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ASYMPTOTIC SUFFICIENCY II: TRUNCATED CASES

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1. Introduction. Asymptotic sufficiency of maximum likelihood (m.l.) estimator in regular cases has been studied by many authors (see Wald [17], LeCam [2], Pfanzagl [12], Michel [8], Suzuki [14], [15], and so on).

In [6], Matsuda showed that for $k \in N = \{1, 2, \cdots\}$ a statistic $T_{n,k} = (T_n, G_n^{(2)}(z_n, T_n), \cdots, G_n^{(k)}(z_n, T_n))$ is asymptotically sufficient up to order $O(n^{-k/2})$. Here $\{T_n\}$ is a sequence of asymptotic m.l. estimators and $G_n^{(m)}(z_n, \theta)$ denotes the m-th derivative relative to θ of the log-likelihood function. In the case $k=1$, $T_{n,1}$ means T_n.

The purpose of this paper is to investigate asymptotic sufficiency of a statistic constructed by m.l. estimators in the following cases. Let $x_1, \cdots, x_n$ be independent and identically distributed random variables with common density $p(x-\theta)$, $-\infty < x, \theta < \infty$, where θ is an unknown translation parameter and $p(x)$ is uniformly continuous and positive only on the interval $(0, \infty)$. We shall consider here two cases.

Case (i): $p(x) \sim \alpha x$ as $x \to +0$, where $\alpha > 0$.

Case (ii): $p(x) \sim \alpha x^{1+\beta}$ as $x \to +0$, where $\alpha, \beta > 0$.

It is assumed that in Case (i) Fisher's information number is infinite. Let $\hat{\theta}_n$ denote m.l. estimator of θ for the sample size n. In this case, Takeuchi [16] and Woodroofe [20] proved the asymptotic normality of $\sqrt{\frac{1}{2}\alpha n \log n}\,(\hat{\theta}_n - \theta)$ and the speed of convergence to the standard normal distribution was given by Matsuda [4]. Moreover, it was shown by. Takeuchi [16] and Weiss and Wolfowitz [19] that $\hat{\theta}_n$ is an asymptotically efficient estimator of θ.

In Case (ii), it is well known that if Fisher's information number J is finite, then the distribution of $\sqrt{Jn}\,(\hat{\theta}_n - \theta)$ converges weakly to the standard normal distribution. The order of convergence to normality is $o(n^{-\nu/2})$ for every $\nu < \beta$ if $\beta \leqq 1$ and $O(n^{-1/2})$ if $\beta > 1$ (see Matsuda [3] and cf. also Pfanzagl [11]).

In both cases, Mita [9] showed that m.l. estimator is asymptotically sufficient up to order $o(1)$. For $n, k \in N$ define $\hat{\theta}_{n,k} = (\hat{\theta}_n, G_n^{(2)}(z_n, \hat{\theta}_n), \cdots, G_n^{(k)}(z_n, \hat{\theta}_n))$, where $\hat{\theta}_{n,1}$ means $\hat{\theta}_n$. We shall show that in Case (i) the statistic $\hat{\theta}_{n,k}$ is asymptotically sufficient up to order $o((\log n)^{-\nu})$ for every $\nu < (k+1)/(k+3)$ and that in Case (ii) $\hat{\theta}_{n,k}$ is asymptotically sufficient up to order $o(n^{-\nu})$ for every

$\nu < \dfrac{\beta(k+1)}{2(k+\beta+3)}$ if $\beta \leqq k(k+3)$ and is asymptotically sufficient up to order $O(n^{-k/2})$ if $\beta > k(k+3)$.

In Section 2 we introduce the results of von Bahr and Esseen [1] and Nagaev [10] which are useful to estimate probabilities of deviations for sums of independent and identically distributed random variables with a restricted moment. Section 3 is devoted to the problem of asymptotic sufficiency in Case (i) and Section 4 to the one in Case (ii).

2. Probabilities of deviations. Let $Y_1, \cdots, Y_n$ be a sequence of random variables (r.v.'s) and put $S_m = \sum_{i=1}^{m} Y_i$, $1 \leqq m \leqq n$. Using the elementary inequality

$$E|S_n|^r \leqq \sum_{i=1}^{n} E|Y_i|^r, \qquad r \leqq 1,$$

it follows from Markov's inequality that for $x > 0$

$$P\{|S_n| \geqq x\} \leqq x^{-r} \sum_{i=1}^{n} E|Y_i|^r, \qquad r \leqq 1. \tag{2.1}$$

If the r.v.'s satisfy the relations

$$E(Y_{m+1}|S_m) = 0 \ \text{ a.s.} \qquad 1 \leqq m \leqq n-1, \tag{2.2}$$

then von Bahr and Esseen [1] showed that

$$E|S_n|^r \leqq 2 \sum_{i=1}^{n} E|Y_i|^r, \qquad 1 \leqq r \leqq 2. \tag{2.3}$$

The condition (2.2) is fulfilled if the r.v.'s are independent and have zero means. In this case, (2.3) together with Markov's inequality implies the following inequality

$$P\{|S_n| \geqq x\} \leqq 2x^{-r} \sum_{i=1}^{n} E|Y_i|^r, \qquad 1 \leqq r \leqq 2, \tag{2.4}$$

for $x > 0$.

Let $Y_1, \cdots, Y_n$ be a sequence of identically distributed independent r.v.'s and $E(Y_i) = 0$, $E(Y_i^2) = 1$. In [10], Nagaev proved the following theorem (cf. Lemma in Michel [8]).

Theorem 2.1. *If $\xi_r = E|Y_i|^r < \infty$, $r > 2$, then*

$$P\{|S_n| > x\} < c\xi_r n x^{-r} \tag{2.5}$$

for $x > 4\sqrt{n \max\left[\log \dfrac{n^{r/2-1}}{K_r \xi_r}, 0\right]}$,

where $K_r = 1 + (r+1)^{r+2} \exp(-r)$ and c is an absolute constant depending only on r.

REMARK 1. It is obvious that Theorem 2.1 remains valid even if $E(Y_i^2) = 0$.

As a consequence of Theorem 2.1 we obtain a result on probabilities of moderate deviations: If $\xi_r<\infty$, $r>2$, then there is a positive constant c such that

$$P\{|S_n|>c\sqrt{n\log n}\}=o(n^{-(r-2)/2}).$$

It is remarked that Theorem 4 in Michel [7] implies the above result under the same condition and Lemma 1 in Pfanzagl [13] gives a uniform version of this result when $r\geqq 3$.

3. Asymptotic sufficiency: Case (i). For $\theta\in\boldsymbol{R}$, let P_θ be a probability measure on the Borel real line $(\boldsymbol{R},\mathscr{B})$. It is assumed that every P_θ is absolutely continuous with respect to the Lebesgue measure μ on $\boldsymbol{R}$ and $dP_\theta/d\mu=p(x-\theta)$. For each $n\in N=\{1,2,\cdots\}$, let $(\boldsymbol{R}^n,\mathscr{B}^n)$ be the Cartesian product of n copies of $(\boldsymbol{R},\mathscr{B})$ and $P_{n,\theta}$ be the product measure of n copies of P_θ. Furthermore, let μ_n denote the product measure of n copies of μ and set $p_n(z_n,\theta)=dP_{n,\theta}/d\mu_n$ for $\theta\in\boldsymbol{R}$ and $z_n=(x_1,\cdots,x_n)\in\boldsymbol{R}^n$.

