega_1^+\wedge 1)$,

iii) $\delta(\mu_{8n-2}\otimes\beta(\Delta_{8n-1}))=\chi(\omega_1^+\wedge\zeta_{8n-1})=H\chi(\omega_1^+\wedge\zeta_{8n-1})$,

iv) $\delta(\gamma_{8n-2}\otimes 1)=\delta(1\otimes\beta(\lambda^{i_1}\rho_{8n})\cdots\beta(\lambda^{i_s}\rho_{8n}))=0\quad(1\leqslant i_1<\cdots<i_s\leqslant 8n-1)$,

v) $\chi(\omega_1^+\wedge\beta(\varepsilon_{8n-1}))=0$.

Using (4.2) we see by Lemmas 4.10 and 4.12 that $\widetilde{KO}_G^*(\Sigma^{7,0}\wedge\mathrm{Spin}(8n-1)_+)$ is generated by $\chi(\omega_1^+\wedge\beta(\lambda^1\rho_{8n-1})^{b_1}\cdots\beta(\lambda^{4n-2}\rho_{8n-1})^{b_{4n-2}}\zeta_{8n-1}^c)$ $(b_1,\cdots,b_{4n-2},c=0,1)$ as a $KO^*(+)$-module. Therefore the parallel argument as Theorem 3.10 yields

Theorem 4.13.

$$KO^*(SO(8n-1)) = \Lambda_{KO^*(+)}(\beta(\lambda^1\rho_{8n-1}), \cdots, \beta(\lambda^{4n-2}\rho_{8n-1}), \beta(\varepsilon_{8n-1})) \otimes_Z (Z\cdot 1 \oplus Z_{2^{4n-1}}\cdot\xi_{8n-1} \oplus Z_2\cdot\zeta_{8n-1})$$

as a $KO^*(+)$*-module with the ring structure given by*

$$\beta(\lambda^k\rho_{8n-1})^2 = \eta_1(\beta(\lambda^2(\lambda^k\rho_{8n-1})) + \binom{8n-1}{k}\beta(\lambda^k\rho_{8n-1})) \quad (1 \leqslant k \leqslant 4n-2)$$

and the relations of Proposition 4.11 *in which* $\otimes_Z$ *is left out.*

Finally we discuss $KO^*(SO(8n+1))$.

Lemma 4.14. i) $I(\xi_{8n+1}) = \gamma_{8n}\otimes 1$,

ii) $I(\beta(\lambda^k\rho_{8n+1})) = 1\otimes\beta(\lambda^k\rho_{8n+1}) + \varepsilon\eta_1\gamma_{8n}\otimes 1, \quad (\varepsilon = 0, 1, 0 \leqslant k \leqslant 8n+1)$,

iii) $I(\beta(\varepsilon_{8n+1})) = (\gamma_{8n}+2)\otimes\beta(\Delta_{8n+1})$.

Proof. Similar to the proofs of Lemma 3.3, i), ii), iv). But we have a supplemental term in ii) by the same reason as in Lemma 4.3, ii). As for vanishing of the corresponding element α in iii) to $\beta(f)$ in the proof of Lemma 3.3, iv), we have $i^*(\alpha)=0$ by the definition of α and Proposition 2.2 where i denotes the inclusion $P^{8n-2}\subset P^{8n}$. Hence we get $\alpha=0$ because $i^*\colon \widetilde{KO}^{-1}(P^{8n}) \to \widetilde{KO}^{-1}(P^{8n-2})$ is an isomorphism. q.e.d.

Proof of Lemma 4.1, iii). Consider the exact sequence of the pair $(\Sigma^{1,0}, \Sigma^{0,0})$ in the equivariant K-theory, then it is clear that $i^*\colon \tilde{K}_G(\Sigma^{1,0}) \to \tilde{K}_G(\Sigma^{0,0}) = R(G)$ is injective and so $\tilde{K}_G(\Sigma^{1,0}) = Z\cdot(1-L)$ as a subgroup of $R(G)$ where i is the inclusion $\Sigma^{0,0}\subset\Sigma^{1,0}$. Similarly $\widetilde{KO}_G(\Sigma^{1,0}) = Z\cdot(1-H)$ as a subgroup of $RO(G)$.

