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Osaka J. Math.
21 (1984), 133-147

UNBOUNDEDNESS OF SAMPLE FUNCTIONS OF STOCHASTIC PROCESSES WITH ARBITRARY PARAMETER SETS, WITH APPLICATIONS TO LINEAR AND l_p-VALUED PARAMETERS*

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(Received August 3, 1982)

1. Introduction and summary

Let T be a pseudometric space, and let $X(t)$, $t \in T$, be a real valued stochastic process on some probability space. There has been much recent interest in conditions for the continuity or boundedness of the sample functions stated in terms of the finite-dimensional distributions of the process and their relation to the pseudometric. On the other hand, one of my interests has been the search for conditions under which $X(t)$, $t \in T$, is unbounded, or has even more drastically irregular behavior. The main tool in this analysis is the local time of the process. The theme of this work has been that the smoothness of the local time implies the irregularity of the sample function. In the current paper, in particular, we use the result that if the local time is an analytic function of its spatial variable, then the sample function spends positive time in every set of positive measure, so that it is unbounded on T. This was used in [2] to supplement the Beljaev dichotomy theorem for stationary Gaussian processes [1]: In the noncontinuous case, the sample functions are often not only unbounded, but have the property of the Carathéodory function [2].

The current work extends the latter results to more general, not necessarily Gaussian processes. Suppose that there is a nonnegative function $d(s, t)$ on T^2 such that $d(s, t) > 0$ for $s \neq t$, and that the density function of the random variable,

$$(1.1) \qquad \frac{X(s)-X(t)}{d(s, t)},$$

is uniformly sufficiently smooth in a precise sense for all $s \neq t$; and suppose that there is a specified nonincreasing function $K(u)$, $u > 0$, determined by the

* This paper represents results obtained at the Courant Institute of Mathematical Sciences, New York University, under the sponsorship of the National Science Foundation, Grant MCS 82 01119.

density of (1.1) such that

$$\int_T \int_T K(d(s, t))d\mu(s)d\mu(t) < \infty \tag{1.2}$$

for some measure μ such that $\mu(T)>0$. Then the local time of $X(t)$, $t \in T$, relative to the measure μ on T, is almost surely an analytic function, so that, in particular, the sample functions are unbounded on T. The intuitive reason for the sufficiency of (1.2) is that the double integral is an energy integral with the potential kernel $K(d(s, t))$, and so the finiteness of the integral for some μ such that $\mu(T)>0$ signifies that T has positive capacity. This implies that the time set T is sufficiently large so that the sample functions have "enough time" to visit every set of positive measure.

The main theorem of the paper is applied to two cases: T is an interval on the real line, and T is an ellipsoid in the Banach space l_p, $p \geq 1$. The latter represents a generalization of the earlier result in [3] where X was Lévy's Brownian motion over Hilbert space, and T an ellipsoid in the latter space.

The sufficient conditions for the analyticity of the local time, and the subsequent unboundedness of the sample functions are shown to be very close to being necessary. In proving this we cite some of the recent work of Weber [9] [10] on sufficient conditions for the boundedness of the sample functions on a compact set T. In stating such conditions, Weber and the other investigators cited in his work have used an extension of the entropy concept which was introduced in the Gaussian case by Strassen and Dudley (see [6]). If $d(s, t)$ is a pseudometric for T, and the latter is compact, then for $\varepsilon>0$, there is a finite set of open balls of radius at most ε such that T is covered by their union; then let $N(\varepsilon)$ be the least number of balls in such a cover. In studying Gaussian processes, the pseudometric $d(s, t)$ is taken to be the standard deviation of $X(s)-X(t)$. The sample functions are shown to be continuous or bounded if the function $N(\varepsilon)$ increases at a sufficiently slow rate as $\varepsilon \downarrow 0$. In the non-Gaussian case, the pseudometric d is not necessarily the incremental standard deviation.

For the convenience of the reader, we sketch the result of Weber of particular relevance to our work. If there is a pseudometric d such that the two-sided tail probabilities of the random variable (1.1) for $s \neq t$ are sufficiently small, and if the corresponding function $N(\varepsilon)$ grows at a sufficiently slow rate for $\varepsilon \downarrow 0$, then the sample functions are almost surely bounded on T. In two typical examples we show that, with the exception of one case, either my sufficient condition for unboundedness holds, or else Weber's sufficient condition for boundedness holds. This indicates that the well known dichotomy theorem for Gaussian processes has modified versions in the area of more general processes. In both of these examples I have, for the purpose of comparison, chosen the estimate of the density of (1.1), required by my theorem, to be comparable to the estimate of the tail probability of (1.1), required by Weber's theorem. However, in order to apply either of the two theorems in general, it is not necessary that there be a relation between the density of (1.1) and the tail probabilities.

