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A HOMOTOPY GROUP OF THE SYMMETRIC SPACE $SO(2n)/U(n)$

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In [1] B. Harris calculated some homotopy groups of the symmetric space $\Gamma_n = SO(2n)/U(n)$. He determined $\pi_{2n+r}(\Gamma_n)$ for $-1 \leqq r \leqq 1$ and for $r=3$, $n \equiv 0 \pmod 4$ except for $r=1$, $n \equiv 2 \pmod 4$. For the last case he made a group extension

$$(1) \qquad 0 \to Z_2 \to \pi_{2n+1}(\Gamma_n) \to Z_{n!/2} \to 0$$

from the homotopy exact sequence of the fibration $\Gamma_n \to \Gamma_{n+1} \to S^{2n}$. The purpose of this note is to show that this extension splits.

Theorem. $\pi_{2n+1}(\Gamma_n) = Z_2 \oplus Z_{n!/2}$ *if* $n \equiv 2 \pmod 4$.

Proof. If $n=2$, then the conclusion is obvious, by (1). Thus we will always assume that $n \equiv 2 \pmod 4$ and $n \geqq 6$.

The rotation group $SO(m)$ and the unitary group $U(m)$ are embedded, respectively, in $SO(m+1)$ and $U(m+1)$ as the upper left hand blocks. We embed $U(m)$ in $SO(2m)$ as the subset of matrices consisting of 2×2 blocks

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

The natural map $SO(2n-1)/U(n-1) \to SO(2n)/U(n) = \Gamma_n$ is a homeomorphism and will be used to identify these spaces. The inclusion map $SO(2n-2) \to SO(2n-1)$ then induces a map between the fibrations:

$$\begin{array}{ccc} U(n-1) & = & U(n-1) \\ \downarrow{\scriptstyle j} & & \downarrow \\ SO(2n-2) & \to & SO(2n-1) \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Gamma_{n-1} & \xrightarrow{i} & \Gamma_n . \end{array}$$

Applying the homotopy functor $\pi_*(-)$ to this, we obtain a commutative diagram with exact columns:

$$
\begin{array}{ccc}
\pi_{2n+1}(\Gamma_{n-1}) & \xrightarrow{i_*} & \pi_{2n+1}(\Gamma_n) \\
\downarrow \partial & & \downarrow \Delta \\
\pi_{2n}(U(n-1)) & = & \pi_{2n}(U(n-1)) \\
\downarrow j_* & & \downarrow \\
\pi_{2n}(SO(2n-2)) & \rightarrow & \pi_{2n}(SO(2n-1)) \\
\downarrow & & \downarrow p_* \\
\pi_{2n}(\Gamma_{n-1}) & \rightarrow & \pi_{2n}(\Gamma_n) \\
\downarrow & & \downarrow \\
\pi_{2n-1}(U(n-1)) & = & \pi_{2n-1}(U(n-1)) .
\end{array}
$$

We already know all the groups except $\pi_{2n+1}(\Gamma_n)$ in the above diagram, as follows:

(2) $\pi_{2n+1}(\Gamma_{n-1})=Z_{n!(24,n-2)/48}$, *by* (8.2) *of* [2];
(3) $\pi_{2n}(U(n-1))=Z_{n!/2}$, *by* Lemma 1.6 *of* [3];
(4) $\pi_{2n}(SO(2n-2))=Z_{12}$, *by* [3];
(5) $\pi_{2n}(SO(2n-1))=Z_2$, *by* [3];
(6) $\pi_{2n}(\Gamma_{n-1})=Z_{(24,n-2)/2}$, *by* (6.2) *of* [2];
(7) $\pi_{2n}(\Gamma_n)=Z_2$, *by* [1];
(8) $\pi_{2n-1}(U(n-1))=0$, *by* Lemma 1.4 *of* [3].

Here (a, b) denotes the greatest common divisor of a and b.

We use the following notations. For a finite abelian group G, G_{ev} and G_{od} denote the 2-primary and the odd components of G, respectively. For a homomorphism $f: G \rightarrow H$, $f_{ev}: G_{ev} \rightarrow H_{ev}$ and $f_{od}: G_{od} \rightarrow H_{od}$ are the restrictions of f to the appropriate busgroups.

By (5), (7) and (8), p_* is an isomorphism, so Δ is an epimorphism. It follows that Δ_{od} is an isomorphism, from (1) and (3), and that (1) splits if and only if Δ has a right inverse. Therefore (1) splits if and only if Δ_{ev} has a right inverse.

Let $n\equiv 2 \pmod 8$. By (4), (6) and (8), the image of j_*, *Image*(j_*), is Z_3 or 0, so ∂_{ev} is an epimorphism. Hence ∂_{ev} is an isomorphism, by (2) and (3). It follows that $(i_*)_{ev}\circ\partial_{ev}^{-1}$ is a right inverse of Δ_{ev}, so that (1) splits.

Let $n\equiv 6 \pmod 8$. By (4), (6) and (8), $(Image(j_*))_{ev}=Z_2$. Hence, by (2) and (3), we have a commutative diagram with exact columns:

$$
\begin{array}{ccc}
(Z_{n!/4})_{ev} & \xrightarrow{(i_*)_{ev}} & (\pi_{2n+1}(\Gamma_n))_{ev} \\
\downarrow \partial_{ev} & & \downarrow \Delta_{ev} \\
(Z_{n!/2})_{ev} & = & (Z_{n!/2})_{ev} \\
\downarrow & & \downarrow \\
Z_2 & & 0 . \\
\downarrow & & \\
0 & &
\end{array}
$$

Suppose that (1) does not split, that is, $\pi_{2n+1}(\Gamma_n)=Z_{n!}$. Then we can choose generators α, β and γ of $(Z_{n!/4})_{ev}$, $(\pi_{2n+1}(\Gamma_n))_{ev}$ and $(Z_{n!/2})_{ev}$, respectively, such that $\partial_{ev}(\alpha)=2\gamma$ and $\Delta_{ev}(\beta)=\gamma$. Since we can write $(i_*)_{ev}(\alpha)=4x\beta$ for some integer x, we have

$$2\gamma=\partial_{ev}(\alpha)=(\Delta_{ev}\circ(i_*)_{ev})(\alpha)=\Delta_{ev}(4x\beta)=4x\gamma\,.$$

Hence $2(2x-1)\gamma=0$. But this is impossible, because the order of γ is a multiple of 8. Therefore (1) splits. This completes the proof.

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