As we saw in the proof of Theorem 3.10 $\tilde{K}^*_G(\Sigma^{1,0}\wedge \mathrm{Spin}(8n+1)_+)$ is generated by $\chi(\tau_1^+\wedge m(b_1, \cdots, b_{4n-1}))$ $(b_1, \cdots, b_{4n-1} = 0, 1)$ where $\chi = (i\wedge 1)^*$ and $m(b_1, \cdots, b_{4n-1}) = \beta(\lambda^1\rho_{8n+1})^{b_1} \cdots, \beta(\lambda^{4n-1}\rho_{8n+1})^{b_{4n-1}}$. Since $i^*(\tau_1^+) = 1-L$ we have $\chi(\tau_1^+\wedge 1) = (1-L)\wedge 1$.

Here we regard $K^*_G(-)$ as a cohomology theory over $R(G)\otimes Z[\mu]/\mu^4=1$ where $\mu\in K^{-2}(+)$ is the Bott class. Let c and r denote the complexification homomorphism and the realification homomorphism respectively. Let $x\in \widetilde{KO}^*_G(\Sigma^{1,0}\wedge \mathrm{Spin}(8n+1)_+)$. Then because of $r(\mu)=\eta_1^2$, $r(\mu^2)=\eta_4$, $r(\mu^3)=0$ and $rc=2$, we see that $2x$ is a linear combination of $2(1-H)\wedge m(b_1, \cdots, b_{4n-1})$, $(1-H)\wedge\eta_1^2 m(b_1, \cdots, b_{4n-1})$ and $(1-H)\wedge\eta_4 m(b_1, \cdots, b_{4n-1})$. Let $\chi = (i\wedge 1)^*\colon \widetilde{KO}^*_G(\Sigma^{1,0}\wedge \mathrm{Spin}(8n+1)_+) \to \widetilde{KO}^*_G(\Sigma^{0,0}\wedge \mathrm{Spin}(8n+1)_+) = \widetilde{KO}^*(SO(8n+1))$. Then we therefore have

$$2\chi(x) = \xi_{8n+1}(2\Sigma\alpha(b_1, \cdots, b_{4n-1})m(b_1, \cdots, b_{4n-1})$$

$$+\Sigma\beta(b_1, \cdots, b_{4n-1})\eta_1^2 m(b_1, \cdots, b_{4n-1})$$
$$+\Sigma\gamma(b_1, \cdots, b_{4n-1})\eta_4 m(b_1, \cdots, b_{4n-1}))$$

for $\alpha(b_1, \cdots, b_{4m-1})$, $\gamma(b_1, \cdots, b_{4n-1})\in Z$ and $\beta(b_1, \cdots, b_{4n-1})=0, 1$ where Σ denotes the finite sum with respect to $(b_1, \cdots, b_{4n-1})$. From this, because of $J(\omega_n^+\wedge x) = -2^{4n-1}\xi_{8n+1}\chi(x)$ and (1.5), we have

$$J(\omega_n^+\wedge x) = 2^{4n-1}\eta_4\xi_{8n+1}\Sigma\gamma(b_1, \cdots, b_{4n-1})m(b_1, \cdots, b_{4n-1})\,.$$

Since $IJ=0$ we have by Lemma 4.14

$$2^{4n-1}\eta_4\xi_{8n+1}\otimes\Sigma\gamma(b_1, \cdots, b_{4n-1})m(b_1, \cdots, b_{4n-1}) = 0\,.$$

By Propositions 2.2 and 2.4, $\eta_4\gamma_{8n}$ is of order 2^{4n} and $KO^*(\mathrm{Spin}(8n+1))$ is torsion free. Thus we see that $\gamma(b_1, \cdots, b_{4n-1})$ are divisible by 2. Therefore $J(\omega_n^+\wedge x)=0$ by (1.5). This proves that J is zero. q.e.d.

As above we regard $\widetilde{KO}_G(\Sigma^{1,0})=Z\cdot(1-H)$ as a subgroup of $RO(G)$ in the following. To calculate $\delta(1\otimes\beta(\Delta_{8n+1}))$ we mimic the proof of Lemma 3.4, i). Then we see that $\bar{\delta}(1\otimes\beta(\Delta_{8n+1}))$ is defined as an element of $\widetilde{KO}_G(\Sigma^{8n+1,0})$. Moreover $\bar{\delta}(1\otimes\beta(\Delta_{8n+1}))=\bar{\delta}(1\otimes\beta(\Delta_{8n}^+))+\bar{\delta}(1\otimes\beta(\Delta_{8n}^-))$ as an element of $\widetilde{KO}_G(\Sigma^{8n,0})$. Hence using Lemma 3.4, i) we have $\bar{\delta}(1\otimes\beta(\Delta_{8n+1}))=(H-1)\wedge\omega_n^+$, that is,

