Ōshima, H.
Osaka J. Math.
21 (1984), 765-772

A REMARK ON JAMES NUMBERS OF STIEFEL MANIFOLDS

Dedicated to Professor Nobuo Shimada on his 60th birthday

HIDEAKI ŌSHIMA

(Received July 13, 1983)

1. Introduction

The purpose of this note is to supply a few relations between the unstable and stable James numbers of Stiefel manifolds.

Let F be the field H of the quaternions or the field C of the complex numbers, and d the dimension of F over the field of the real numbers. Let $G(F^n)$ be the symplectic group $Sp(n)$ or the unitary group $U(n)$ according as F is H or C. The stunted quasi-projective space $Q_{n,k}=Q_n/Q_{n-k}$ is a subspace of the Stiefel manifold $O_{n,k}=G(F^n)/G(F^{n-k})$ (see e.g. [8]). There exist the quotient maps q_r: $Q_{n,k}\to Q_{n,k-r}$ and p_r: $O_{n,k}\to O_{n,k-r}$. Let i': $Q_{n,k}\to O_{n,k}$ be the inclusion map. Then $i'\circ q_r=p_r\circ i'$ and i': $Q_{n,1}\to O_{n,1}$ is the identity map of the $(dn-1)$-dimensional sphere S^{dn-1}.

Applying the homotopy functor $\pi_{dn-1}(\ )$ and the stable homotopy functor $\pi^s_{dn-1}(\ )$ to q_{k-1} and p_{k-1}, we define the unstable James numbers (see [7]) $Q\{n, k\}=Q_F\{n, k\}$, $O\{n, k\}=O_F\{n, k\}$ and the stable James numbers $Q^s\{n, k\}=Q^s_F\{n, k\}$, $O^s\{n, k\}=O^s_F\{n, k\}$ by the following equations:

$$q_{k-1*}\pi_{dn-1}(Q_{n,k}) = Q\{n, k\}\pi_{dn-1}(S^{dn-1}),$$
$$p_{k-1*}\pi_{dn-1}(O_{n,k}) = O\{n, k\}\pi_{dn-1}(S^{dn-1}),$$
$$q_{k-1*}\pi^s_{dn-1}(Q_{n,k}) = Q^s\{n, k\}\pi^s_{dn-1}(S^{dn-1}),$$
$$p_{k-1*}\pi^s_{dn-1}(O_{n,k}) = O^s\{n, k\}\pi^s_{dn-1}(S^{dn-1});$$

whenever $1\leqq k\leqq n$. As easily seen (see e.g. [12]), we have

(1.1) $Q^s\{n, k\}\mid Q\{n, k\}$, $O^s\{n, k\}\mid O\{n, k\}$, $O\{n, k\}\mid Q\{n, k\}$, $Q^s\{n, k\}\mid Q^s\{n, k+1\}$, $Q\{n, k\}\mid Q\{n, k+1\}$ *and* $O\{n, k\}\mid O\{n, k+1\}$;

where $a\mid b$ means that b is a multiple of a. In [12] we proved

(1.2) $$Q^s\{n, k\} = O^s\{n, k\}.$$

The stable James number $O^s\{n, k\}$ has been investigated by various authors, but the unstable ones $Q\{n, k\}$, $O\{n, k\}$ have been done not so much (see e.g. [7], [13], [15], [18]). By [2], [3], [4] we have

$$(1.3)\qquad O_H\{n, n\} = \begin{cases} 2\cdot(2n-1)! & \text{if } n \text{ is even} \\ (2n-1)! & \text{if } n \text{ is odd;} \end{cases}$$
$$O_C\{n, n\} = O_C\{n, n-1\} = (n-1)!\,.$$

Our first result is an easy consequence of the results of Mukai [10], [11].

