

## AN APPROXIMATION THEOREM FOR MARKOV PROCESSES

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### 1. Introduction

In [1], Watanabe proved that for every Markov process  $X$ , under some conditions, there exists a sequence of regular step processes (R.S.P.)  $X^n$  such that the resolvents of  $X^n$  converge weakly to the resolvent of  $X$ . Under some supplementary conditions we shall prove that the distributions of  $X^n$  converge to the distribution of  $X$ . An intuitive description of  $X^n$  is as follows:  $X$  and  $X^n$  start from the same state  $x_0$  (we mean that  $X_0$  and  $X_0^n$  have the same distribution). If  $X$  remains closed to  $x_0$  for a time  $T_n$  (that is,  $d(x_0, X_t) < \frac{1}{n}$  for all  $t < T_n$  and  $X_{T_n} = x_1$ , with  $d(x_0, x_1) \geq \frac{1}{n}$ ), then  $X_t^n = x_0$  for all  $t < D_n$ , with  $D_n$  an exponentially distributed holding time with same mean value as  $T_n$  ( $T_n$  is generally not exponentially distributed). Then  $X^n$  jumps in  $x_1$  (we mean that  $X_{T_n}$  and  $X_{D_n}^n$  have the same distribution), and so on.

The rigorous construction of  $X^n$  and Watanabe's result are presented in the beginning of the paper. The theorem following this construction is the main result of the paper.

### 2. Main results

Let  $E$  be a locally compact with countable base space (L.C.C.B.),  $\mathcal{U}$  an open base and  $d$  any metric of  $E$ . For each  $n$  we can choose the system  $U_i^n$ ,  $i \in N$  and  $V_i^n$ ,  $i \in N$  of sets in  $\mathcal{U}$  satisfying the following conditions:

- (1) Each  $\bar{U}_i^n$  is compact and  $d(U_i^n) < \frac{1}{n}$  ( $d(A) = \sup (d(x, y); x, y \in A)$ );
- (2)  $V_i^n \subseteq U_i^n$ ;
- (3)  $\bigcup_i V_i^n = E$ ;
- (4) For every compact set  $K$  only a finite number of  $V_i^n$  intersect with  $K$ .

Let  $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$  be a standard process with state space  $E$  and  $(U_\alpha)_{\alpha > 0}$  be the resolvent of  $X$ . We now define  $\sigma_\alpha^n$  by

$$\begin{aligned} \sigma_1^n(\omega) &= \inf(t; X_t(\omega) \notin U_1^n) && \text{if } X_0(\omega) \in V_i^n - \bigcup_{j < i} V_j^n \\ \sigma_k^n(\omega) &= \sigma_{k-1}^n + \sigma_1^n \circ \theta_{\sigma_{k-1}^n} && \text{for } k > 1. \end{aligned}$$

The following result is Lemma 3.3 in Watanabe's work. Let  $X$  be a standard process such that  $U_\alpha(C_b(E)) \subseteq C_b(E)$  for  $\alpha > 0$  and  $\sup_{x \in E} U_0(x, E) < \infty$ . Then the following assertions hold:

(i) For each  $n$ ,  $q_n(x) = [E^x(\sigma_1^n)]^{-1}$  and  $\Pi_n(x, A) = P^x(X_{\sigma_1^n} \in A)$  represents the parameters of a R.S.P.

The corresponding R.S.P. are denoted by  $X^n$  and the resolvent of  $X^n$  by  $U_\alpha^{(n)}$ .