Let k be a positive integer equal to or greater than 2. We shall impose the following Condition A_k on $p(x)$.

Condition A_k

(i) $p(x)$ is a uniformly continuous density which vanishes on $(-\infty,0)$ and is positive on $(0,\infty)$.

(ii) $p(x)$ is $(k+1)$-times continuously differentiable on $(0,\infty)$.

Let $g(x)=\log p(x)$ for $x>0$ and $g^{(m)}(x)$ be the m-th derivative of $g(x)$.

(iii) For some $\alpha\in(0,\infty)$ and $\gamma\in(0,\infty)$
$p(x)=\alpha x+O(x^{1+\gamma})$, $g^{(1)}(x)=x^{-1}+O(x^{\gamma-1})$, $g^{(2)}(x)=-x^{-2}+O(x^{\gamma-2})$, $g^{(3)}(x)=O(x^{-3})$ and $g^{(k+1)}(x)=O(x^{-k-1})$ as $x\to+0$.

(iv) For every $t\geqq 0$, there exists $\eta>0$ such that

$$\int_0^\infty|g(x+t)|^{1+\eta}p(x)\,d\mu<\infty\,.$$

(v) For some $M>0$

$$\int_M^\infty|g^{(1)}(x)|^3p(x)\,d\mu<\infty\,.$$

(vi) For every $a>0$, there exist $\delta>0$ and $\eta>0$ such that

(a) $$\int_a^\infty\sup_{|u|\leqq\delta}|g^{(2)}(x+u)|^{1+\eta}p(x)\,d\mu<\infty\,,$$

(b) $$\int_a^\infty\sup_{|u|\leqq\delta}|g^{(3)}(x+u)|\,p(x)\,d\mu<\infty\,,$$

(c) $$\int_a^\infty\sup_{|u|\leqq\delta}|g^{(k+1)}(x+u)|\,p(x)\,d\mu<\infty\,.$$

Let $M_n=\min(x_1, \cdots, x_n)$ and $G_n(z_n, t)=\sum_{i=1}^{n} g(x_i-t)$ for $t<M_n$. Condition (i) insures that m.l. estimators of θ for the sample size n exist in the interval $(-\infty, M_n)$. Let $\{\hat{\theta}_n; n\in N\}$ be a sequence of m.l. estimators. Woodroofe [20] remarked that condition (i) and

$$\int_0^\infty -g(x)\,p(x)\,d\mu<\infty$$

imply all assumptions of Wald [18] and that, moreover, if $g(x)$ is continuously differentiable, then $\{\hat{\theta}_n\}$ will form a consistent sequence of roots of the likelihood equation

$$\hat{\theta}_n<M_n \quad \text{and} \quad G_n^{(1)}(z_n, \hat{\theta}_n)=0\,.$$

We shall use $a_n=\sqrt{\frac{1}{2}\,\alpha n(\log n+\log\log n)}$ rather than $\sqrt{\frac{1}{2}\,\alpha n\log n}$ as the convergence order of m.l. estimator to the true parameter θ (see [4] and [5]).

Since θ is a translation parameter, we restrict our attention to the case that $\theta=0$. The following lemma is the same as Lemma 5 in [5] except that conditions (v) (b) and (v) (c) in [5] can be replaced by weaker conditions (vi) (a) and (vi) (b) because of (2.4).

Lemma 3.1. *Let conditions* (i)–(iii), (vi) (a) *and* (vi) (b) *be satisfied for* $k=2$. *Then for every* $s\in(0, 1)$ *there exists* $c>0$ *such that*

$$P_{n,0}\{\sup_{|t|\leqq 2b_n(s)}|a_n^{-2}G_n^{(2)}(z_n, t)+1|\geqq c(\log n)^{-s}\}=O((\log n)^{s-1})\,,$$

where $b_n(s)=a_n^{-1}(\log n)^{s/2}$.

REMARK 2. It seems to be impossible to improve Lemma 3.1 (see Remark in [5]).

Using (2.4) instead of Chebyshev's inequality in the proofs of Lemma 1 and Lemma 2 of [4], we obtain the following Lemma 3.2 and Lemma 3.3, respectively.

Lemma 3.2. *Let conditions* (i)–(iii) *and* (vi) (a) *be satisfied for* $k=2$. *Then for sufficiently small* $\varepsilon>0$, *there are events* D_n, $n\in N$ *for which* $P_{n,0}\{(D_n)^c\}=o((\log n)^{-1})$ *and* $z_n\in D_n$ *implies*

$$\sup_{-\varepsilon\leqq t<M_n} n^{-1}G_n^{(2)}(z_n, t)<-1\,.$$

Lemma 3.3. *Let conditions* (i)–(iii) *and* (iv) *be satisfied for* $k=2$. *Then for every* $\varepsilon>0$

$$P_{n,0}\{|\hat{\theta}_n|\geqq\varepsilon\}=o((\log n)^{-1})\,.$$

Lemma 3.4. *Let Condition* A_2 *be satisfied. Then for every* $s\in(0, 1)$

$$P_{n,0}\{a_n|\hat{\theta}_n|\geqq \log\log n\} = O((\log n)^{s-1}).$$

Proof. We shall use ideas related to Woodroofe [20]. It follows from Lemma 3.2 and Lemma 3.3 that

$$(3.1)\quad P_{n,0}\{a_n\hat{\theta}_n\leqq -\log\log n\} = P_{n,0}\{\sum_{i=1}^{n} g^{(1)}(x_i+a_n^{-1}\log\log n)\geqq 0\}+o((\log n)^{-1}),$$

$$(3.2)\quad P_{n,0}\{a_n\hat{\theta}_n\geqq \log\log n\} = P_{n,0}\{\sum_{i=1}^{n} g^{(1)}(x_i-a_n^{-1}\log\log n)\leqq 0,\ M_n>a_n^{-1}\log\log n\}+o((\log n)^{-1}).$$

Using Lemma 3.1 and the equality

$$\sum_{i=1}^{n} g^{(1)}(x_i+a_n^{-1}\log\log n) = \sum_{i=1}^{n} g^{(1)}(x_i)+a_n^{-1}\log\log n \sum_{i=1}^{n} g^{(2)}(x_i+u_n)$$

with $u_n\in(0,\ a_n^{-1}\log\log n)$, (3.1) implies that

$$P_{n,0}\{a_n\hat{\theta}_n\leqq -\log\log n\}\leqq P_{n,0}\{a_n^{-1}\sum_{i=1}^{n} g^{(1)}(x_i)\geqq \frac{1}{2}\log\log n\}+O((\log n)^{s-1}).$$

A similar argument shows that (3.2) implies

$$P_{n,0}\{a_n\hat{\theta}_n\geqq \log\log n\}\leqq P_{n,0}\{a_n^{-1}\sum_{i=1}^{n} g^{(1)}(x_i)\leqq -\frac{1}{2}\log\log n\}+O((\log n)^{s-1}).$$

Lemma 3 in [4], together with the fact

$$\Phi(-\frac{1}{2}\log\log n) = o((\log n)^{-1}),$$

leads to the desired result. Here $\Phi(x)$ denotes the cumulative distribution function of the standard normal distribution.