Theorem 1. (i) *$O_H\{n, n\} = O_H^s\{n, n\} = a\cdot Q_H\{n, n\}$, where $a=1$ if n is even, $a=1$ or $1/2$ if n is odd.*

(ii) *$O_C\{n, n\} = O_C^s\{n, n\} = Q_C\{n, n\} = O_C\{n, n-1\} = O_C^s\{n, n-1\} = Q_C\{n, n-1\}$.*

Let E^∞: $\pi_r(\ \ )\to\pi_r^s(\ \ )$ be the stabilization homomorphism. Since $Q_{n,k}$ and $O_{n,k}$ are $(d(n-k+1)-2)$-connected (see e.g. [8]), it follows from Freudenthal suspension theorem that E^∞: $\pi_{dn-1}(Q_{n,k}) \to \pi_{dn-1}^s(Q_{n,k})$ and E^∞: $\pi_{dn-1}(O_{n,k}) \to \pi_{dn-1}^s(O_{n,k})$ are surjective whenever $n\geqq 2k-1$. Thus $Q\{n, k\} = Q^s\{n, k\}$ and $O\{n, k\} = O^s\{n, k\}$ if $n\geqq 2k-1$. As seen in [13], if $n<2k-1$, then $O\{n, k\} \neq O^s\{n, k\}$ in general.

We consider the case $n=2k-2$. Since $(O_{2k-2,k}, Q_{2k-2,k})$ is $(2dk-d-3)$-connected (see e.g. [8]) and $d(2k-2)-1\leqq 2dk-d-3$, it follows that i'_*: $\pi_{d(2k-2)-1}(Q_{2k-2,k}) \to \pi_{d(2k-2)-1}(O_{2k-2,k})$ is surjective, so that

$$(1.4)\qquad O\{2k-2, k\} = Q\{2k-2, k\}\,.$$

Our second result is

Theorem 2. (iii) *If $F=H$ or $F=C$ and k is odd, then $O\{2k-2, k\} = O^s\{2k-2, k\}$.*

(iv) *If $F=C$ and k is even, then $O\{2k-2, k\}/O^s\{2k-2, k\} = 1$ or 2.*

REMARK 1. In [13] we proved (iv) by a different method from the one in this note, and showed that $O_C\{2k-2, k\}/O_C^s\{2k-2, k\}$ is 1 if $k=2, 6$ and it is 2 if $k=4, 8$.

REMARK 2. In [13] we did not determine $O_H\{8, 5\}$. Now (iii) says that $O_H\{8, 5\} = O_H^s\{8, 5\}$ which was calculated in [12].

REMARK 3. I know of no case where $O_H\{n, k\} \neq O_H^s\{n, k\}$.

2. Proof of Theorem 1

The assertions are trivial when $n=1$. So we assume that $n\geqq 2$.

Let E denote both the reduced suspension functor in the category of pointed spaces and the suspension homomorphism in homotopy groups. For a continuous map f: $S^r\to X$, we denote the order of f in $\pi_r(X)$ and $\pi_r^s(X)$ by $\sharp f$

and $\#E^\infty f$, respectively.

As well known (see e.g. [8]), we have a CW-decomposition

$$Q_n = Q_{n,n} = e^0 \cup e^{d-1} \cup e^{2d-1} \cup \cdots \cup e^{dn-1}$$

such that Q_m is a subcomplex of Q_n provided $m<n$, so

(2.1) $$Q_{n,k} = e^0 \cup e^{d(n-k+1)-1} \cup \cdots \cup e^{dn-1}.$$

Let $T'_{n-1}: (B^{dn-1}, S^{dn-2}) \to (Q_n, Q_{n-1})$ be a characteristic map of the top cell, and let $T_{n-1}: S^{dn-2} \to Q_{n-1}$ be the restriction of T'_{n-1} to S^{dn-2}, the boundary of the disk B^{dn-1}. Let also $T'_{n-1,k-1}=q_{n-k}\circ T'_{n-1}: (B^{dn-1}, S^{dn-2}) \to (Q_{n,k}, Q_{n-1,k-1})$ and $T_{n-1,k-1}=q_{n-k}\circ T_{n-1}: S^{dn-2} \to Q_{n-1,k-1}$.