(ii)  $X^n$  is an approximation of  $X$  in the following sense:

$$(5) \quad \lim_n U_\alpha^{(n)} f(x) = U_\alpha f(x) \quad \text{for every } x \in E \text{ and } f \in C_b(E).$$

We shall need instead of (5) a stronger result. For any compact set  $K$ ,  $\alpha > 0$  and  $f \in C_b(E)$

$$(6) \quad \lim_n U_\alpha^{(n)} f(x) = U_\alpha f(x) \quad \text{uniformly for } x \in K.$$

Considering the proof of Lemma 3.1 in [1] it is obvious that to obtain (6) it is sufficient to prove the following condition:

(c) For any compact set  $L \subseteq E$ ,  $\alpha > 0$  and  $\varepsilon > 0$ , we may choose a compact set  $K$  such that  $L \subseteq K$  and

$$(7) \quad \overline{\lim}_n \sup_{x \in L} U_\alpha^{(n)}(x; \mathcal{C}K) \leq \varepsilon.$$

Through this paper we shall consider on  $E$  a metric  $d$  such that  $B_h(x) = \{y; d(x, y) < h\}$  has compact closure for any  $h > 0$  and  $x \in E$ . For  $h > 0$  we define  $T_h^k$ ,  $k \in N$  by

$$\begin{aligned} T_h &= \inf(t; d(X_0, X_t) > h), \\ T_h^1 &= T_h \quad \text{and} \quad T_h^{k+1} = T_h^k + T_h \circ \theta_{T_h^k}. \end{aligned}$$

We shall also consider the function

$$q(h) = \sup_{x \in E} [E^x(T_h)]^{-1}.$$

We note that  $q(h)$  and  $q_n(x)$  are distinct notations. The function  $h \rightarrow q(h)$  is monotone, so we may choose, for every  $n$ ,  $h_n$  and  $d_n$  such that  $\lim_n h_n = 0$ ,  $d_n < h_n$  and

$$(8) \quad \lim_n \frac{q(h_n - d_n)}{q(h_n + d_n)} = 1.$$

Now we shall choose the above mentioned  $U_i^n$  and  $V_i^n$  in the following

particular form:  $U_i^n = B_{h_n}(x_i)$  and  $V_i^n = B_{d_n}(x_i)$  with  $x_i, i \in N$ , chosen such that condition (4) is fulfilled.  $\sigma_i^n$  will be defined like above with respect to this system of sets. The following two inequalities will be useful in what it follows

$$(9) \quad T_{h_n-d_n} \leq \sigma_1^n \leq T_{h_n+d_n}.$$

Now we assume that the following condition holds for  $X$ . There is some  $a > 0$  such that for every  $h > 0$

$$\sup_{x \in E} E^x(T_h) \leq a \inf_{x \in E} E^x(T_h).$$

Then, by (9) we may conclude that

$$(10) \quad aq(h_n-d_n) \geq q_n(x) \geq q(h_n+d_n).$$

Now we are able to formulate the result of the paper.

**Theorem.** *Let  $X$  be a standard process with state space  $E$  such that*

- (i)  $U_\alpha(C_b(E)) \subseteq C_b(E)$  for every  $\alpha > 0$ .
- (ii)  $\lim_{t \rightarrow 0} P_t f(x) = f(x)$  uniformly on  $E$ , for every  $f \in C_c(E)$ .
- (iii)  $\sup_{x \in E} U_0(x, E) < \infty$ .
- (iv) *There is some  $a > 0$  such that for every  $h > 0$*

$$\sup_{x \in E} E^x(T_h) \leq a \inf_{x \in E} E^x(T_h).$$

- (v) *There is some  $c > 0$  such that for every  $x \in E$  and  $h > 0$*

$$E^x(D_h^z) \geq c \|E^{(\cdot)}(D_h^z)\|$$

with

$$\|E^{(\cdot)}(\varphi)\| = \sup_{y \in E} E^y(\varphi) \quad \text{and}$$

$$(11) \quad D_h^z(\omega) = \inf(t; X_t(\omega) \notin B_h(x)).$$

Then,  $\lim_n P_n^\mu = P$  for every  $\mu$  (probability measure on  $E$ ). (We denote by  $P_n^\mu$  the distribution of the R.S.P.  $X^n$  which has initial measure  $\mu$ ).

We note that condition (iv) implies

$$(vi) \quad \lim_{h \rightarrow 0} \sup_{x \in E} E^x(T_h) = 0.$$

That is because for any  $x \in E, \lim_{h \rightarrow 0} E^x(T_h) = 0$ .