Lemma 3.1 and Lemma 3.4 yield the following lemma.

Lemma 3.5 (cf. Lemma 6 in [5]). *Let Condition A_2 be satisfied. Then for every $s\in(0,1)$ there exists $c>0$ such that*

$$P_{n,0}\{\sup_{|t|\leqq b_n(s)} |a_n^{-2}G_n^{(2)}(z_n,\ \hat{\theta}_n+t)+1|\geqq c(\log n)^{-s}\} = O((\log n)^{s-1}).$$

We shall investigate an asymptotic behavior of $a_n^{-k-1}G_n^{(k+1)}(z_n,\ t)$, $|t-\hat{\theta}_n|\leqq b_n(s)$, with $k\geqq 2$.

Lemma 3.6. *Let conditions* (i)–(iii) *and* (vi) (c) *be satisfied for some $k\geqq 2$. Then for every $s\in(0,1)$ there exists $c>0$ such that*

$$P_{n,0}\{\sup_{|t|\leqq 2b_n(s)} |a_n^{-k-1}G_n^{(k+1)}(z_n,\ t)|\geqq c(\log n)^{-(k+1)s/2}\} = O((\log n)^{s-1}).$$

Proof. Since $P_{n,0}\{M_n\leqq 2b_n(s)\}=O((\log n)^{s-1})$, we may assume that $M_n>2b_n(s)$. Then $G_n^{(k+1)}(z_n,\ t)=(-1)^{k+1}\sum_{i=1}^{n} g^{(k+1)}(x_i-t)$ for $|t|\leqq 2b_n(s)$. Let $a>0$ be so small that $p(x)\leqq 2\alpha x$ and $|g^{(k+1)}(x)|\leqq Lx^{-k-1}$ for $0<x<2a$ where L is a positive constant, and choose $\delta>0$ to satisfy condition (vi) (c). Then we have

$$|a_n^{-k-1}G_n^{(k+1)}(z_n, t)| \leqq La_n^{-k-1}\textstyle\sum_0^a (x_i-2b_n(s))^{-k-1} + a_n^{-k-1}\sum_a^\infty \sup_{|u|\leqq\delta} |g^{(k+1)}(x_i+u)| \tag{3.3}$$

for $|t|\leqq 2b_n(s)$ and all sufficiently large n. Here $\sum_u^v$ denotes the summation over $i\leqq n$ satisfying $u\leqq x_i<v$.

To evaluate the first term above define $\{Y_{ni};\ i=1, \cdots, n\}$ by

$$\begin{aligned} Y_{ni} &= (x_i-2b_n(s))^{-k-1}, \quad &&\text{if } 3b_n(s)\leqq x_i<a, \\ &= 0 \quad &&\text{if } x_i<3b_n(s) \text{ or } a\leqq x_i. \end{aligned}$$

Since $E(Y_{n1}^2)=O(b_n(s)^{-2k})$, it follows from Chebyshev's inequality that

$$P_{n,0}\{|a_n^{-k-1}\sum_{i=1}^n (Y_{ni}-E(Y_{ni}))| \geqq \frac{1}{2}(\log n)^{-(k+1)s/2}\} = O((\log n)^{s-1}).$$

Moreover, using $a_n^{-k-1}\sum_{i=1}^n E(Y_{ni})=o((\log n)^{-(k+1)s/2})$ we obtain

$$P_{n,0}\{|a_n^{-k-1}\sum_{i=1}^n Y_{ni}| \geqq (\log n)^{-(k+1)s/2}\} = O((\log n)^{s-1}).$$

This, together with the fact $P_{n,0}\{\sum_{i=1}^n Y_{ni} \neq \sum_0^a (x_i-2b_n(s))^{-k-1}\} = O((\log n)^{s-1})$, implies

$$P_{n,0}\{|a_n^{-k-1}\textstyle\sum_0^a (x_i-2b_n(s))^{-k-1}| \geqq (\log n)^{-(k+1)s/2}\} = O((\log n)^{s-1}).$$

It remains to estimate the second term on the righthand side of (3.3). It follows from Markov's inequality and condition (vi) (c) that

$$P_{n,0}\{a_n^{-k-1}\textstyle\sum_a^\infty \sup_{|u|\leqq\delta} |g^{(k+1)}(x_i+u)| \geqq (\log n)^{-(k+1)s/2}\} = o((\log n)^{s-1}).$$

This completes the proof.

REMARK 3. It is easily seen that in the case $k\geqq 2$ the distribution of $n^{-(k+1)/2}\sum_{i=1}^n g^{(k+1)}(x_i)$ converges weakly to a stable law with characteristic exponent $2/(k+1)$. By the same reason as of Remark 2, we cannot expect to improve Lemma 3.6.

The following lemma immediately follows from Lemma 3.4 and Lemma 3.6.

Lemma 3.7. *Let Condition A_k be satisfied for some $k\geqq 2$. Then for every $s\in(0, 1)$ there exists $c>0$ such that*

$$P_{n,0}\{\sup_{|t|\leqq b_n(s)} |a_n^{-k-1}G_n^{(k+1)}(z_n, \hat{\theta}_n+t)| \geqq c(\log n)^{-(k+1)s/2}\} = O((\log n)^{s-1}).$$

At first we study asymptotic sufficiency of m.l. estimator $\hat{\theta}_n$.

Theorem 3.1. *If Condition A_2 holds, then there exists a sequence of families of probability measures $\{Q_{n,\theta};\ \theta\in\boldsymbol{R}\}$, $n\in N$, such that*

(a) *for each $n\in N$, $\hat{\theta}_n$ is sufficient for $\{Q_{n,\theta};\ \theta\in\boldsymbol{R}\}$*

(b) *for every $\nu<1/2$*

$$\sup_{\theta\in\boldsymbol{R}} \|P_{n,\theta}-Q_{n,\theta}\| = o((\log n)^{-\nu}) .$$

Proof. Let $I_A(\cdot)$ be the indicator function of a set A. We define

$$q_n^*(z_n,\ \theta) = I_{V_{n,\theta}\cap W_n}(z_n) \exp \{G_n(z_n,\ \hat{\theta}_n)-\frac{1}{2}\ a_n^2(\hat{\theta}_n-\theta)^2\} ,$$

$$q_n(z_n,\ \theta) = c_n(\theta)\ q_n^*(z_n,\ \theta) ,$$

where

$$V_{n,\theta} = \{z_n\in\boldsymbol{R}^n;\ a_n|\hat{\theta}_n-\theta|<\log\log n\} ,$$

$$W_n = \{z_n\in\boldsymbol{R}^n;\ \sup_{|t|\leq b_n(1/2)} |a_n^{-2}G_n^{(2)}(z_n,\ \hat{\theta}_n+t)+1|<c(\log n)^{-1/2}\}$$

and $c_n(\theta)=[\int_{\boldsymbol{R}^n}q_n^*(z_n,\ \theta)\ d\mu_n]^{-1}$. Here the constant c in W_n is determined by Lemma 3.5 with $s=1/2$.