Applying $\pi^s_*(\ )$ to the cofibre sequence

$$S^{dn-2} \xrightarrow{T_{n-1,k-1}} Q_{n-1,k-1} \to Q_{n,k} \xrightarrow{q_{k-1}} S^{dn-1},$$

we obtain the exact sequence

$$\pi^s_{dn-1}(Q_{n,k}) \xrightarrow{q_{k-1*}} \pi^s_{dn-1}(S^{dn-1}) \xrightarrow{(ET_{n-1,k-1})_*} \pi^s_{dn-1}(EQ_{n-1,k-1}).$$

It follows from the cell structure of $Q_{n-1,k-1}$ that $\pi^s_{dn-1}(EQ_{n-1,k-1})$ is finite, so $\#E^\infty T_{n-1,k-1}$ is finite. Hence the exactness implies that

(2.2) $$Q^s\{n, k\} = \#E^\infty T_{n-1,k-1}.$$

Next we see the unstable case. Consider the homotopy exact sequence of the pair $(Q_{n,k}, Q_{n-1,k-1})$:

$$\pi_{dn-1}(Q_{n,k}) \xrightarrow{j_*} \pi_{dn-1}(Q_{n,k}, Q_{n-1,k-1}) \xrightarrow{\partial} \pi_{dn-2}(Q_{n-1,k-1}).$$

By definition $\partial(T'_{n-1,k-1})=T_{n-1,k-1}$. Let $q': (Q_{n,k}, Q_{n-1,k-1}) \to (S^{dn-1}, *)$ be the collapsing map. Then $q'_*(T'_{n-1,k-1})$ generates $\pi_{dn-1}(S^{dn-1})$. If $n>k$ or $F=H$, then, by Blakers–Massey [1], $q'_*: \pi_{dn-1}(Q_{n,k}, Q_{n-1,k-1}) \to \pi_{dn-1}(S^{dn-1})$ is an isomorphism, so $T'_{n-1,k-1}$ generates $\pi_{dn-1}(Q_{n,k}, Q_{n-1,k-1})$. Since $q'_*\circ j_*=q_{k-1*}$, it follows that the order of $T_{n-1,k-1}$ is equal to the order of the cokernel of q_{k-1*}: $\pi_{dn-1}(Q_{n,k}) \to \pi_{dn-1}(S^{dn-1})$ provided $n>k$ or $F=H$. Hence the following lemma implies that

(2.3) $$Q\{n, k\} = \#T_{n-1,k-1} \quad \text{if } n>k \text{ or } F=H.$$

Lemma (2.4). *The order of $T_{n-1,k-1}$ is finite if $n>k$ or $F=H$.*

Since $T_{n-1,k-1}=q_{n-k}\circ T_{n-1}$ and since $T_{n-1,k-1}=q_{n-k-1}\circ T_{n-1,n-2}$ if $n>k$, it is sufficient for proving (2.4) to show that $\#T_{n-1}$ is finite if $F=H$, and $\#T_{n-1,n-2}$ is finite if $F=C$.

The rest of this section is devoted to the proofs of (2.4) and Theorem 1.

We consider the case $F=H$. In [11] Mukai proved that $\#T_{n-1}=\#E^{\infty}T_{n-1}=2\cdot(2n-1)!$ if n is even; $\#E^{\infty}T_{n-1}=(2n-1)!$ and $\#T_{n-1}/(2n-1)!=1$ or 2 if n is odd. Hence we obtain (2.4) and (i) follows from (1.1), (1.2), (1.3), (2.2), (2.3).