The proof will go as follows: In the first part we establish the similarities between  $X$  and  $X^n$ . We refer to Appendix 1 which presents the law of large numbers in two forms which are appropriate to our deal. The first two Lemmas assure that we may use the results in Appendix 1. We use it in Lemma

3 which is essential for the whole proof. Roughly speaking this lemma establishes the similarity between the "time" of  $X^n$  and the time of  $X$ . Lemma 4 is a simple remark which assures that the "space" of  $X^n$  and the "space" of  $X$  coincide. These similarities are used in all the following, in order to evaluate quantities referring to  $X^n$  by their analogues with respect to  $X$ .

In the second part of the proof we establish the tightness of the sequence  $P_n^x$ ,  $n \in \mathbb{N}$ . The last part deals with the convergence of the marginal distributions. We use here Watanabe's result in his stronger form (6). To do it we prove first (c), and then we refer to Appendix 3 which enables us to check the convergence of the marginal distributions by the convergence of the resolvents.

### 3. Proofs

We first define, for all  $h > 0$ ,

$$(12) \quad \tilde{F}_h(t) = \inf_{x \in \mathbb{B}} P^x(T_h < t) = 1 - \sup_{x \in \mathbb{B}} P^x(T_h > t).$$

$\tilde{F}_h$  is infimum of a family of increasing functions which are right continuous and have left hand limits, then so is  $\tilde{F}_h$ . Next, it is obvious that  $\tilde{F}_h(0) = 0$ , and so, in order to show that  $\tilde{F}_h$  is a distribution function on  $R_+$ , it will suffice to see that

$$(13) \quad \lim_{t \rightarrow \infty} \tilde{F}_h(t) = 1.$$

By Chebyshev's inequality  $P^x(T_h > t) \leq \frac{1}{t} E^x(T_h)$  and so

$$\tilde{F}_h(t) \geq 1 - \frac{1}{t} \sup_{x \in \mathbb{B}} E^x(T_h).$$

Because  $U_0(x, E) = E^x\left(\int_0^\infty \mathbf{1}_E \circ X_t dt\right) = E^x(\zeta)$  with  $\zeta = \inf(t; X_t = \Delta)$ , by (ii),  $\sup_{x \in \mathbb{B}} E^x(T_h) \leq \sup_{x \in \mathbb{B}} E^x(\zeta) < \infty$  and so (13) is proved.

We denote by  $F_h$  the distribution on  $R_+$  corresponding to  $\tilde{F}_h$ . It is obvious that for any  $t$  and  $x$

$$(14) \quad F_h([0, t]) \leq P^x(T_h \leq t) \quad \text{and} \quad F_h((t, \infty)) = \sup_{x \in \mathbb{B}} P^x(T_h > t).$$

In order to simplify notations we shall denote

$$(15) \quad Y_k^n = X_{\sigma_k^n} \quad \text{and} \quad Z_k^n = X_{\tau_k^n}.$$

with  $\tau_1 = \inf(t; X_t^n \neq X_0^n)$  and  $\tau_k = \tau_{k-1} + \tau_1 \circ \theta_{\tau_{k-1}}$ . Next, let us put

$$t_n = 2 \sup_{x \in \mathbb{B}} \|E^{(\cdot)}(D_{h_n}^x)\| \quad (\text{see (11)}).$$

The following relations will be used to prove Lemma 1

$$(16) \quad \begin{aligned} (a) \quad & \lim_n t_n = 0, \\ (b) \quad & F_{h_n}((k \cdot t_n, \infty)) \leq \left(\frac{1}{2}\right)^k. \end{aligned}$$