2. The main result

Let $X(t)$, $t\in T$, be a real valued stochastic process, where T is a subset of some measure space. For arbitrary s, $t\in T$, with $s\neq t$, let $p(x; s, t)$ be the density of $X(s)-X(t)$ at x, which we assume exists. Then the density of the random variable (1.1) is

$$d(s, t)p(xd(s, t); s, t)\,, \qquad \text{for} \quad s\neq t\,.$$

Our hypothesis will be stated as conditions on the smoothness and growth of this function of x. More exactly, we will suppose that the function has an analytic extension to a strip in the complex plane, and that the latter extension satisfies specified growth requirements. The function d is not necessarily a pseudometric, but it is required to be nonnegative and tend to 0 for $s-t\to 0$. The intuitive meaning of such an assumption on the density of (1.1) is that the values of $X(s)-X(t)$ remain very diffuse for s near t, which explains the unboundedness of the sample functions. The formal hypotheses are now stated:

HYPOTHESIS A. *$p(x; s, t)$ has an extension $p(z; s, t)$ to the complex z-plane for $s\neq t$ which is analytic in the strip $|\operatorname{Im} z|<c$, for some $c>0$ which is independent of (s, t).*

In what follows, $h(u)$, $u\geq 0$, is a nonnegative, nondecreasing function, and $d(s, t)$ is a measurable function on T^2 such that $d(s, t)\geq 0$, and $d(s, t)=0$ for $s=t$.

HYPOTHESIS B. *There exist functions h and d of the type described above such that*

$$\int_{-\infty}^{\infty}|p(-ix+y; s, t)|^2 dy\leq \frac{1}{d(s, t)}h\left(\frac{x}{d(s, t)}\right) \tag{2.1}$$

for all $s\neq t$ and all $0\leq x<c$.

HYPOTHESIS C. *$p(x; s, t)$ is a positive definite function of x for $s\neq t$, and there exist h and d such that*

$$|p(-ix; s, t)|\leq \frac{1}{d(s, t)}h\left(\frac{x}{d(s, t)}\right) \tag{2.2}$$

for all $s \neq t$ *and* $0 \leq x < c$.

For the convenience of the reader, let us briefly recall the definition of the local time. Let $x(t)$, $t \in T$ be a real valued measurable function on the measure space T endowed with the measure μ. For every measurable subset $I \subset T$, and linear Borel set A, define $\nu(A, I) = \mu\{t: t \in I, x(t) \in A\}$. If for fixed I, $\nu(\cdot, I)$ is absolutely continuous as a measure on sets A, then its Radon-Nikodym derivative, denoted $\alpha_I(x)$, is called the local time of $x(t)$ relative to (I, μ). It is obvious from the definition that if the local time exists relative to a measurable set T, then it also exists relative to every measurable subset $I \subset T$. In our work on stochastic processes, the sample function $X(t)$ plays the role of the function $x(t)$.

Theorem 2.1. *Let* μ *be a positive finite measure on* T. *Then under Hypothesis A and B or Hypothesis A and C, if*

$$\int_T \int_T \frac{1}{d(s, t)} h\left(\frac{c}{d(s, t)}\right) d\mu(s) d\mu(t) < \infty , \tag{2.3}$$

for some $c > 0$, *then the local time* $\alpha_T(x)$ *of* $X(t)$ *relative to* (T, μ) *exists and is analytic in the strip* $|\mathrm{Im}\, z| < \frac{1}{2}c$.

Proof. Assume Hypothesis A and B. We aim to establish

$$E\left\{\int_{-\infty}^{\infty} \left|\int_T e^{iuX(s)} d\mu(s)\right|^2 e^{2xu} du\right\} < \infty , \tag{2.4}$$

for all $x < \frac{1}{2}c$. By [3], formula (6.1), this condition is sufficient for the conclusion of our theorem. The fact that the domain of analyticity is $|\mathrm{Im}\, z| < \frac{1}{2}c$ follows from the argument in [2].