We see the case $F=C$. Let P_n be the $(n-1)$-dimensional complex projective space, and let P_n^+ be the union of P_n and a base point. We then have $Q_n=E(P_n^+)$ and $Q_{n,n-1}=EP_n$ (see e.g. [8]). Note that there is a homotopy equivalence $E(P_n^+)\simeq EP_n\vee S^1$ which makes the following triangle commutative up to homotopy:

$$\begin{array}{ccc} Q_n=E(P_n^+) & \simeq & EP_n\vee S^1 \\ {\scriptstyle q_1}\searrow & & \swarrow{\scriptstyle p} \\ & Q_{n,n-1}=EP_n & \end{array}$$

where p is the projection. Hence q_1 has a left homotopy inverse, so

(2.5) $\qquad q_{1*}\colon \pi_{2n-1}(Q_n)\to\pi_{2n-1}(Q_{n,n-1})$ *is surjective.*

Let $SU(n)$ be the special unitary group and let $h\colon O_{n,n-1}=U(n)/(U(1)\times 1_{n-1})\to SU(n)$ be the homeomorphism defined by

$$h(A \bmod U(1)\times 1_{n-1}) = A\begin{pmatrix} 1/|A| & & & & \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 1 \end{pmatrix}.$$

Note that $h\circ i'\colon EP_n=Q_{n,n-1}\to SU(n)$ is the inclusion map defined in [20]. Hence in the following commutative diagram $h_*\circ i'_*$ is surjective by Proposition 4.2 of [16].

$$\begin{array}{ccccc} \pi_{2n-1}(Q_n) & \xrightarrow{i'_*} & \pi_{2n-1}(U(n)) & & \\ {\scriptstyle q_{1*}}\downarrow & & \cong\downarrow{\scriptstyle p_{1*}} & & \\ \pi_{2n-1}(Q_{n,n-1}) & \xrightarrow{i'_*} & \pi_{2n-1}(O_{n,n-1}) & \underset{\cong}{\xrightarrow{h_*}} & \pi_{2n-1}(SU(n)). \end{array}$$

It follows that the lower i'_* is surjective and so is the upper i'_* from (2.5). Thus we have

(2.6) $\qquad Q_C\{n,n\} = O_C\{n,n\}.$

On the other hand we can take $T_{n-1,n-2}=E\gamma_{n-1}$ where $\gamma_{n-1}\colon S^{2n-3}\to P_{n-1}$ is the canonical S^1-fibration. It is well known (see e.g. [10]) that $\#E\gamma_{n-1}=\#E^{\infty}\gamma_{n-1}=(n-1)!$. Thus we have (2.4) and

(2.7) $\qquad Q_C\{n,n-1\} = Q_C^s\{n,n-1\} = (n-1)!$

by (2.2), (2.3). Therefore (ii) follows from (1.1), (1.2), (1.3), (2.6), (2.7). This completes the proofs of (2.4) and Theorem 1.

3. EHP-sequence

Let X, Y be r-connected CW-complexes which have exactly one vertex $*$, and let $f: X \to Y$ be a continuous map with $f(*)=*$. We then have a diagram consisting of the exact EHP-sequences for $i \leqq 3r+1$ (see e.g. [9], [19]):

$$\begin{array}{ccccccccc} \pi_i(X) & \xrightarrow{E} & \pi_{i+1}(EX) & \xrightarrow{H} & \pi_{i+1}(E(X \wedge X)) & \xrightarrow{P} & \pi_{i-1}(X) & \xrightarrow{E} & \cdots \\ \downarrow f_* & & \downarrow (Ef)_* & & \downarrow (E(f \wedge f))_* & & \downarrow f_* & & \\ \pi_i(Y) & \xrightarrow{E} & \pi_{i+1}(EY) & \xrightarrow{H} & \pi_{i+1}(E(Y \wedge Y)) & \xrightarrow{P} & \pi_{i-1}(Y) & \xrightarrow{E} & \cdots \end{array}$$

In the next section we shall use

Lemma (3.1). *The above diagram commutes.*

By using Theorem 5.3 of [6] and following faithfully the construction of the EHP-sequence, we can prove (3.1). We omit the details.

4. Proof of Theorem 2

For an abelian group A, A/Tor denotes the quotient group of A by its torsion subgroup, and π: $A \to A/Tor$ denotes the quotient homomorphism. Let Z be the infinite cyclic group.