Because  $T_h = D_h^x$   $P^x$  a.s., (a) is a consequence of (v) and (vi). To prove (b) we shall use exercise (10.25) in [2]: If  $\alpha(t) = \sup_{x \in \mathbb{B}} P^x(D_U > t)$ , then

$$(17) \quad P^x(D_U > k \cdot t) \leq \alpha(t)^k$$

( $U$  is a measurable set and  $D_U = \inf\{t; X_t \in U\}$ ). Next we consider  $t = t_n$  and  $U = CB_{h_n}(x)$  then  $D_U = D_{h_n}^x$  and by Chebyshev's inequality we obtain

$$P^y(D_{h_n}^x > t_n) \leq \frac{1}{t_n} E(D_{h_n}^x).$$

We take the supremum over all  $y \in E$  and considering the definition of  $t_n$  and  $\alpha$  we conclude that

$$\alpha(t_n) \leq \frac{1}{2}.$$

By (14) and (17) we get (b)

$$F_{h_n}((k \cdot t_n, \infty)) = \sup_{x \in \mathbb{B}} P^x(D_{h_n}^x > k \cdot t_n) \leq \left(\frac{1}{2}\right)^k.$$

**Lemma 1.** (a) For every  $a > 0$

$$\lim_n \int_a^\infty z F_{h_n}(dz) \cdot \left[ \int_0^\infty z F_{h_n}(dz) \right]^{-1} = 0.$$

$$(b) \quad \int_0^\infty z F_{h_n}(dz) \leq \frac{4}{c} \sup_{x \in \mathbb{B}} E^x(T_{h_n}) = \frac{4}{c} [q(h_n)]^{-1}$$

where  $c$  is defined in (v).

Proof. Let be  $k_n \in \mathbb{N}$  such that  $k_n \leq \frac{a}{t_n} < k_n + 1$ ; we have

$$\int_a^\infty z F_{h_n}(dz) \leq (k_n + 1) F_{h_n}([k_n \cdot t_n, \infty)) + t_n \sum_{k > k_n} F_{h_n}([k \cdot t_n, \infty)).$$

By (16) we get

$$(18) \quad \int_a^\infty z F_{h_n}(dz) \leq t_n (k_n + 2) 2^{-k_n}.$$

It is obvious that for every  $x \in E$

$$\int_0^\infty z F_{h_n}(dz) \geq \int T_{h_n} P^x(d\omega)$$

and so, by (v)

$$\int_0^\infty z F_{h_n}(dz) \geq \sup_{x \in \mathbb{R}} E^x(T_{h_n}) \geq c \sup_{x \in \mathbb{R}} \|E^{(\cdot)}(D_{h_n}^x)\| = \frac{c}{2} t_n.$$

The right continuity of the trajectories assures that  $D_{h_n}^x > 0$   $P^x$  a.s. (see (11)) and then  $t_n > 0$  and we may write

$$\left[ \int_a^\infty z F_{h_n}(dz) \right] \cdot \left[ \int_0^\infty z F_{h_n}(dz) \right]^{-1} \leq t_n (k_n + 2) 2^{-k_n} \cdot \left[ \frac{c}{2} t_n \right]^{-1} = \frac{c}{2} (k_n + 2) 2^{-k_n}.$$

The last term vanishes when  $n \rightarrow \infty$  and (a) is proved. Now we have

$$\int_0^\infty z F_{h_n}(dz) \leq t_n F_{h_n}((0, \infty)) + t_n \sum_{k=1}^\infty F_{h_n}((k \cdot t_n, \infty)) \leq \frac{4}{c} \sup_{x \in \mathbb{R}} E^x(T_{h_n})$$

and (b) is also proved.

**Lemma 2.** For every  $k \in N$  and  $l > 0$ ,

- (a)  $P^\mu(\sigma_k^n > l) \leq F_{h_n + d_n}^{*k}(l, \infty)$ ,
- (b)  $P_n^\mu(\tau_k \leq l) \leq e_{\alpha}^{*k}(h_n - d_n)(0, l)$ ,

(a is defined in (iv) and  $e_\alpha$  is the exponential distribution with parameter  $\alpha$ ).

