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ON A WEAKLY UNKNOTTED 2-SPHERE IN A SIMPLY-CONNECTED 4-MANIFOLD

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Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

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Introduction

The purpose of this note is to present the following criterion for unknotting in a weak sense which gives us a simple geometric proof of Theorem of Kawauchi stated below.

Theorem 1. *Let M be a smooth 1-connected 4-manifold and S^2 a smoothly embedded 2-sphere in M. Suppose that $\pi_1(M-S^2)\cong Z$ and $S^2\simeq 0$ in M. Then, S^2 is unknotted in $M\#(\#_n S^2\times S^2)$ for some $n\geqq 0$.*

Here S^2 is called unknotted if there is a smoothly embedded D^3 which is bounded by S^2. As a corollary we shall give a proof of Theorem of Kawauchi. His original proof uses the partial Poincaré duality associated to infinite cyclic covering (see [3], [4] and Suzuki [9, Th. 8.6]). Other proofs are founded in [1], [8] and [10].

Corollary (Theorem of Kawauchi). *Let S^2 be a smoothly embedded 2-sphere in the 4-sphere S^4. Suppose that $\pi_1(S^4-S^2)\cong Z$. Then, it is algebraically unknotted, i.e. $S^4-S^2\simeq S^1$.*

Is a smooth 2-knot with $\pi_1(S^4-S^2)\cong Z$ unknotted? This is a unsolved question. We stabilize the problem by making connected sum of the ambient manifold with $\#(S^2\times S^2)$ and another stabilization may be done by making connected sum of the embedded manifold S^2 with trivially embedded $\#(S^1\times S^1)$. There is a result due to [2].

Theorem 2 (Hosokawa-Kawauchi [2]). *Under the same assumption of Theorem 1, S^2 surgered by attaching n trivially embedded 1-handles is unknotted in M for some $n\geqq 0$.*

We refer the reader to [ibid] for the precise meaning of trivial (=trivially embedded) 1-handles and unknottedness of surfaces. We shall give also a proof of Theorem of Kawauchi using this theorem in the last section.

1. Proof of Theorem 1

Since $S^2 \simeq 0$ in M, we have $S^2 \times D^2 \subset M$. And $* \times \partial D^2 \subset M - S^2$ gives a generator of $\pi_1(M - S^2) \cong Z$. This implies that there exists a map f: $M - S^2 \times \mathring{D}^2 \to S^1$ which is an extension of the projection $S^2 \times \partial D^2 \to S^1$. We make f transversely regular at a point of S^1 and get a connected smooth 3-manifold $N \subset M$ such that $\partial N = S^2$ in M.

In case M has a spin structure, we can restrict the spin structure of M on N and extend it over $N \cup D^3$, because the spin structure is determined by a framing of the stable tangent bundle over the 2-skelton (cf. Milnor [7]). Since the 3-dimensional spin cobordism group vanishes [ibid], we have a smooth spin cobordism $(W^4; N^3, D^3)$ relative to the boundary. We may assume that W^4 is the union of the elementary cobordisms consisting of one of 1-handles, 2-handles and 3-handles in this order. The elementary cobordism $N \times I \cup$ (1-handle) is easily embedded in M and the spin structure on the other boundary is compatible with that of M. By an inductive argument on the number of 1-handles, the level manifold N_1 just above all the 1-handles is embedded in M and $\partial N_1 = S^2$. Remark that the spin structure of $N_1 \subset W$ is compatible with that of $N_1 \subset M$. The elementary cobordism $N_1 \times I \cup$(2-handle) cannot be embedded in M but can be embedded in $M \# (S^2 \times S^2)$. In fact, we take $S^1 \subset N_1$ which is the boundary of the axis of the 2-handle. Then, $S^1 \simeq 0$ in $M - S^2$, because S^1 does not link with S^2 and $\pi_1(M - S^2) \cong Z$. The framing of $S^1 \times D^3$ is uniquely determined by the spin structure of N_1 and the surgery along this framed $S^1 \times D^3$ changes $M - S^2$ into $(M - S^2) \# (S^2 \times S^2)$ because of the choice of the spin structure. Of course, the spin structure on the other boundary is compatible with that of $M \# (S^2 \times S^2)$. The level manifold N_2 just above all the 2-handles is embedded in $M \# (\underset{k}{\#} S^2 \times S^2)$ and $\partial N_2 = S^2$, where k is equal to the number of the 2-handles of (W, N). We note that there is a diffeomorphism h: $(\underset{l}{\#} S^1 \times S^2 - \mathring{D}^3, \partial) \to (N_2, \partial)$, where l is the number of 3-handles of (W, N) i.e. 1-handles of (W, D^3). Take the component S^1 of $S^1 \times S^2$ and consider $h(S^1) \subset N_2 \subset M \# (\underset{k}{\#} S^2 \times S^2)$. As before, $h(S^1) \simeq 0$ in $M \# (\underset{k}{\#} S^2 \times S^2) - S^2$. The spin structure of N_2 induces a framing of the tubular neighborhood of $h(S^1)$ so that the surgery along $h(S^1)$ changes $M \# (\underset{k}{\#} S^2 \times S^2) - S^2$ into $(M \# (\underset{k}{\#} S^2 \times S^2) - S^2) \# S^2 \times S^2$. Then $N_2' \cong (\underset{l-1}{\#} S^1 \times S^2 - \mathring{D}^3)$ is easily embedded in $M \# (\underset{k+1}{\#} S^2 \times S^2)$ such that $\partial N_2' = S^2$. By induction we get a smooth submanifold N_3 of $M \# (\underset{k+l}{\#} S^2 \times S^2)$ such that $\partial N_3 = S^2$ and N_3 is diffeomorphic to D^3. This means that S^2 is unknotted in $M\#(\#S^2\times S^2)$.