By (2.1) we have

$$\pi^s_{dn-1}(Q_{n,k})/Tor \cong Z . \tag{4.1}$$

It follows that $Q^s\{n, k\} \neq 0$ from (2.2) and that $Q\{n, k\} \neq 0$ from (2.3), (2.6), (1.3). Thus we have

Lemma (4.2). $\pi \circ E^\infty \neq 0$: $\pi_{dn-1}(Q_{n,k}) \to \pi^s_{dn-1}(Q_{n,k})/Tor$.

From now on we denote $Q_{2k-2,k}$ by Q. By (1.1), (1.2), (1.4), (4.1) and (4.2), Theorem 2 is equivalent to

Proposition (4.3). *Let $n=2k-2$. Then the image of $\pi \circ E^\infty$: $\pi_{dn-1}(Q) \to \pi^s_{dn-1}(Q)/Tor$ is $a \cdot \pi^s_{dn-1}(Q)/Tor$, where $a=1$ if $F=H$ or k is odd, $a=1$ or 2 if $F=C$ and k is even.*

Proof. We consider the case $F=C$ only, because we can prove the assertion for the case $F=H$ by a similar but slightly easier method to the following one.

If $k=2$, then the assertion is trivial by Theorem 1. So we assume that $k \geqq 3$. By (2.1) we have

$$Q = e^0 \cup e^{2k-3} \cup e^{2k-1} \cup \cdots \cup e^{4k-5} ,$$

and so

(4.4) $$Q \wedge Q = e^0 \cup e^{4k-6} \cup e^{4k-4} \cup e^{4k-4} \cup \cdots \cup e^{8k-10}.$$

Let i: $S^{2k-3}=e^0 \cup e^{2k-3}=Q_{k-1,1} \rightarrow Q$ be the inclusion. Since Q is $(2k-4)$-connected, it follows that

(4.5) $E^\infty: \pi_{4k-2}(E^3Q) \rightarrow \pi^s_{4k-5}(Q)$ *is an isomorphism, and*

(4.6) $E: \pi_{4k-3}(E^2Q) \rightarrow \pi_{4k-2}(E^3Q)$ *is surjective.*

By (3.1) we have the commutative diagram:

$$\begin{array}{ccccccc}
\pi_{4k-1}(E^3S^{2k-3}) & \xrightarrow{H} & \pi_{4k-1}(E^5(S^{2k-3}\wedge S^{2k-3})) \cong \pi_{4k-1}(S^{4k-1}) \cong Z & & & & \\
\downarrow (E^3 i)_* & & \downarrow (E^5(i\wedge i))_* & & & & \\
\pi_{4k-1}(E^3Q) & \underset{H}{\rightarrow} & \pi_{4k-1}(E^5(Q\wedge Q)) & \underset{P}{\rightarrow} & \pi_{4k-3}(E^2Q) & \underset{E}{\rightarrow} & \pi_{4k-2}(E^3Q).
\end{array}$$

By (4.4) and Blakers–Massey [1], $\pi_{4k}(E^5(Q\wedge Q), S^{4k-1}) \cong \pi_{4k-1}(E^5(Q\wedge Q), S^{4k-1}) \cong 0$, so the above $(E^5(i\wedge i))_*$ is an isomorphism. As well known (see e.g. Proposition 2.7 of [17]), the upper H is not zero, hence so is the lower H. Thus the image of P is finite, so that, by (4.6), E induces an isomorphism

(4.7) $$\tilde{E}: \pi_{4k-3}(E^2Q)/Tor \cong \pi_{4k-2}(E^3Q)/Tor.$$

Consider the EHP-sequence:

$$\pi_{4k-2}(E^3(Q\wedge Q)) \xrightarrow{P} \pi_{4k-4}(EQ) \xrightarrow{E} \pi_{4k-3}(E^2Q) \xrightarrow{H} \pi_{4k-3}(E^3(Q\wedge Q)).$$