Since the integrand in (2.4) is nonnegative, Fubini's theorem permits us to interchange expectation and integration over u. Since $\mu(T)$ is finite, we may also move E inside the double integral after taking the square of the modulus:

$$\int_{-\infty}^{\infty} \int_T \int_T \hat{p}(u; s, t) d\mu(s) d\mu(t) e^{2xu} du ,$$

where $\hat{p}(u; s, t) = E \exp(iu(X(t) - X(s)))$. As the modulus of a characteristic function, $|\hat{p}(u; s, t)|$ is even in u; therefore, the integral above is at most equal to 2 times

$$\int_0^{\infty} \int_T \int_T |\hat{p}(u; s, t)| d\mu(s) d\mu(t) e^{2xu} du .$$

For fixed x, $x < \frac{1}{2}c$, let x' be an arbitrary number such that $x < x' < \frac{1}{2}c$. Then, by the Cauchy-Schwarz inequality, the square of the integral above is at most equal to

$$\int_0^\infty \left(\int_T \int_T |\hat{p}(u; s, t)| d\mu(s) d\mu(t)\right)^2 e^{4x'u} du \int_0^\infty e^{-4u(x'-x)} du .$$

By the moment inequality, the latter is at most equal to $\mu^2(T)/4(x'-x)$ times

$$\int_0^\infty \int_T \int_T |\hat{p}(u; s, t)|^2 d\mu(s) d\mu(t) e^{4x'u} du .$$

By Fubini's theorem, the latter is equal to

$$\int_T \int_T \left\{\int_0^\infty |\hat{p}(u; s, t)|^2 e^{4x'u} du\right\} d\mu(s) d\mu(t) .$$

According to the theorem of Paley and Wiener [8], page 7, the inner integral is at most equal to

$$\int_{-\infty}^\infty |p(-i2x'+y; s, t)|^2 dy;$$

hence, the finiteness of the preceding multiple integral for $x'<\frac{1}{2}c$ follows from (2.1) and (2.3). This completes the proof of (2.4).

Next assume Hypothesis A and C. Then (2.2) and (2.3) imply

$$\int_T \int_T |p(-ix; s, t)| d\mu(s) d\mu(t) < \infty ,$$

for $0 \leq x < c$. In [4] we showed that the latter is sufficient for the conclusion of the theorem in the case where μ is linear Lebesgue measure, and the bivariate density function of $X(s)$ and $X(t)$ is a positive definite kernel. The former is not a real restriction, as the proof extends to an arbitrary measure space (T, μ). The positive definiteness of the joint density was used only as a sufficient condition for the positive definiteness of the density of the increment. The latter forms the hypothesis of the present theorem.

REMARK. Note that the integral appearing in (2.1) is of fundamental importance in the theory of Hardy functions; see, for example, Hoffman [7].

In order to demonstrate the sharpness of our conditions, we will employ some recent results of Weber [9], which, for the convenience of the reader, we will briefly summarize. Let $d(s, t)$ be a pseudometric for T, and suppose that T is compact. Let $\Phi = \Phi(x)$ be a nonincreasing function such that

$$\sup_{s \neq t} P\left(\frac{|X(s)-X(t)|}{d(s, t)} > x\right) \leq \Phi(x) ,$$

for $x>0$, and then the function $R(x)$ as

$$R(x) = x \int_{\Phi^{-1}(1/x)}^\infty \Phi(u) du + \Phi^{-1}(1/x) . \tag{2.5}$$

Weber showed that the sample functions are bounded on T if

$$\int_0^{\mathrm{diam}(T)} R(N(u))du<\infty \,, \tag{2.6}$$

where $N(u)$ is the covering number defined in Section 1.

Since the function R is defined indirectly in terms of Φ, it is useful to obtain an estimate for it more directly in terms of Φ. We deduce the following result: If

$$\limsup_{u\to\infty} \frac{\int_u^\infty \Phi(y)dy}{u\Phi(u)}<\infty \,, \tag{2.7}$$

then

$$\limsup_{u\to\infty} R(u)/\Phi^{-1}(1/u)<\infty \,. \tag{2.8}$$

Indeed, by (2.5), we have

$$R(1/\Phi(u)) = \frac{\int_u^\infty \Phi(y)dy}{\Phi(u)}+u \,,$$

which, by (2.7), is at most a constant multiple of u.