Proof. For  $F$  and  $G$  distributions on  $R_+$

$$F * G(l, \infty) = \int_0^\infty F(l-t, \infty) G(dt)$$

and so, if  $F$  and  $F'$  are such that  $F(s, \infty) \leq F'(s, \infty)$  for every  $s \in R_+$ , then

$$(19) \quad F * G(l, \infty) \leq F' * G(l, \infty).$$

To prove (a) we proceed by induction on  $k$ . For  $k=1$ , (a) is (12). Using the strong Markov property for two variables functions we get

$$P^\mu(\sigma_k^n > l) = P(\sigma_{k-1}^n + \sigma_1^n \circ \theta_{\sigma_{k-1}^n} > l) = \int P^{Y_k^n(\omega)}(\sigma_1^n > l - \sigma_{k-1}^n(\omega)) P(d\omega)$$

(see (15)). By (9) and (12), for every fixed  $\omega$

$$P^{Y_{k-1}^n(\omega)}(\sigma_1^n > l - \sigma_{k-1}^n(\omega)) \leq P^{Y_{k-1}^n(\omega)}(T_{h_n + d_n} > l - \sigma_{k-1}^n(\omega)) \leq F_{h_n + d_n}(l - \sigma_{k-1}^n(\omega), \infty)$$

and so

$$P(\sigma_k^n > l) \leq P^\mu \circ (\sigma_{k-1}^n)^{-1} * F_{h_n + d_n}(l, \infty).$$

Now, using the induction hypothesis and (19) we finish the proof.

In order to prove (b) we obtain in the same way as above

$$P_n^\mu(\tau_k \leq l) = \int P_n^{Z_{k-1}^n(\omega)}(\tau_1 \leq l - \tau_{k-1}(\omega)) P_n^\mu(d\omega)$$

see ((15)). With respect to  $P_n^{Z_{k-1}^n(\omega)}$ ,  $\tau_1$  is exponentially distributed with parameter  $q_n(Z_{k-1}^n(\omega)) \leq aq(h_n - d_n)$  (see (10)). Then

$$P_n^{Z_{k-1}^n(\omega)}(\tau_1 \leq l - \tau_{k-1}(\omega)) \leq e_{aq(h_n - d_n)}(0, l - \tau_{k-1}(\omega)) \quad \text{for every } \omega.$$

We conclude that

$$P_n^\mu(\tau_k \leq l) \leq P_n^\mu \circ \tau_k^{-1} * e_{aq(h_n - d_n)}(0, l)$$

and the proof finishes like above.

**Lemma 3.** For fixed  $k > 0$  and  $\delta > 0$  we define

$$k' = 16kac^{-1}, \quad \delta' = 32\delta ac^{-1} \quad (c \text{ defined in (v) and } a \text{ in (iv)}),$$

$$k_n \in N \text{ such that } k_n \leq 2kq(h_n - d_n)a < k_n + 1,$$

$$l_n \in N \text{ such that } l_n \leq 2q(h_n - d_n)a < l_n + 1,$$

$$A_n = \{\omega; \tau_i - \tau_j > \delta \text{ for every } i, j \text{ such that } l_n \leq i - j \leq k_n\},$$

$$B_n = \{\omega; \sigma_i^n - \sigma_j^n < \delta' \text{ for every } i, j \text{ such that } 0 < i - j \leq l_n\},$$

(We shall use these notations throughout all the rest of the paper). Then

(a)  $\lim_n P_n^\mu(\tau_{k_n} < k) = 0,$

(b)  $\lim_n P^\mu(\sigma_{k_n}^n > k') = 0,$

(c)  $\lim_n P_n^\mu(CA_n) = 0,$

(d)  $\lim_n P^\mu(CB_n) = 0.$

We note that all these limits are uniform with respect to the family  $\{\mu; \mu \text{ probability measure on } E\}$ .

(e)  $\overline{\lim}_n \sup_{x \in E} E^x(\sigma_{k_n}^n; \sigma_{k_n}^n > k') = 0.$

The idea of this lemma is that both  $\sigma_{k_n}^n$  and  $\tau_{k_n}$  are sums of little quantities with the same mean value. If we take  $k_n$  (the number of terms in the sum) such that  $k_n a_n \sim l$  ( $a_n$  is the mean value), then  $\sigma_{k_n}^n \sim l$  and  $\tau_{k_n} \sim l$ . This is the idea of the law of large numbers and to prove the lemma we refer to Appendix 1, which presents appropriate forms of this law.