By (4.4) and Blakers–Massey [1], $\pi_{4k-3}(E^3(Q\wedge Q), S^{4k-3}) \cong \pi_{4k-2}(E^3(Q\wedge Q), S^{4k-3}) \cong 0$, so $E^3(i\wedge i)$ induces a surjection $(Z_2 \cong)\ \pi_{4k-2}(S^{4k-3}) \rightarrow\rightarrow \pi_{4k-2}(E^3(Q\wedge Q))$ and an isomorphism $(Z\cong)\ \pi_{4k-3}(S^{4k-3}) \cong \pi_{4k-3}(E^3(Q\wedge Q))$. Thus it follows that

(4.8) $\pi_{4k-2}(E^3(Q\wedge Q))$ *is finite, and*

(4.9) $$\pi_{4k-3}(E^3(Q\wedge Q)) \cong Z.$$

The kernel of E is finite by (4.8). The cokernel of E is torsion free by (4.9), while it is finite by (4.1), (4.2), (4.5), (4.7), hence it is zero, so E is surjective. Thus E induces an isomorphism

(4.10) $$\tilde{E}: \pi_{4k-4}(EQ)/Tor \cong \pi_{4k-3}(E^2Q)/Tor.$$

By (3.1) we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & Z_2\{\eta_{4k-5}\} & & \\
 & & & & \| & & \\
\pi_{4k-5}(S^{2k-3}) & \xrightarrow{E} & \pi_{4k-4}(S^{2k-2}) & \xrightarrow{H} & \pi_{4k-4}(S^{4k-5}) & \xrightarrow{P} & \pi_{4k-6}(S^{2k-3}) \\
\downarrow i_* & & \downarrow (Ei)_* & & \downarrow (E(i\wedge i))_* & & \downarrow i_* \\
\pi_{4k-5}(Q) & \xrightarrow{E} & \pi_{4k-4}(EQ) & \xrightarrow{H} & \pi_{4k-4}(E(Q\wedge Q)) & \xrightarrow{P} & \pi_{4k-6}(Q).
\end{array}$$

Here η_2: $S^3\to S^2$ is the Hopf map and $\eta_m=E^{m-2}\eta_2$: $S^{m+1}\to S^m$ for $m\geqq 2$. By (4.4) and Blakers–Massey [1], $\pi_{4k-4}(E(Q\wedge Q),\ S^{4k-5})\cong 0$. Thus $(E(i\wedge i))_*$ is surjective, so $\pi_{4k-4}(E(Q\wedge Q))\cong Z_2$ or 0. Hence the cokernel of the lower E is Z_2 or 0. Since $\pi_{4k-4}(EQ)/Tor\cong Z$ by (4.1), (4.5), (4.7), (4.10), it follows that the image of the homomorphism $\tilde{E}$: $\pi_{4k-5}(Q)/Tor\to\pi_{4k-4}(EQ)/Tor$ induced by E is $a\cdot\pi_{4k-4}(EQ)/Tor$, where $a=1$ or 2. Thus the assertion of (4.3) for k even follows from (4.5), (4.7) and (4.10). We can prove (4.3) for k odd by showing that $\pi_{4k-4}(E(Q\wedge Q))$ is Z_2 if k is even and 0 if k is odd. But we will take a different method which can be applied to the case $F=H$.

As well known (see e.g. [19]), $P(\eta_{4k-5})=[l_{2k-3},\ \eta_{2k-3}]$, the Whitehead product, where l_{2k-3} is the identity map of S^{2k-3}. It follows from [5] that $[l_{2k-3},\ \eta_{2k-3}]=0$ if and only if k is odd. We show that $\tilde{E}$: $\pi_{4k-5}(Q)/Tor\to\pi_{4k-4}(EQ)/Tor$ is surjective if k is odd. Then the assertion of (4.3) for k odd follows from (4.5), (4.7) and (4.10).