3. Application to a process on a real interval

Let $X(t)$, $0\le t\le 1$, be a real stochastic process satisfying the conditions of either Hypothesis A and B or A and C, for some functions h and d. Put

$$g(t) = \inf_{|s-s'|\ge t} d(s, s') \,, \tag{3.1}$$

and let μ be linear Lebesgue measure. Then (2.3) holds for an arbitrary subset T of $[0, 1]$ if

$$\int_0^1 \frac{1}{g(t)} h\left(\frac{c}{g(t)}\right) dt<\infty \,. \tag{3.2}$$

Let us now specialize the result of Weber in Section 2 to the case $T=[0, 1]$. Define the function $g(t)$, in contrast to (3.1), as

$$g(t) = \sup_{|s-s'|\le t} d(s, s') \,. \tag{3.3}$$

Let N_d be the covering number function for the interval $[0, 1]$ in the pseudometric d. Since an interval of length ε in the Euclidian metric is of length at most $g(\varepsilon)$ in the d-pseudometric, we have

$$N_d(g(\varepsilon))\le \#\ \textit{intervals of Euclidian length at most } \varepsilon \textit{ which cover } [0, 1]\le 2/\varepsilon \,.$$

It follows that

$$N_d(\varepsilon) \leqq 2/g^{-1}(\varepsilon)\,. \tag{3.4}$$

If the estimate (2.8) is valid, then (2.6) holds if

$$\int_0^1 \Phi^{-1}(\tfrac{1}{2}g^{-1}(u))du<\infty\,. \tag{3.5}$$

We conclude that the sample functions are unbounded under condition (3.2) and bounded under condition (3.5).

Consider the class of processes on [0, 1] whose increment distributions satisfy (3.2) with $h(x)=c\exp(bx^\gamma)$ for some positive c, b and γ, and where $g(t)$ in both (3.1) and (3.3) is asymptotically, for $t\to 0$, a constant multiple of $|\log t|^{-\theta}$, for some $\theta>0$. Assume also that Φ may be chosen of the form $\Phi(x)=c\exp(-bx^\gamma)$, for some positive c, b and γ, where γ is the same as for $h(x)$ but where c and b are not necessarily the same. Condition (2.7) holds for this Φ, and so (3.5) is sufficient for sample function boundedness. An elementary calculation shows that (3.2) holds if

$$\theta<1/\gamma\,. \tag{3.6}$$

On the other hand, we have $g^{-1}(u)=\exp(-ku^{-1/\theta})$, for some $k>0$, and $\Phi^{-1}(y)=|b^{-1}\log(y/c)|^{1/\gamma}$, so that (3.5) holds if

$$\theta>1/\gamma\,. \tag{3.7}$$

This result was proved in the particular case of a stationary Gaussian process in [2]. There the functions h and Φ are of the exponential forms above with $\gamma=2$, and the pseudometric d is the standard deviation of the increment. As in that particular case, the nature of the sample functions at the "boundary" $\theta=1/\gamma$ is undetermined.

4. Preliminary analytic results for the case where T is an ellipsoid in l_p

The rest of this paper is devoted to the conditions for boundedness and unboundedness in the case of the ellipsoid. In this section, we obtain some purely analytical results which are needed for our calculations.

Lemma 4.1. *Let $F(x)$ be a distribution function with support in $[0, \infty)$, and such that*

$$m=\int_0^\infty x dF(x)<\infty\,. \tag{4.1}$$

Put

$$f(s)=\int_0^\infty e^{-sx}dF(x)\,;$$

then, for every $q>1$, *there exist* $b_1>0$, *and* $b_2>0$ *such that*

$$\prod_{n=1}^{\infty} f(n^{-q}y) \le b_1 \exp(-b_2 y^{1/q}), \tag{4.2}$$

for all $y \ge 0$.