Proof. (a) By Lemma 2.b

$$P_n^\mu(\tau_{k_n} < k) \leq e_{dq(h_n-d_n)}^{*k}(0, k).$$

$$\lim_n aq(h_n-d_n)^{-1}k_n = 2k$$

and so, by Lemma 3, Appendix 1 we get

$$\lim_n e_{dq(h_n-d_n)}^{*k}(0, k) = 0 \quad (\text{independent of } \mu).$$

(b) By Lemma 2.a,

$$P_n^\mu(\sigma_{k_n}^n > k') \leq F_{h_n+d_n}^{*k_n}(k', \infty).$$

By Lemma 1.b,

$$M(F_{h_n+d_n}) \leq \frac{4}{c}(q(h_n+d_n))^{-1}$$

and so

$$M(F_{h_n+d_n}) \leq \frac{4}{c}(2kq(h_n-d_n)a)(q(h_n+d_n))^{-1}$$

and by (8) we obtain

$$\overline{\lim}_n M(F_{h_n+d_n})k_n \leq \frac{4}{c} \cdot 2ka = \frac{8ka}{c} = \frac{k'}{2}.$$

Lemma 1.a assures that conditions in Lemma 1 and Corollary 2, Appendix 1 are fulfilled and so  $\overline{\lim}_n F_{h_n+d_n}(k', \infty) = 0$  and (b) is proved.

To prove (c) we note that for  $i, j$  with  $i-j \geq l_n$ ,  $\tau_i - \tau_j \geq \tau_{j+l_n} - \tau_j$  and so  $CA_n \subseteq \bigcup_{j \leq k_n} (\tau_{j+l_n} - \tau_j < \delta)$ . Then  $P_n^\mu(CA_n) \leq \sum_{j \leq k_n} P_n^\mu(\tau_{j+l_n} - \tau_j < \delta)$ . By the Markov property

$$P_n^\mu(\tau_{j+l_n} - \tau_j < \delta) = E_n^\mu(PZ_j^n(\tau_{l_n} < \delta))$$

which is dominated by  $e_{dq(h_n-d_n)}^{*l_n}(0, \delta)$  (see Lemma 2.b) and so

$$P_n^\mu(CA_n) \leq k_n e_{dq(h_n-d_n)}^{*l_n}(0, \delta)$$

$$\lim_n \frac{k_n}{l_n} = \frac{k}{\delta} < \infty \quad \text{and} \quad \lim_n \frac{l_n}{aq(h_n-d_n)} = 2\delta,$$

and so, by Lemma 3, Appendix 1 the term in the right of the above inequality vanishes when  $n \rightarrow \infty$ .

To prove (d) we note that  $\omega \in CB_n$  implies that there is some  $i, j \leq k_n$  such that  $0 \leq i-j \leq l_n+1$  and  $\sigma_i^n - \sigma_j^n > \delta'$ .

Then, there is some  $p \leq \frac{k_n}{l_n+1}$  such that

$$\sigma_{p(l_n+1)}^n - \sigma_{(p-1)(l_n+1)}^n > \frac{\delta'}{2}.$$

We conclude that

$$P^\mu(CB_n) \leq \sum P^\mu\left(\left(\sigma_{p(l_n+1)}^n - \sigma_{(p-1)(l_n+1)}^n\right) > \frac{\delta'}{2}\right)$$

with the sum over  $p < \frac{k_n}{l_n+1}$ . By the strong Markov property and Lemma 2.a we obtain

$$P^\mu\left(\left(\sigma_{p(l_n+1)}^n - \sigma_{(p-1)(l_n+1)}^n\right) > \frac{\delta'}{2}\right) \leq F_{h_n+d_n}^{*(l_n+1)}\left(\frac{\delta'}{2}, \infty\right)$$

for every  $p$ , and so

$$P^\mu(CB_n) \leq \frac{k_n}{l_n+1} F_{h_n+d_n}^{*(l_n+1)}\left(\frac{\delta'}{2}, \infty\right).$$

The proof ends like to the points (b) and (c).