For every $A\in\mathscr{B}^n$, let $Q_{n,\theta}\{A\}=\int_A q_n(z_n,\theta)d\mu_n$ and $Q^*_{n,\theta}\{A\}=\int_A q_n^*(z_n,\theta)d\mu_n$. According to the factorization theorem, for each $n\in N$ $\hat{\theta}_n$ is sufficient for $\{Q_{n,\theta};\ \theta\in\boldsymbol{R}\}$.

Next we have

$$\begin{aligned}(3.4)\qquad \|P_{n,\theta}-Q^*_{n,\theta}\| &= \int_{\boldsymbol{R}^n}|p_n(z_n,\ \theta)-q_n^*(z_n\ ,\theta)|\,d\mu_n \\ &\leq \int_{V_{n,\theta}\cap W_n}|1-q_n^*(z_n,\ \theta)/p_n(z_n,\ \theta)|\,p_n(z_n,\ \theta)d\mu_n \\ &\quad +P_{n,\theta}\{(V_{n,\theta})^c\}+P_{n,\theta}\{(W_n)^c\} .\end{aligned}$$

It follows from Lemma 3.4 that $\sup_{\theta\in\boldsymbol{R}} P_{n,\theta}\{(V_{n,\theta})^c\}=o((\log n)^{-\nu})$. And Lemma 3.5 implies that $\sup_{\theta\in\boldsymbol{R}} P_{n,\theta}\{(W_n)^c\}=O((\log n)^{-1/2})$. It remains to estimate the first term on the righthand side of (3.4). Since

$$G_n(z_n,\ \theta) = G_n(z_n,\ \hat{\theta}_n)+\frac{1}{2}(\theta-\hat{\theta}_n)^2G_n^{(2)}(z_n,\ \theta_n^*)$$

with $|\theta_n^*-\hat{\theta}_n|\leq|\hat{\theta}_n-\theta|$, we have for $z_n\in V_{n,\theta}\cap W_n$

$$\begin{aligned}\left|1-\frac{q_n^*(z_n,\ \theta)}{p_n(z_n,\ \theta)}\right| &= |1-\exp\{-[G_n(z_n,\ \theta)-G_n(z_n,\ \hat{\theta}_n)+\frac{1}{2}a_n^2(\hat{\theta}_n-\theta)^2]\}| \\ &\leq c(\log\log n)^2(\log n)^{-1/2} \\ &= o((\log n)^{-\nu}) .\end{aligned}$$

Here we used the inequality $|1-\exp(x)|\leqq 2|x|$ for sufficiently small x. Thus we obtain

$$\sup_{\theta\in\boldsymbol{R}} \|P_{n,\theta}-Q^*_{n,\theta}\| = o((\log n)^{-\nu}).$$

Since

$$\sup_{\theta\in\boldsymbol{R}} |1-c_n(\theta)^{-1}| = \sup_{\theta\in\boldsymbol{R}} |P_{n,\theta}\{\boldsymbol{R}^n\}-Q^*_{n,\theta}\{\boldsymbol{R}^n\}| = o((\log n)^{-\nu}),$$

we have

$$\begin{aligned}\sup_{\theta\in\boldsymbol{R}} \|P_{n,\theta}-Q_{n,\theta}\| &\leqq \sup_{\theta\in\boldsymbol{R}} \|P_{n,\theta}-Q^*_{n,\theta}\|+\sup_{\theta\in\boldsymbol{R}} \|Q^*_{n,\theta}-Q_{n,\theta}\| \\ &\leqq o((\log n)^{-\nu})+\sup_{\theta\in\boldsymbol{R}} |1-c_n(\theta)^{-1}| \\ &= o((\log n)^{-\nu}).\end{aligned}$$

This completes the proof.

We can also show higher order asymptotic sufficiency of the statistic $\hat{\theta}_{n,k} =(\hat{\theta}_n, G_n^{(2)}(z_n, \hat{\theta}_n), \cdots, G_n^{(k)}(z_n, \hat{\theta}_n))$.

Theorem 3.2. *If Condition A_k holds for some $k\geqq 2$, then there exists a sequence of families of probability measures $\{R_{n,\theta};\ \theta\in\boldsymbol{R}\}$, $n\in N$, such that*

(a) *for each $n\in N$, the statistic $\hat{\theta}_{n,k}$ is sufficient for $\{R_{n,\theta};\ \theta\in\boldsymbol{R}\}$*

(b) *for every $\nu<(k+1)/(k+3)$*

$$\sup_{\theta\in\boldsymbol{R}} \|P_{n,\theta}-R_{n,\theta}\| = o((\log n)^{-\nu}).$$

Proof. Let

$$r_n^*(z_n, \theta) = I_{V_{n,\theta}\cap W_{n,k}}(z_n) \exp\{G_n(z_n, \hat{\theta}_n)+\sum_{m=2}^{k}\frac{(\theta-\hat{\theta}_n)^m}{m!} G_n^{(m)}(z_n, \hat{\theta}_n)\},$$

$$r_n(z_n, \theta) = \bar{c}_n(\theta) r_n^*(z_n, \theta),$$

where

$$W_{n,k} = \{z_n\in\boldsymbol{R}^n;\ \sup_{|t|\leqq b_n(2/(k+3))} |a_n^{-k-1}G_n^{(k+1)}(z_n, \hat{\theta}_n+t)|<c(\log n)^{-(k+1)/(k+3)}\},$$

$$\bar{c}_n(\theta)= [\int_{\boldsymbol{R}^n} r_n^*(z_n, \theta)\, d\mu_n]^{-1}$$

and $V_{n,\theta}$ is the same as in the proof of Theorem 3.1. Here the constant c in $W_{n,k}$ is determined by Lemma 3.7 with $s=2/(k+3)$. Moreover, define $R_{n,\theta}\{A\} = \int_A r_n(z_n, \theta)\, d\mu_n$ and $R^*_{n,\theta}\{A\}=\int_A r_n^*(z_n, \theta)\, d\mu_n$ for every $A\in\mathscr{B}^n$. Then it follows from the factorization theorem that $\hat{\theta}_{n,k}$ is sufficient for $\{R_{n,\theta};\ \theta\in\boldsymbol{R}\}$.