Proof. For every ε, $0<\varepsilon<m$, there exists $\delta>0$ such that

$$f(s) \le 1 - s(m-\varepsilon), \qquad \text{for} \quad 0 \le s < \delta\,; \tag{4.3}$$

this is a consequence of (4.1). It follows that

$$\begin{aligned} \prod_{n=1}^{\infty} f(n^{-q}y) &\le \prod_{\{n:\, n^{-q}y<\delta\}} f(n^{-q}y) \\ &\le \prod_{n>(y/\delta)^{1/q}} [1-(m-\varepsilon)yn^{-q}] \\ &\le \exp\{-y(m-\varepsilon) \sum_{\{n>(y/\delta)^{1/q}\}} n^{-q}\}\,. \end{aligned} \tag{4.4}$$

The sum in the exponent is at least equal to

$$\int_{(y/\delta)^{1/q}+1}^{\infty} x^{-q}dx\,, \qquad \text{or} \quad \frac{1}{(q-1)}\left\{\left(\frac{y}{\delta}\right)^{1/q}+1\right\}^{1-q},$$

which, for sufficiently large y, is at least equal to

$$\frac{1}{2(q-1)}(y/\delta)^{(1-q)/q}\,.$$

Thus, the last member of (4.4) is at most equal to

$$\exp\left[-(m-\varepsilon)\frac{\delta^{(q-1)/q}y^{1/q}}{2(q-1)}\right].$$

This establishes (4.2) with $b_1=1$ and for all large y. This immediately implies the more general bound (4.2).

Lemma 4.2. *Let* $F(x)$ *be a distribution function on the nonnegative reals whose Laplace-Stieltjes transform satisfies*

$$\int_0^{\infty} e^{-st}\, dF(t) \le b_1 \exp(-b_2 s^{1/q})\,, \qquad \text{for} \quad s \ge 0\,, \tag{4.5}$$

where b_1 *and* b_2 *are positive constants, and* $q>1$. *Then there exist* $c_1>0$ *and* $c_2>0$ *such that*

$$F(x) \le c_1 \exp(-c_2 x^{-(q-1)^{-1}})\,, \qquad \text{for} \quad x>0\,. \tag{4.6}$$

Proof. It is elementary that

$$F(x) \le e^{sx}\int_0^x e^{-sy}dF(y) \le e^{sx}\int_0^{\infty} e^{-sy}dF(y)\,,$$

and the latter, by (4.5), is at most equal to

$$b_1 \exp(sx - b_2 s^{1/q}) . \tag{4.7}$$

Since $s \geq 0$ is arbitrary, we choose that value of s which minimizes the exponent in (4.7), namely,

$$s = \left(\frac{b_2}{qx}\right)^{q/(q-1)},$$

and here the exponent in (4.7) takes the value

$$x^{-(q-1)^{-1}} b^{q/(q-1)} q^{-(q-1)^{-1}} (1-q)/q ,$$

where the constant is negative because $q > 1$.

Lemma 4.3. *Let $F(x)$ be a distribution function satisfying an inequality of the form* (4.6). *Let $H(t)$ $t > 0$, be a positive nondecreasing function such that $H(0+) = 0$. If, for $\omega > 0$,*

$$\lim_{y \to 0} H(\omega/y) F(y) = 0 , \tag{4.8}$$

then,

$$\int_0^\infty H(\omega/y) dF(y) \leq c_1 \int_0^\infty \exp(-c_2 (x/\omega)^{(q-1)^{-1}}) dH(x) . \tag{4.9}$$

Proof. By integration by parts and the condition $H(0+) = 0$, and the assumption (4.8), it follows that

$$\int_0^\infty H(\omega/y) dF(y) = -\int_0^\infty F(y) d_y H(\omega/y) ,$$

and the latter, by a change of variable, is equal to

$$\int_0^\infty F(\omega/y) dH(y) .$$

By (4.6), the expression above is dominated by the right hand member of (4.9).

Finally, we state the elementary result,

Lemma 4.4. *Let $H(x)$ be a nondecreasing nonnegative function such that for some $\gamma > 0$ and $\delta > 0$,*

$$\lim_{x \to \infty} H(x) e^{-\delta x^\gamma} = 0; \tag{4.10}$$

then, for every $p > -1$, $\delta' > \delta$,

$$\int_0^\infty x^p e^{-\delta' x^\gamma} H(x) dx < \infty . \tag{4.11}$$

5. Estimation of the energy integral for a class of measures on l_p

In this application we take the underlying time parameter set to be the real vector space l_p, for $p\geq 1$. We define a general class of probability measures on the space. Then we state conditions on the function h and on the function d which are sufficient for the finiteness of the energy integral, taken over the whole space. In the next section we show that the measure is positive on the class of ellipsoids under consideration. This implies the analyticity of the local time relative to the ellipsoid, and so the sample function is unbounded on it.