$$\delta(1\otimes\beta(\Delta_{8n+1})) = (H-1)\wedge 1\,. \tag{4.15}$$

Let $\delta^*\colon KO^{-2}(P^{8n}) = \widetilde{KO}_G^{-2}(S_+^{8n+1,0}) \to \widetilde{KO}_G^{-1}(\Sigma^{8n+1,0})$ be the coboundary homomorphism of the exact sequence of the pair $(B_+^{8n+1,0}, S_+^{8n+1,0})$ in the equivariant KO-theory. Then it follows that one generator $\bar{\nu}_{8n}$ of $\widetilde{KO}^{-2}(P^{8n})$ as in Proposition 2.2 is characterized by $\delta^*(\bar{\nu}_{8n})=\eta_1(H-1)\wedge\omega_n^+$. Therefore we obtain

$$\delta(\bar{\nu}_{8n}\otimes 1) = \eta_1(H-1)\wedge 1\,. \tag{4.16}$$

From (4.14) and (4.15) it follows that $\delta(\eta_1\otimes\beta(\Delta_{8n+1})+\bar{\nu}_{8n}\otimes 1)=0$. Hence there exists an element ν_{8n+1} of $KO^*(SO(8n+1))$ such that

$$I(\nu_{8n+1}) = \eta_1\otimes\beta(\Delta_{8n+1})+\bar{\nu}_{8n}\otimes 1\,. \tag{4.17}$$

By the injectivity of I and Lemma 4.14 we then have

Proposition 4.18. *In $KO^*(SO(8n+1))$ the following relations hold:*

i) $\xi_{8n+1}^2 = -2\xi_{8n+1}$

ii) $\beta(\varepsilon_{8n+1})^2 = \nu_{8n+1}^2 = 0$

iii) $\eta_1^2\nu_{8n+1} = 2^{4n-1}\eta_4\xi_{8n+1},\ \eta_4\nu_{8n+1} = 0$

iv) $\beta(\varepsilon_{8n+1})v_{8n+1} = \eta_1^2\xi_{8n+1}\beta(\lambda^2\Delta_{8n+1})$

v) $\xi_{8n+1}v_{8n+1} = \eta_1\beta(\varepsilon_{8n+1})$

vi) $\xi_{8n+1}\beta(\varepsilon_{8n+1}) = 0$

Here we note that we have the similar remark to $\beta(\lambda^2\Delta_{8n+1})$ as $\beta(\lambda^2\Delta_{8n}^+)$, that is, $\eta_1\beta(\lambda^2\Delta_{8n+1})$ is represented as a linear combination of $\eta_1^i\xi_{8n+1}^a\beta(\lambda^1\rho_{8n+1})^{b_1} \cdots \beta(\lambda^{4n-1}\rho_{8n+1})^{b_{4n-1}}$ (a, b_1, $\cdots$, $b_{4n-1}=0$, 1, $i=1$, 2). Using this and (4.2), by Lemma 4.14, (4.17), (4.15) and (4.16) we obtain

Lemma 4.19. i) $\delta(1\otimes\beta(\Delta_{8n+1})) = (H-1)\wedge 1$,

ii) $\delta(\bar{\nu}_{8n}\otimes 1) = \eta_1(H-1)\wedge 1$,

iii) $\delta(\bar{\nu}_{8n}\otimes\beta(\Delta_{8n+1})) = (H-1)\wedge v_{8n+1}$,

iv) $\delta(\gamma_{8n}\otimes 1) = \delta(1\otimes\beta(\lambda^{i_1}\rho_{8n+1}) \cdots \beta(\lambda^{i_s}\rho_{8n+1})) = 0$
$(1\leqslant i_1< \cdots <i_s\leqslant 8n+1)$,

v) $(H-1)\wedge\beta(\varepsilon_{8n+1}) = 0$.

From this we see that $\widetilde{KO}_G^*(\Sigma^{1,0}\wedge \mathrm{Spin}(8n+1)_+)$ is generated by $(1-H)\wedge\beta(\lambda^1\rho_{8n+1})^{b_1} \cdots \beta(\lambda^{4n-1}\rho_{8n+1})^{b_{4n-1}}v_{8n+1}^c$ (b_1, $\cdots$, b_{4n-1}, $c=0$, 1) as a $KO^*(+)$-module.