Using the Taylor expansion

$$G_n(z_n, \theta) = G_n(z_n, \hat{\theta}_n) + \sum_{m=2}^{k} \frac{(\theta-\hat{\theta}_n)^m}{m!} G_n^{(m)}(z_n, \hat{\theta}_n) + \frac{(\theta-\hat{\theta}_n)^{k+1}}{(k+1)!} G_n^{(k+1)}(z_n, \theta_n^*)$$

with $|\theta_n^* - \hat{\theta}_n| \leqq |\hat{\theta}_n - \theta|$, we have for $z_n \in V_{n,\theta} \cap W_{n,k}$

$$\begin{aligned}\left|1-\frac{r_n^*(z_n, \theta)}{p_n(z_n, \theta)}\right| &= \left|1-\exp\left\{-\left[G_n(z_n, \theta)-G_n(z_n, \hat{\theta}_n)-\sum_{m=2}^{k}\frac{(\theta-\hat{\theta}_n)^m}{m!}G_n^{(m)}(z_n, \hat{\theta}_n)\right]\right\}\right| \\ &\leqq \frac{2c}{(k+1)!}(\log\log n)^{k+1}(\log n)^{-(k+1)/(k+3)} \\ &= o((\log n)^{-\nu}).\end{aligned}$$

Hence an argument analogous to the proof of Theorem 3.1 shows that Lemma 3.4 and Lemma 3.7 imply

$$\sup_{\theta\in R} \|P_{n,\theta} - R_{n,\theta}^*\| = o((\log n)^{-\nu}),$$

which leads to

$$\sup_{\theta\in R} \|P_{n,\theta} - R_{n,\theta}\| = o((\log n)^{-\nu}).$$

This completes the proof.

EXAMPLES (Woodroofe [20]). Let

$$f(x) = r[\Gamma(2/r)]^{-1}\, x \exp(-x^r), \quad x>0 \quad \text{for some } r>0,$$

or

$$f(x) = [r(1+r)]^{-1}\, x(1+x)^{-2-r}, \quad x>0 \quad \text{for some } r>0,$$

then Condition A_k is satisfied for every $k \geqq 2$.

4. Asymptotic sufficiency: Case (ii). We continue to use the same notations as in Section 3. We shall need the following Condition $B(\beta, k)$ on $p(x)$ where β is a positive number and k is a positive integer.

Condition $B(\beta, k)$

(i) $p(x)$ is a uniformly continuous density which vanishes on $(-\infty, 0)$ and is positive on $(0, \infty)$.

(ii) $p(x)$ is $(k+2)$-times continuously differentiable on $(0, \infty)$.

(iii) $p(x)=O(x^{\beta+1})$, $g^{(1)}(x)=O(x^{-1})$, $g^{(2)}(x)=O(x^{-2})$, $g^{(3)}(x)=O(x^{-3})$, $g^{(k+1)}(x)=O(x^{-k-1})$ and $g^{(k+2)}(x)=O(x^{-k-2})$ as $x\to+0$.
Moreover, $g^{(2)}(x)\leqq 0$ for sufficiently small $x>0$.

Let $\rho_1=\min\left[\frac{k+2}{2}, \frac{(\beta+2)(k+3)}{2(k+\beta+3)}\right]$, $\rho_2=\min\left[k+2, \frac{\beta+2}{k+1}\right]$ and $\rho_3=\min\left[\frac{k+2}{2}, \frac{(\beta+2)(k+3)}{(k^2+5k+\beta+6)}\right]$. It is clear that $\rho_2\leqq 2\rho_3\leqq 2\rho_1$.

(iv) For every $t\geqq 0$

$$\int_0^\infty |g(x+t)|^{\rho_1} p(x)\, d\mu < \infty.$$

(v) There exists $M>0$ such that

(a) $$\int_M^\infty |g^{(1)}(x)|^{2\rho_1} p(x)\, d\mu < \infty ,$$

(b) $$\int_M^\infty |g^{(2)}(x)|^{\rho_1} p(x)\, d\mu < \infty ,$$

(c) $$\int_M^\infty |g^{(k+1)}(x)|^{\rho_2} p(x)\, d\mu < \infty .$$

(vi) For every $a>0$, there exists $\delta>0$ such that

(a) $$\int_a^\infty \sup_{|u|\leqq\delta} |g^{(3)}(x+u)|^{\rho_1} p(x)\, d\mu < \infty ,$$

(b) $$\int_a^\infty \sup_{|u|\leqq\delta} |g^{(k+2)}(x+u)|^{\rho_3} p(x)\, d\mu < \infty .$$

Let $J=-\int_0^\infty g^{(2)}(x)\, p(x)\, d\mu$. Conditions (i)–(iii) and (v) (b) guarantee that J is finite. Moreover, we need

(vii) $J>0$.

According to condition (iii), it may be expected that $g^{(k+1)}(x)$ has the absolute moment of order r for every $r<(\beta+2)/(k+1)$, but we will not always require this. That is, conditions (i)–(iii) and (v) (c) insure that

$$E|g^{(k+1)}(\cdot)|^r<\infty \text{ for every } r<(\beta+2)/(k+1), \quad \text{if } \beta\leqq k(k+3), \tag{4.1}$$
$$E|g^{(k+1)}(\cdot)|^{k+2}<\infty , \quad \text{if } \beta>k(k+3) .$$

We define λ^* and ν^* as follows

$$\lambda^* = \frac{(k+1)(k+3)}{2(k+\beta+3)}, \quad \text{for } \beta<k(k+3), \tag{4.2}$$

$$\nu^* = \min\left[\frac{k}{2}, \frac{\beta(k+1)}{2(k+\beta+3)}\right], \quad \text{for } \beta>0,\ k\in N. \tag{4.3}$$

It is noticed that $\lambda^*>1/2$, and that $\nu^*<k/2$ for $\beta<k(k+3)$ and $\nu^*=k/2$ for $\beta\geqq k(k+3)$.

(4.1) together with (2.1), (2.4) and (2.5) implies

$$P_{n,0}\{|\sum_{i=1}^n g^{(k+1)}(x_i)|\geqq n^{\lambda^*}\} = o(n^{-\nu}), \quad 0<\beta\leqq k-1, \tag{4.4}$$

$$P_{n,0}\{|\sum_{i=1}^n [g^{(k+1)}(x_i)-E(g^{(k+1)}(\cdot))]|\geqq n^{\lambda^*}\} = o(n^{-\nu}), \quad k-1<\beta<k(k+3),$$

$$P_{n,0}\{|\sum_{i=1}^n [g^{(k+1)}(x_i)-E(g^{(k+1)}(\cdot))]|\geqq c\sqrt{n\log n}\} = o(n^{-\nu}), \quad \beta=k(k+3),$$

$$P_{n,0}\{|\sum_{i=1}^n [g^{(k+1)}(x_i)-E(g^{(k+1)}(\cdot))]|\geqq c\sqrt{n\log n}\} = o(n^{-k/2}), \quad \beta>k(k+3),$$

where $\nu<\nu^*$ and c is a positive constant independent of ν.

Hereafter, we shall use $c>0$ as a generic constant independent of $n\in N$.

Lemma 4.1. *Let conditions* (i)–(iii) *and* (vi) (b) *be satisfied for some* $\beta>0$ *and* $k\in N$. *Then for every* $L>0$ *there exists* $c>0$ *such that for* $\nu<\nu^*$

$$P_{n,0}\{\sup_{|t|\leqq Ld_n}|\sum_{i=1}^n[g^{(k+1)}(x_i+t)-g^{(k+1)}(x_i)]|\geqq cn^{\lambda^*}\}=o(n^{-\nu}),\quad \beta<k(k+3),$$

and

$$P_{n,0}\{\sup_{|t|\leqq Ld_n}|\sum_{i=1}^n[g^{(k+1)}(x_i+t)-g^{(k+1)}(x_i)]|\geqq cnd_n\}=O(n^{-k/2}),\ \beta\geqq k(k+3),$$

where $d_n=n^{-1/2}\sqrt{\log n}$ *and the supremum is understood to be infinite if* $M_n\leqq Ld_n$.