Let $\{\xi_n, n\geq 1\}$ be a sequence of independent random variables with a common distribution such that $E|\xi_1|^p<\infty$. For arbitrary $q>1$, the random sequence $X=\{n^{-q/p}\xi_n\}$ belongs to l_p because $E\sum n^{-q}|\xi_n|^p=E|\xi_1|^p\sum n^{-q}<\infty$. Let μ be the probability measure on l_p induced by the distribution of X.

We will consider functions $d(s, t)$ on $(l_p)^2$ of the following form; Let $w(u)$, $u\geq 0$, be a nondecreasing function such that $w(0)=0$ and $w(\infty)=\infty$. For arbitrary s and t, put

$$d(s, t) = w(||s-t||^p)\,, \tag{5.1}$$

where $||\cdot||$ is the usual l_p norm. d is not necessarily a pseudometric.

Now let Y be another random point in l_p with the same distribution as X and independent of it, so that it has the representation $Y=\{n^{-q/p}\xi'_n\}$ where ξ'_n has the same distribution as ξ_n and where ξ_i, ξ'_j, $i, j\geq 1$, are mutually independent. Then the d-distance between X and Y is the function w at

$$\sum_{n=1}^{\infty} n^{-q}|\xi_n-\xi'_n|^p\,. \tag{5.2}$$

Let h be the function defined in Hypothesis B or C; then, for fixed $c>0$, define

$$H(x)=\frac{1}{w(1/x)}\,h\left(\frac{c}{w(1/x)}\right), \tag{5.3}$$

which is a nondecreasing function of x. If $T=l_p$, then condition (2.3) assumes the form

$$EH\left(\frac{1}{\sum_{n\geq 1} n^{-q}|\xi_n-\xi'_n|^p}\right)<\infty\,. \tag{5.4}$$

Lemma 5.1. *If there are numbers $\gamma>0$ and $\delta>0$ such that the function $H(x)$ satisfies*

$$\lim_{x\to\infty} H(x)e^{-\delta x^{\gamma}} = 0\,, \tag{5.5}$$

then,

$$EH\left(\frac{\omega}{\sum_{n\geq 1} n^{-q}|\xi_n-\xi_n'|^p}\right)<\infty\,, \tag{5.6}$$

for $q=1+1/\gamma$, and all sufficiently small $\omega>0$.

Proof. Put $f(s)=E\exp(-s|\xi_1-\xi_1'|^p)$; then $\sum_{n\geq 1} n^{-q}|\xi_n-\xi_n'|^p$ has a distribution function G with the Laplace-Stieltjes transform

$$\int_0^\infty e^{-st}dG(t)=\prod_{n=1}^\infty f(n^{-q}s)\,.$$

By Lemma 4.1, there exist constants b_1 and b_2 such that the latter product is at most equal to

$$b_1\exp(-b_2s^{1/q})\,. \tag{5.7}$$

Now the expected value in (5.6) may be written as

$$\int_0^\infty H(\omega/t)dG(t)\,. \tag{5.8}$$

Now we apply Lemma 4.3 with G in the role of F. First we note that, by the definition of H in (5.3) we have $H(0+)=0$. Next, it follows from Lemma 4.2 and the bound (5.7) that

$$G(y)\leq c_1\exp(-c_2y^{-(q-1)^{-1}})\,;$$

furthermore, (5.5) implies that

$$H(\omega/y)\leq\exp(\delta\omega^\gamma y^{-\gamma})\,,$$

for all sufficiently small y; therefore, the condition corresponding to (4.8) is satisfied if $\delta\omega^\gamma$ is chosen to be smaller than c_2. (Here we also use the fact that $q=1+1/\gamma$.) Since the conditions in the hypothesis of Lemma 4.3 are satisfied, we may apply the conclusion (4.9) to (5.8), and infer that the latter is at most equal to

$$c_1\int_0^\infty\exp(-c_2(x/\omega)^\gamma)dH(x)\,.$$

By integration by parts, it follows from Lemma 4.4 that the integral above is finite if ω is chosen so small that $c_2\omega^{-\gamma}>\delta$.