Theorem 4.20.

$$KO^*(SO(8n+1)) = \Lambda_{KO^*(+)}(\beta(\lambda^1\rho_{8n+1}), \cdots, \beta(\lambda^{4n-1}\rho_{8n+1}), \beta(\varepsilon_{8n+1})) \otimes_Z(Z\cdot 1\oplus Z_{2^{4n}}\cdot\xi_{8n+1}\oplus Z_2\cdot v_{8n+1})$$

as a $KO^*(+)$-module with the ring structure given by

$$\beta(\lambda^k\rho_{8n+1})^2 = \eta_1(\beta(\lambda^2(\lambda^k\rho_{8n+1}))+\binom{8n+1}{k}\beta(\lambda^k\rho_{8n+1})) \quad (1\leqslant k\leqslant 4n-1)$$

and the relations of Proposition 4.18 *in which $\otimes_Z$ is left out.*

Proof. It suffices to prove that $KO^*(SO(8n+1))$ is generated by $\beta(\lambda^k\rho_{8n+1})$ $(1\leqslant k\leqslant 4n-1)$, $\beta(\varepsilon_{8n+1})$, ξ_{8n+1} and v_{8n+1}. Because the rest is proved in the similar way as in the preceding theorems.

Consider the exact sequence of the pair $(\Sigma^{1,0}\wedge \mathrm{Spin}(8n+1)_+, \Sigma^{0,0}\wedge \mathrm{Spin}(8n+1)_+)$ with a commutative diagram

$$\begin{array}{ccccccc} \cdots \to \widetilde{KO}_G^*(\Sigma^{1,0}\wedge \mathrm{Spin}(8n+1)_+) & \xrightarrow{\chi} & KO^*(SO(8n+1)) & \xrightarrow{\pi^*} & KO^*(\mathrm{Spin}(8n+1)) \to \cdots \\ & & c\downarrow & & c\downarrow \\ & & K^*(SO(8n+1)) & \underset{\pi^*}{\to} & K^*(\mathrm{Spin}(8n+1)). \end{array}$$

Let $m=m(b_1, \cdots, b_{4n-1})$ be as in Proof of Lemma 4.1, iii). Take any $x\in KO^*(SO(8n+1))$, then we see by Proposition 2.4 that $\pi^*(x)$ is represented as a linear combination of $m\beta(\Delta_{8n+1})^c$, $\eta_1 m\beta(\Delta_{8n+1})^c$, $\eta_1^2 m\beta(\Delta_{8n+1})^c$ and $\eta_4 m\beta(\Delta_{8n+1})^c$

($c=0, 1$). Compare $c\pi^*(x)$ with $\pi^*c(x)$ using Theorem 3.10 and Proposition 2.3. Then we see that $\pi^*(x)$ is a linear combination of $2^c m\beta(\Delta_{8n+1})^c$, $\eta_1 m\beta(\Delta_{8n+1})^c$, $\eta_1^2\beta(\Delta_{8n+1})^c$ and $\eta_4 m\beta(\Delta_{8n+1})^c$, because of $c(\eta_1)=c(\eta_1^2)=0$, $c(\eta_4)=2\mu^2$ and $\pi^*(\beta(\mathcal{E}_{8n+1}))=2\beta(\Delta_{8n+1})$ where $\mu\in K^{-2}(+)$ is the Bott class. Clearly $2^c m\beta(\Delta_{8n+1})^c$, $\eta_1 m$, $\eta_1^2 m$ and $2^c\eta_4 m\beta(\Delta_{8n+1})^c$ belong to Im π^* and also $\eta_1^i m\beta(\Delta_{8n+1})$ ($i=1, 2$) do so because of $\pi^*(v_{8n+1})=\eta_1\beta(\Delta_{8n+1})$ by (4.17). Therefore if $\pi^*(x)$ has not $a\eta_4\beta(\Delta_{8n+1})$ (a odd) as a monomial, then it follows from the remark following Lemma 4.19 that x belongs to the right-hand side of the required equality.

If $\pi^*(x)$ has such a monomial as above then we see that there exists an element $\tilde{x}$ of $KO^*(SO(8n+1))$ satisfying $c(\tilde{x})=\mu^2 m\beta(\mathcal{E}_{8n+1})+\mu\xi_{8n+1}y$ for some $y\in K^*(SO(8n+1))$. Here again we consider the following exact sequence of Atiyah [2], (3.4)

$$\cdots \to KO^{1-q}(X) \to KO^{-q}(X) \xrightarrow{c} K^{-q}(X) \xrightarrow{\delta} KO^{2-q}(X) \to \cdots$$

when $X=SO(8n+1)$. Since $\delta(\mu z)=r(z)$ for $z\in K^{2-q}(X)$, we have $\eta_1^2 m\beta(\mathcal{E}_{8n+1})=\xi_{8n+1}r(y)$. But applying I to this equality and using Theorem 3.10, Lemma 4.14, Propositions 2.2 and 2.4 we see that such a relation does not hold in $KO^*(SO(8n+1))$. q.e.d.

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