Proof. Let $a>0$ be so small that $p(x)<cx^{\beta+1}$ and $|g^{(k+2)}(x)|<cx^{-k-2}$ for $0<x<2a$. Since $P_{n,0}\{M_n\leqq 2Ld_n\}=o(n^{-\nu^*})$, we may assume that $M_n>2Ld_n$. Using the equality

$$\sum_{i=1}^n g^{(k+1)}(x_i+t)=\sum_{i=1}^n g^{(k+1)}(x_i)+\sum_{i=1}^n\int_0^t g^{(k+2)}(x_i+u)\,du$$

we have for sufficiently large n and $|t|\leqq Ld_n$

$$(4.5)\qquad |\sum_{i=1}^n[g^{(k+1)}(x_i+t)-g^{(k+1)}(x_i)]|\leqq|t|[c\textstyle\sum_0^a(x_i-Ld_n)^{-k-2}+\sum_a^\infty\sup_{|u|\leqq\delta}|g^{(k+2)}(x_i+u)|],$$

where δ is determined by condition (vi) (b). For $i=1,\cdots,n$ let us define

$$(4.6)\qquad \begin{aligned}U_{ni}&=(x_i-Ld_n)^{-k-2}, &&2Ld_n\leqq x_i<a,\\ &=0, &&\text{otherwise.}\end{aligned}$$

Since $E|U_{n1}|^r\leqq c<\infty$ for every $n\in N$ and $r<(\beta+2)/(k+2)$, it follows from (2.1), (2.4) or (2.5) that

$$P_{n,0}\{\sum_{i=1}^n U_{ni}\geqq d_n^{-1}n^{\lambda^*}\}=o(n^{-\nu^*}),\qquad \beta\leqq k,$$

$$P_{n,0}\{|\sum_{i=1}^n[U_{ni}-E(U_{ni})]|\geqq d_n^{-1}n^{\lambda^*}\}=o(n^{-\nu^*}),\quad k<\beta<k(k+3),$$

$$P_{n,0}\{|\sum_{i=1}^n[U_{ni}-E(U_{ni})]|\geqq n\}=o(n^{-k/2}),\qquad \beta\geqq k(k+3).$$

This implies that

$$P_{n,0}\{\textstyle\sum_0^a(x_i-Ld_n)^{-k-2}\geqq 2d_n^{-1}n^{\lambda^*}\}=o(n^{-\nu^*}),\quad \beta<k(k+3),$$
$$P_{n,0}\{\textstyle\sum_0^a(x_i-Ld_n)^{-k-2}\geqq cn\}=o(n^{-k/2}),\qquad \beta\geqq k(k+3).$$

Taking account of condition (vi) (b), a similar argument shows that

$$P_{n,0}\{\textstyle\sum_a^\infty\sup_{|u|\leqq\delta}|g^{(k+2)}(x_i+u)|\geqq 2d_n^{-1}n^{\lambda^*}\}=o(n^{-\nu}),\quad \beta<k(k+3),$$

$$P_{n,0}\{\sum_a^\infty \sup_{|u|\leqq\delta} |g^{(k+2)}(x_i+u)|\geqq cn\} = O(n^{-k/2}), \qquad \beta\geqq k(k+3).$$

Thus, (4.5) implies the desired assertion.

REMARK 4. If $\beta\geqq k(k+3)$, then for every $L>0$ there are events H_n, $n\in N$ for which $P_{n,0}\{(H_n)^c\}=O(n^{-k/2})$ and $z_n\in H_n$ implies

$$\sup_{|t|\leqq Ld_n} |\sum_{i=1}^n [g^{(k+1)}(x_i+t)-g^{(k+1)}(x_i)]| \leqq cn|t| .$$

Lemma 4.2. *Let conditions* (i)–(iii), (v) (b), (vi) (a) *and* (vii) *be satisfied for some* $\beta>0$ *and* $k\in N$. *Then for sufficiently small* $\varepsilon>0$ *there are events* D_n^*, $n\in N$ *for which* $P_{n,0}\{(D_n^*)^c\}=O(n^{-\nu^*})$ *and* $z_n\in D_n^*$ *implies*

$$\sup_{-\varepsilon\leqq t<M_n} G_n^{(2)}(z_n, t)<-Jn/5 .$$

Proof. Let $a>0$ be so small that $g^{(2)}(x)\leqq 0$ for $0<x<2a$ and $\int_0^a g^{(2)}(x)\, p(x)\, d\mu>-J/5$. Then the event $M_n\leqq\varepsilon$ implies that

$$\begin{aligned}\sup_{-\varepsilon\leqq t<M_n} G_n^{(2)}(z_n, t) &\leqq \sup_{-\varepsilon\leqq t<M_n} \sum_a^\infty g^{(2)}(x_i-t)\\ &\leqq \sum_a^\infty g^{(2)}(x_i)+\varepsilon \sum_a^\infty \sup_{|u|\leqq\varepsilon} |g^{(3)}(x_i+u)| .\end{aligned}$$

Let $\varepsilon=\min\,[\delta, \frac{J}{5}(\int_a^\infty \sup_{|u|\leqq\delta} |g^{(3)}(x+u)|\, p(x)d\mu)^{-1}]$ with δ, $\delta<a$, satisfying condition (vi) (a). Then

$$M_n\leqq\varepsilon, \quad \sum_a^\infty g^{(2)}(x_i)-n\int_a^\infty g^{(2)}(x)\, p(x)\, d\mu<Jn/5$$

and

$$\sum_a^\infty \sup_{|u|\leqq\delta} |g^{(3)}(x_i+u)| -n\int_a^\infty \sup_{|u|\leqq\delta} |g^{(3)}(x+u)|\, p(x)\, d\mu<\frac{Jn}{5\varepsilon}$$

imply

$$\sup_{-\varepsilon\leqq t<M_n} G_n^{(2)}(z_n, t)<-Jn/5 .$$

Since $\rho_1>1$, it follows from (2.4) or (2.5) that

$$P_{n,0}\{\sum_a^\infty g^{(2)}(x_i)-n\int_a^\infty g^{(2)}(x)\, p(x)\, d\mu\geqq Jn/5\} = O(n^{-\nu^*}),$$

$$P_{n,0}\{\sum_a^\infty \sup_{|u|\leqq\delta} |g^{(3)}(x_i+u)| -n\int_a^\infty \sup_{|u|\leqq\delta} |g^{(3)}(x+u)|\, p(x)\, d\mu\geqq\frac{Jn}{5\varepsilon}\} = O(n^{-\nu^*}).$$

Lemma 4.2 follows easily.

The following lemma is proved in the same manner as Lemma 2 in [4] except that (2.4) or (2.5) is used instead of Chebyshev's inequality.