6. Conditions for the unboundedness of the sample function on the ellipsoid

Following Chevet [5] we define an r-ellipsoid in l_p, where $r\geq 1$. Let $\{b_n, n\geq 1\}$ be a nonincreasing sequence of positive numbers, and consider the set of sequences $x=(x_n)$ in l_p such that

$$\sum_{n=1}^{\infty}(|x_n|/b_n)^r \leq 1 . \tag{6.1}$$

This a called an r-ellipsoid with semi-axes $\{b_n\}$.

Lemma 6.1. *If ξ_1 has a density function which is continuous and positive in some neighborhood of the origin, and if $E|\xi_1|^r<\infty$, and if*

$$\sum_{n=1}^{\infty}(n^{q/p}b_n)^{-r}<\infty , \tag{6.2}$$

then the measure μ defined in Section 5 assigns positive probability to the ellipsoid (6.1).

Proof. We show that the coordinates $x_n=n^{-q/p}\xi_n$ of the random point X satisfy

$$\sum_{n=1}^{\infty}\frac{(n^{-q/p}|\xi_n|)^r}{b_n^r} \leq 1 \tag{6.3}$$

with positive probability. First of all, the expected value of the sum above is equal to

$$E|\xi_1|^r \sum_{n=1}^{\infty}(n^{q/p}b_n)^{-r} ,$$

which, by (6.2), is finite. Next, for any $m\geq 2$, we write the sum in (6.3) as the sum of two subsums,

$$\sum_{n=1}^{\infty} = \left(\sum_{n=1}^{m}+\sum_{n=m+1}^{\infty}\right)\left(\frac{n^{-q/p}|\xi_n|}{b_n}\right)^r \tag{6.4}$$

where the two sums on the right are independent. The first sum, over indices $1\leq n\leq m$, has a density function which is continuous and positive in an interval $[0, \varepsilon]$ for some $\varepsilon>0$, because it is the convolution of density functions which, by hypothesis, have this property. On the other hand, for every $\varepsilon>0$ and $\delta>0$, there exists an integer $m\geq 1$ such that

$$P\left(\sum_{n=m+1}^{\infty}\frac{n^{-qr/p}|\xi_n|^r}{b_n^r}>\varepsilon\right)\leq \delta .$$

It follows that the density of the left hand member of (6.3) exists and is the convolution of a density which is positive and continuous on $[0, \varepsilon]$ with a density whose integral over $[0, \varepsilon]$ is at least equal to $1-\delta$. Therefore, the convolution is positive and continuous on $[0, \varepsilon]$, and so (6.3) has positive probability.

Theorem 6.1. *Let $X(t)$, $t\in l_p$, be a real stochastic process satisfying the* *conditions of either Hypothesis A and B or Hypothesis A and C, with d(s, t) of the form* (5.1), *and where the function H, defined in terms of h by* (5.3) *for some c, satisfies the condition* (5.5) *for some* δ *and* γ. *If the semi-axes of the ellipsoid T, defined by* (6.1), *satisfy the condition* (6.2) *with*

$$q = 1+1/\gamma\,, \tag{6.5}$$

then the local time $\alpha_T(x)$ *exists and has an analytic extension* $\alpha_T(z)$ *in the strip* $|\mathrm{Im}\, z|<c$. *Thus, in particular, the sample function is unbounded.*

Proof. Let μ be defined as in Section 5, and let μ_ε be the measure defined in the same way as μ except that the random variables ξ_n are replaced by ξ_n/ε, for arbitrary $\varepsilon>0$. It is obvious that μ_ε inherits the properties of μ: It is a measure on l_p which assigns positive probability to the ellipsoid (6.1). If $\omega=\varepsilon^p$, then the energy integral in (5.4) with respect to μ_ε is identical with the energy integral in (5.6) with respect to the measure μ. Thus if condition (5.5) holds, then (5.6) holds, and so does (5.4) for the measure μ_ε.

The condition on the density function of ξ_1 required in the hypothesis of Lemma 6.1 can always be fulfilled; indeed, ξ_1 can be taken with a standard normal distribution. The conclusion of the theorem is now a consequence of Theorem 2.1 and Lemma 6.1.

We remark that the method of proof is a generalization of the method used in [3] for the local time of the Lévy Brownian motion over an ellipsoid in l_2.

7. An example illustrating the sharpness of the conditions for unboundedness

In this section we present a general class of processes on l_p and a class of ellipsoids and show, by comparison to Weber's conditions, that neither his nor our conditions can be appreciably improved.