In the other case that $w_2(M)\neq 0$, we have only to remark that the surgery along the trivial circle with any framing gives us $M\#(S^2\times S^2)$. Since the closed 3-manifold $N\cup D^3$ is orientable and the tangent bundle is trivial, there is a spin structure on $N\cup D^3$ and any choice of the spin structure on $N\cup D^3$ leads to the same proof as above. q.e.d.

2. Proof of Corollary

Let $\tilde{E}$ be the universal covering space of $E=S^4-S^2$. Then, $E\#(\underset{n}{\#}S^2\times S^2)$ is diffeomorphic to $S^1\times R^3\#(\underset{n}{\#}S^2\times S^2)$ by Theorem 1. Hence, we have $H_*(\tilde{E};Z)=0$ for $*\geqq 3$ and there is an isomorphism as $Z[Z]$-modules, $\alpha\colon H_2(\tilde{E};Z)\oplus(Z[Z])^{2n}\to(Z[Z])^{2n}$, where $Z[Z]$ is the group ring of Z over Z. (From this fact the argument in the last paragraph of [5] completes the proof. We present here a little modified one.) Let $\beta=p\circ\alpha^{-1}$ where $p\colon H(\tilde{E};Z)\oplus(Z[Z])^{2n}\to(Z[Z])^{2n}$ is the projection onto the 2nd factor. Since β is a surjection onto a free $Z[Z]$-module, there exists a $Z[Z]$-module homomorphism γ such that $\beta\circ\gamma=\mathrm{id}$. But, since $Z[Z]$ can be embedded in a field $Q(t)$, the right inverse matrix γ over $Q(t)$ is also a left inverse of β. In particular, β is an injection and so is $p=\beta\circ\alpha$. Hence, $H_2(\tilde{E};Z)=0$, which implies that $\tilde{E}$ is contractible and E has the homotopy type of S^1. q.e.d.

3. Further discussions

3.1. In Theorem 1, $\pi_1(M-S^2)\cong Z$ implies $S^2\simeq 0$ in M if M is a closed manifold. In fact, $[f(S^2)]\cap[S^2]=0$ for any immersion $f\colon S^2\to M$, because we can assume that $f(S^2)$ and S^2 intersect transversally and hence the algebraic intersection number times generator of $\pi_1(M-S^2)=H_1(M-S^2)$ is zero. By the fact that $\pi_2(M)=H_2(M)$ and the Poincaré duality this means $S^2\simeq 0$ in M.

3.2. Theorem of Kawauchi is valid for the locally flat topological 2-knot S^2 if it has a normal micro-bundle. In this case $S^2\times D^2$ is embedded in S^4 so that the interior of $\bar{E}=S^4-S^2\times\mathring{D}^2$ is homeomorphic to E. Then, Kawauchi's proof can be applied to $\bar{E}$ to get $E\simeq S^1$. Our method is also applicable. In fact, we consider an embedding of S^1 parallel to ∂D^2 in Int $\bar{E}$. Since $H^i(\bar{E}-S^1,\partial\bar{E})=H^i(S^1\times S^2\times[0,+\infty),S^1\times S^2\times 0)=0$ for any i, the non-compact 4-manifold $\bar{E}-S^1$ admits a smooth structure relative to the boundary $\partial\bar{E}$ (see [6, V. 1.4.1]). So we get a smooth embedding of S^2 into a 1-connected smooth 4-manifold M which is homeomorphic to S^4-S^1 such that $\pi_1(M-S^2)\cong Z$ and $S^2\simeq 0$ in M^4. By Theorem 1 S^2 is unknotted in $M\#(\underset{n}{\#}S^2\times S^2)$. Then $E\#(\underset{n}{\#}S^2\times S^2)$ is homeomorphic to $S^1\times R^3\#(\underset{n}{\#}S^2\times S^2)$. This implies $\bar{E}\simeq S^1$ by the argument of §2.

3.3. Proof of Corollary by using Theorem 2: Let $T(n)$ be S^2 surgered by attaching n trivially embedded 1-handles in S^4. Then by Theorem 2 we can assume that $T(n)$ is unknotted. By 3.3 of [5], $S^4-T(n)$ has the homotopy type of $S^1\vee(\bigvee_{2n} S^2)$. On the other hand we see in the same way that $S^4-T(n) \simeq E\vee(\bigvee_{2n} S^2)$. Now the same argument as in §2 leads to the conclusion of Corollary.

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