Lemma 4.3. *Let conditions* (i)–(iii) *and* (iv) *be satisfied for some* $\beta>0$ *and* $k\in N$. *Then for every* $\varepsilon>0$

$$P_{n,0}\{|\hat{\theta}_n|\geqq\varepsilon\} = O(n^{-\nu*}) .$$

Lemma 4.4. *Let conditions* (i)–(iii), (v)(b), (vi)(a) *and* (vii) *be satisfied for some* $\beta>0$ *and* $k\in N$. *Then for every* $L>0$

$$P_{n,0}\{\sup_{|t|\leqq Ld_n}|\sum_{i=1}^n [g^{(2)}(x_i+t)+J]|\geqq Jn/2\} = O(n^{-\nu*}) .$$

Proof. Since $E|g^{(2)}(\cdot)|^{\nu*+1}<\infty$, (2.4) or (2.5) implies that

$$P_{n,0}\{|\sum_{i=1}^n [g^{(2)}(x_i)+J]|\geqq Jn/6\} = O(n^{-\nu*}) .$$

It remains to estimate the right side of (4.5) with $k=1$. Let U_{ni} be the same as in (4.6) with $k=1$. It is easy to see that

$$\begin{aligned}
&E|U_{n1}|^r\leqq c<\infty \text{ for } r<(\beta+2)/3\leqq 1 , && \text{if } \beta\leqq 1 ,\\
&E|U_{n1}|^r\leqq c<\infty \text{ for } 1<r<(\beta+2)/3 , && \text{if } 1<\beta<k(k+3) ,\\
&E|U_{n1}|^{(k+3)/3}\leqq c<\infty , && \text{if } \beta\geqq k(k+3) .
\end{aligned}$$

Accordingly, from (2.1), (2.4) or (2.5) we obtain

$$P_{n,0}\{cLd_n\sum_{i=1}^n U_{ni}\geqq Jn/6\} = o(n^{-\nu*}) ,$$

so that

$$P_{n,0}\{cLd_n\textstyle\sum_0^a (x_i-Ld_n)^{-3}\geqq Jn/6\} = o(n^{-\nu*}) .$$

Moreover, condition (vi)(a) together with (2.4) or (2.5) gives us

$$P_{n,0}\{Ld_n\textstyle\sum_a^\infty \sup_{|u|\leqq\delta}|g^{(3)}(x_i+u)|\geqq Jn/6\} = o(n^{-\nu*}) .$$

Thus the lemma follows.

Lemma 4.5. *Let conditions* (i)–(iv), (v)(a), (v)(b), (vi)(a) *and* (vii) *be satisfied for some* $\beta>0$ *and* $k\in N$. *Then there exists* $L>0$ *such that*

$$P_{n,0}\{|\hat{\theta}_n|\geqq Ld_n\} = O(n^{-\nu*}) .$$

Proof. It follows from Lemma 4.2 and Lemma 4.3 that for every $L>0$

$$P_{n,0}\{\hat{\theta}_n\leqq -Ld_n\} = P_{n,0}\{\sum_{i=1}^n g^{(1)}(x_i+Ld_n)\geqq 0\}+O(n^{-\nu*}) , \tag{4.7}$$

$$P_{n,0}\{\hat{\theta}_n\geqq Ld_n\} = P_{n,0}\{\sum_{i=1}^n g^{(1)}(x_i-Ld_n)\leqq 0,\ M_n>Ld_n\}+O(n^{-\nu*}) . \tag{4.8}$$

Using the equality

$$\sum_{i=1}^n g^{(1)}(x_i+Ld_n) = \sum_{i=1}^n g^{(1)}(x_i)+Ld_n\sum_{i=1}^n g^{(2)}(x_i+u_n)$$

with $u_n\in(0, Ld_n)$, Lemma 4.4 implies that

$$P_{n,0}\{\sum_{i=1}^n g^{(1)}(x_i+Ld_n)\geqq 0\}\geqq P_{n,0}\{\sum_{i=1}^n g^{(1)}(x_i)\geqq\frac{JL}{2}\sqrt{n\log n}\}+O(n^{-\nu*}) . \tag{4.9}$$

Since $E|g^{(1)}(\cdot)|^{2p_1}<\infty$, it follows from (2.5) that

$$P_{n,0}\{\sum_{i=1}^{n} g^{(1)}(x_i)\geqq\frac{JL}{2}\sqrt{n\log n}\} = o(n^{-\nu^*}) \tag{4.10}$$

for some large $L>0$. Thus relations (4.7), (4.9) and (4.10) imply

$$P_{n,0}\{\hat{\theta}_n\leqq -Ld_n\} = O(n^{-\nu^*}).$$

By a similar argument, (4.8) implies

$$P_{n,0}\{\hat{\theta}_n\geqq Ld_n\} = O(n^{-\nu^*}).$$

The following lemma is an immediate consequence of (4.4), Lemma 4.1 and Lemma 4.5.

Lemma 4.6. *Suppose that Condition $B(\beta, k)$ holds for some $\beta>0$ and $k\in N$.*

If $\beta\leqq k(k+3)$, then there exist $L>0$ and $c>0$ such that for $0<\nu<\nu^*$

$$P_{n,0}\{\sup_{|t|\leqq Ld_n}|G_n^{(k+1)}(z_n, \hat{\theta}_n+t)|\geqq cn^{\lambda^*}\} = o(n^{-\nu}), \qquad 0<\beta\leqq k-1,$$

$$P_{n,0}\{\sup_{|t|\leqq Ld_n}|G_n^{(k+1)}(z_n,\hat{\theta}_n+t)-E(G_n^{(k+1)}(\cdot,0))|\geqq cn^{\lambda^*}\} = o(n^{-\nu}),\quad k-1<\beta<k(k+3),$$

$$P_{n,0}\{\sup_{|t|\leqq Ld_n}|G_n^{(k+1)}(z_n,\hat{\theta}_n+t)-E(G_n^{(k+1)}(\cdot,0))|\geqq c\sqrt{n\log n}\} = o(n^{-\nu}),\quad \beta=k(k+3),$$

where λ^* and ν^* are defined by (4.2) *and* (4.3), *respectively.*

If $\beta>k(k+3)$, then there exist $L>0$ and $c>0$ such that

$$P_{n,0}\{\sup_{|t|\leqq Ld_n}|G_n^{(k+1)}(z_n, \hat{\theta}_n+t)-E(G_n^{(k+1)}(\cdot, 0))|\geqq c\sqrt{n\log n}\} = O(n^{-k/2}).$$

Now we shall discuss asymptotic sufficiency of the statistic $\hat{\theta}_{n,k}=(\hat{\theta}_n, G_n^{(2)}(z_n, \hat{\theta}_n), \cdots, G_n^{(k)}(z_n, \hat{\theta}_n))$, $k\in N$, where $\hat{\theta}_{n,1}$ means $\hat{\theta}_n$.

Theorem 4.1. *If Condition $B(\beta, k)$ holds for some $\beta>0$ and $k\in N$, then there exists a sequence of families of probability measures $\{Q_{n,\theta}^k;\ \theta\in\boldsymbol{R}\}$, $n\in N$, such that*

(a) *for each $n\in N$, the statistic $\hat{\theta}_{n,k}$ is sufficient for $\{Q_{n,\theta}^k;\ \theta\in\boldsymbol{R}\}$*

(b) $\sup_{\theta\in\boldsymbol{R}}\|P_{n,\theta}-Q_{n,\theta}^k\| = o(n^{-\nu})$ *for $0<\nu<\nu^*$,*

where $\nu^*=\min\left[\frac{k}{2},\ \frac{\beta(k+1)}{2(k+\beta+3)}\right]$.