We will take the distance function d in (5.1) to be

$$d(s, t) = \|s-t\|^{\alpha}\,, \qquad \text{for some} \quad \alpha\le 1\,, \tag{7.1}$$

which is the same as choosing $w(u)=u^{\alpha/p}$. We take the function h in the statement of Hypothesis B or C to be $h(x)=d\exp(bx^{\gamma})$ for some positive b, d and γ. Then the function $H(x)$ in (5.3) is of the form

$$H(x) = dx^{\alpha/p}\exp(bc^{\gamma}x^{\alpha\gamma/p})\,; \tag{7.2}$$

thus, the condition (5.5) is satisfied for $\delta>bc^{\gamma}$, and with $\alpha\gamma/p$ in place of γ.

Suppose that the sequence b_n satisfies the condition

$$\liminf_{n\to\infty} b_n n^s>0\,, \tag{7.3}$$

for some $s>0$. Then the sufficient condition (6.2) of Theorem 6.1 holds with $q=1+p/(\alpha\gamma)$ if $sr-r(1+p/(\alpha\gamma))/p<-1$, or equivalently,

$$s<1/\alpha\gamma+1/p-1/r\,. \tag{7.4}$$

Let us now derive the conditions for the boundedness of the sample function by means of Weber's results described in Section 2. We take d in (7.1) as the pseudometric, and $\Phi(x)=d'\exp(-b'x^{\gamma})$, with the same γ as before. Let $N_d(\varepsilon)$ be the covering number of the ellipsoid (6.1). It will now be obtained from results of Chevet [5]. Let λ be the exponent of convergence of the sequence $\{b_n\}$, that is,

$$\lambda=\inf(\lambda':\sum_{n\geq 1} b_n^{\lambda'}\text{ converges})\,.$$

If $\lambda<\infty$, then, if ρ is the exponent of entropy of the ellipsoid in l_p, we have, in our notation,

$$\rho=(\lambda^{-1}+r^{-1}-p^{-1})^{-1}\,. \tag{7.5}$$

This means that the covering number $N(\varepsilon)$, computed with respect to the metric of the l_p norm, satisfies

$$\rho=\limsup_{\varepsilon\to 0}\frac{\log\log N(\varepsilon)}{\log(1/\varepsilon)}\,.$$

This implies, for arbitrary $\theta>1$,

$$N(\varepsilon)\leq\exp(\varepsilon^{-\theta\rho})\,, \tag{7.6}$$

for all sufficiently small $\varepsilon>0$.

Let ρ_d be the exponent of entropy of the ellipsoid in the d-metric. Since the latter is the α-power of the l_p metric, a ball of radius ε in the latter metric is of radius ε^{α} in the d-metric; thus, it follows from the definition of the exponent of entropy that

$$\rho_d=\rho/\alpha\,. \tag{7.7}$$

It follows from (7.5), (7.6) and (7.7) that

$$N_d(\varepsilon)\leqq\exp(\varepsilon^{-\alpha^{-1}(\lambda^{-1}+r^{-1}-p^{-1})\theta})\,, \tag{7.8}$$

for all small $\varepsilon>0$. As in Section 3, Weber's R function is of the form $|b'^{-1}\log(d'/x)|^{1/\gamma}$, so that the condition (2.6) for boundedness becomes

$$\int_0^{\mathrm{diam}(T)} u^{-\theta\alpha^{-1}\gamma^{-1}(\lambda^{-1}+r^{-1}-p^{-1})^{-1}}du<\infty\,, \tag{7.9}$$

which holds if the expression in the exponent above is greater than -1, that is,

$$\lambda^{-1} > (\alpha\gamma\theta)^{-1} + p^{-1} - r^{-1} .$$

But there exists a $\theta > 1$ such that the latter holds if

$$\lambda^{-1} > (\alpha\gamma)^{-1} + p^{-1} - r^{-1} . \tag{7.10}$$

We conclude from the arguments above that the sample function is unbounded if (7.4) holds and where s satisfies (7.3); and that the sample function is bounded if (7.10) holds and where λ is the exponent of convergence. In particular, if $b_n \sim$ constant n^{-s}, for $n \to \infty$, then there is boundedness if (7.10) holds with $\lambda^{-1} = s$, and unboundedness if the opposite inequality (7.4) holds. The case where equality holds is open. This is an extension of the result in [3]; see the last example, and the note added in proof.

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