Let k be odd. Then there is an element x in $\pi_{4k-4}(S^{2k-2})$ such that $H(x)=\eta_{4k-5}$ by exactness. Hence $H((Ei)_*(x))=E(i\wedge i)_*(H(x))=E(i\wedge i)_*(\eta_{4k-5})$ which generates $\pi_{4k-4}(E(Q\wedge Q))$. Choose y in $\pi_{4k-4}(EQ)$ such that $\pi(y)$ generates the infinite cyclic group $\pi_{4k-4}(EQ)/Tor$. If $H(y)=0$, then there exists y' in $\pi_{4k-5}(Q)$ such that $E(y')=y$, so $\tilde{E}(\pi(y'))=\pi(y)$ and $\tilde{E}$ is surjective. If $H(y)\neq 0$, then $\pi_{4k-4}(E(Q\wedge Q))\cong Z_2$ which is generated by $H(y)$. Hence $H(y)=H((Ei)_*(x))$ and there exists y'' in $\pi_{4k-5}(Q)$ such that $E(y'')=y-(Ei)_*(x)$. Since $\pi_{4k-4}(S^{2k-2})$ is finite as seen in [14], it follows that $(Ei)_*(x)$ has a finite order and $\tilde{E}(\pi(y''))=\pi(y-(Ei)_*(x))=\pi(y)$, so that $\tilde{E}$ is surjective. This completes the proofs of (4.3) and hence of Theorem 2.

References

[1] A.L. Blakers and W.S. Massey: *The homotopy groups of a triad.* II. Ann. of Math. **55** (1952), 192–201.

[2] R. Bott: *The space of loops on a Lie group*, Michigan Math. J. **5** (1958), 35–61.

[3] R. Bott: *The stable homotopy of the classical groups*, Ann. of Math. **69** (1959), 313–337.

[4] B. Harris: *Some calculations of homotopy groups of symmetric spaces*, Trans. Amer. Math. Soc. **106** (1963), 174–184.

[5] P.J. Hilton and J.H.C. Whitehead: Note on the Whitehead product, Ann. of Math. **58** (1953), 429–442.

[6] I.M. James: *Reduced product spaces*, Ann. of Math. **62** (1955), 170–197.

[7] I.M. James: *Cross-sections of Stiefel manifolds*, Proc. London Math. Soc. **8** (1958), 536–547.

[8] I.M. James: The topology of Stiefel manifolds, London Math. Soc. Lecture Note Series 24, Cambridge Univ. Press, Cambridge, 1976.

[9] M. Mimura and H. Toda: Homotopy theory (in Japanese), Kinokuniya Sūgaku Sōsho 3, Kinokuniya, Tokyo, 1975.

[10] J. Mukai: *The S^1-transfer map and homotopy groups of suspended complex projective spaces*, Math. J. Okayama Univ. **24** (1982), 179–200.
[11] J. Mukai: *The order of the attaching class of the suspended quaternionic quasi-projective space*, preprint.
[12] H. Ōshima: *On stable James numbers of stunted complex or quaternionic projective spaces*, Osaka J. Math. **16** (1979), 479–504.
[13] H. Ōshima: *Some James numbers of Stiefel manifolds*, Math. Proc. Cambridge Philos. Soc. **92** (1982), 139–161.
[14] J.P. Serre: *Homologie singulière des espaces fibrés*, Ann. of Math. **54** (1951), 425–505.
[15] F. Sigrist: *Groupes d'homotopie des variétés de Stiefel complexes*, Comment. Math. Helv. **43** (1968), 121–131.
[16] H. Toda: *A topological proof of theorems of Bott and Borel-Hirzebruch for homotopy groups of unitary groups*, Mem. Coll. Sci. Univ. Kyoto **32** (1959), 103–119.
[17] H. Toda: Composition methods in homotopy groups of spheres, Ann. of Math. Studies 49, Princeton Univ. Press, Princeton, 1962.
[18] G. Walker: *Estimates for the complex and quaternionic James numbers*, Quart. J. Math. **32** (1981), 467–489.
[19] G.W. Whitehead: Elements of homotopy theory, Graduate Texts in Math. 61, Springer, Berlin, 1978.
[20] I. Yokota: *On the homology of classical Lie groups*, J. Inst. Polytech. Osaka City Univ. **8** (1957), 93–120.

Department of Mathematics
Osaka City University
Sumiyoshi-ku, Osaka 558
Japan