Proof. The proof of the theorem is analogous to those of Theorem 3.1 and Theorem 3.2. Let

$$q_{n,k}^*(z_n, \theta) = I_{V_{n,\theta}^*\cup W_{n,k}^*}(z_n)\exp\{G_n(z_n, \hat{\theta}_n)-\frac{J}{2}n(\theta-\hat{\theta}_n)^2\}, \qquad \text{if } k=1,$$

$$= I_{V^*_{n,\theta} \cap W^*_{n,k}}(z_n) \exp\{G_n(z_n, \hat\theta_n) + \sum_{m=2}^{k} \frac{(\theta-\hat\theta_n)^m}{m!} G_n^{(m)}(z_n, \hat\theta_n)\}\,, \quad \text{if } k \geqq 2 \text{ and } 0 < \beta \leqq k-1\,,$$

$$= I_{V^*_{n,\theta} \cap W^*_{n,k}}(z_n) \exp\{G_n(z_n, \hat\theta_n) + \sum_{m=2}^{k} \frac{(\theta-\hat\theta_n)^m}{m!} G_n^{(m)}(z_n, \hat\theta_n) + \frac{(\theta-\hat\theta_n)^{k+1}}{(k+1)!} E(G_n^{(k+1)}(\cdot, 0))\}\,, \quad \text{if } k \geqq 2 \text{ and } \beta > k-1\,,$$

where

$$V^*_{n,\theta} = \{|\hat\theta_n - \theta| < Ld_n\}\,,$$

$$W^*_{n,k} = \{\sup_{|t| \leqq Ld_n} |G_n^{(k+1)}(z_n, \hat\theta_n + t)| < cn^{\lambda^*}\}\,, \quad \text{if } 0 < \beta \leqq k-1\,,$$

$$= \{\sup_{|t| \leqq Ld_n} |G_n^{(k+1)}(z_n, \hat\theta_n + t) - E(G_n^{(k+1)}(\cdot, 0))| < cn^{\lambda^*}\}\,, \quad \text{if } k-1 < \beta < k(k+3)\,,$$

$$= \{\sup_{|t| \leqq Ld_n} |G_n^{(k+1)}(z_n, \hat\theta_n + t) - E(G_n^{(k+1)}(\cdot, 0))| < c\sqrt{n \log n}\}, \quad \text{if } \beta \geqq k(k+3)\,.$$

Here $L>0$ and $c>0$ are determined by Lemma 4.5 and Lemma 4.6. Moreover, let $q_{n,k}(z_n, \theta)$ be the normalizing of $q^*_{n,k}(z_n, \theta)$ and let $Q^k_{n,\theta}$ be a probability measure with the density $q_{n,k}(z_n, \theta)$. Since the remaining part of the proof runs parallel to the lines of the corresponding part of the proofs of Theorem 3.1 and Theorem 3.2, we shall omit it.

In the case $\beta > k(k+3)$, we can improve Theorem 4.1, if condition (v) (b) in $B(\beta, k)$ is replaced by a stronger condition (v) (b)′.

(v) (b)′ There exists $M>0$ such that

$$\int_M^\infty |g^{(2)}(x)|^{k+2} p(x)\, d\mu < \infty\,.$$

In this case, condition (v) (b)′ with conditions (i)–(iii) implies $E|g^{(2)}(\cdot)|^{k+2} < \infty$. This leads to a stronger result than Lemma 4.4: For every $L>0$ there exists $c>0$ such that

$$(4.11) \qquad P_{n,0}\{\sup_{|t| \leqq Ld_n} |\sum_{i=1}^{n} [g^{(2)}(x_i + t) + J]| \geqq c\sqrt{n \log n}\} = o(n^{-k/2})\,.$$

Lemma 4.7. *If Condition* $B(\beta, k)$ *with* (v) (b)′ *replacing* (v) (b) *holds for* $\beta > k(k+3)$, *then*

$$P_{n,0}\{|\hat\theta_n + J^{-1} n^{-1} \sum_{i=1}^{n} g^{(1)}(x_i)| \geqq n^{-1} (\log n)^{3/2}\} = O(n^{-k/2})\,.$$

Proof. By a Taylor expansion we obtain

$$\hat\theta_n + J^{-1} n^{-1} \sum_{i=1}^{n} g^{(1)}(x_i) = \hat\theta_n + J^{-1} n^{-1} \{\sum_{i=1}^{n} [g^{(1)}(x_i) - g^{(1)}(x_i - \hat\theta_n)]\}$$

$$= J^{-1} n^{-1} \hat\theta_n \sum_{i=1}^{n} [g^{(2)}(x_i - \theta_n^*) + J]$$

with $|\theta_n^*-\theta|\leqq|\hat{\theta}_n-\theta|$. This, together with Lemma 4.5 and (4.11), implies the desired assertion.

Theorem 4.2. *If Condition* $B(\beta, k)$ *with* (v) (b)′ *replacing* (v) (b) *holds for* $\beta>k(k+3)$, *then Theorem* 4.1 *holds with the following* (b)′ *instead of* (b):

(b)′ $$\sup_{\theta\in R}\|P_{n,\theta}-Q^k_{n,\theta}\|=O(n^{-k/2}).$$

Because of Remark 4 and Lemma 4.7, Theorem 4.2 can be shown in quite the same way as in the proof of Theorem 2 in [6].

REMARK 5. Suppose $k=1$. From Theorem 4.1 and 4.2 it follows that if $\beta\leqq 4$, then m.l. estimator $\hat{\theta}_n$ is asymptotically sufficient up to order $o(n^{-\nu})$ for every $\nu<\beta/(4+\beta)$ and that if $\beta>4$, then $\hat{\theta}_n$ is asymptotically sufficient up to order $O(n^{-1/2})$.

REMARK 6. Theorems 4.1 and 4.2 still hold even if a sequence of m.l. estimators $\{\hat{\theta}_n\}$ is replaced by $\{T_n\}$ with the following properties: There exist positive constants π_1 and π_2 (depending on ν^*) such that

(1) $$\sup_{\theta\in R} P_{n,\theta}\{z_n\in \boldsymbol{R}^n;\, n^{1/2}|T_n(z_n)-\theta|\geqq(\log n)^{\pi_1}\}=O(n^{-\nu^*})$$

(2) $$\sup_{\theta\in R} P_{n,\theta}\{z_n\in \boldsymbol{R}^n;\, n^{\nu^*}|\sum_{i=1}^n g^{(1)}(x_i-T_n(z_n))|\geqq(\log n)^{\pi_2}\}=O(n^{-\nu^*}).$$

$\{T_n\}$ with properties (1) and (2) is called a sequence of *asymptotic m.l. estimators* of order $O(n^{-\nu^*})$ (see Michel [8] and Matsuda [6]).

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