To prove (e) we note that Lemma 2.a implies that

$$E^x(\sigma_{k_n}^n; \sigma_{k_n}^n > k') = \int_{(z > k')} z P^x \circ \sigma_{k_n}^{n-1}(dz) \leq \int_{(z > k')} z F_{h_n+d_n}^{*k_n}(dz).$$

Because

$$\lim_n F_{h_n+d_n}^{*k_n} = \varepsilon_k \quad \text{with } k < k'$$

the term in the right of the above inequality vanishes under  $\overline{\lim}_n \sup_x$  and (e) is proved.

**Lemma 4.**  $(Y_k^n, k \in N)$  has, with respect to  $P^\mu$  the same distribution as  $(Z_k^n, k \in N)$  with respect to  $P_n^\mu$  (see (15)).

Proof. That is because both of them are Markov chains with initial distribution  $\mu$  and kernel

$$\Pi_n(x, dy) = P^x \circ X_{\sigma_1}^{-1}(dy).$$

Now, in order to prove the relative compactness of the sequence  $P_n^\mu, n \in N$ , we shall use Theorem 2, page 429 in [3] which we write down for processes with time  $[0, \infty)$ . The tightness of  $P_n, n \in N$ , is equivalent to the following conditions

(1) for every  $\varepsilon > 0$  and  $k > 0$  there is some compact set  $K \subseteq E$  such that  $\lim_n P_n^\mu(\omega; \text{there is some } t < k \text{ such that } X_t \notin K) \leq \varepsilon$  (We shall shorten the above expression by writing “ $(\omega; (\exists)t \leq k, X_t \notin K)$ ”).

(2)  $\lim_{\delta \rightarrow 0} \overline{\lim}_n P_n^\mu(\omega; W_{k', \delta}''(X^n) > \varepsilon) = 0$  for every  $k > 0, \varepsilon > 0,$

(3)  $\lim_{\delta \rightarrow 0} \overline{\lim}_n P_n^\mu(\omega; W_{[0, \delta]}(X^n) > \varepsilon) = 0$  for every  $\varepsilon > 0,$

(4)  $\lim_{\delta \rightarrow 0} \overline{\lim}_n P_n^\mu(\omega; W_{[k-\delta, k]}(X^n) > \varepsilon) = 0$  for every  $\varepsilon > 0, k > 0,$

$$W_{[a,b]}(X) = \sup (d(X_t, X_s); a \leq t < s < b),$$

$$W'_{k,\delta}(X) = \sup (\min (d(X_{t'}, X_t), d(X_t, X_{t''})))$$

with the supremum taken over all  $t', t, t''$  such that  $0 \vee (t-\delta) \leq t' < t < t'' \leq k \wedge (t+\delta)$ .

To prove (1) it will suffice to show that for any  $k > 0$ ,  $\varepsilon > 0$  and any compact set  $K \subseteq E$ , there is some  $k' > 0$  such that:

$$\overline{\lim}_n P_n^\mu(\omega; (\exists) t < k, X_t^n \notin K) \leq P^\mu(\omega; (\exists) t \leq k', X_t \notin K)$$

and then we refer to the tightness of  $P^\mu$  itself. By Lemma 3, (a) we know that

$$\overline{\lim}_n P_n^\mu(\omega; (\exists) t < k, X_t^n \notin K)$$

$$= \overline{\lim}_n P_n^\mu(\omega; (\exists) t \leq k, X_t^n \notin K, \tau_{k_n} > k) \leq \overline{\lim}_n P_n^\mu(\omega; (\exists) j < k_n, Z_j^n \notin K)$$

( $k_n$  is chosen, with respect to  $k$ , like in Lemma 3).

By Lemma 4 we know that the last term in the above inequality is equal to  $\overline{\lim}_n P_n^\mu(\omega; (\exists) j \leq k_n, Y_j^n \notin K)$  and by Lemma 3, (b) that is

$$\overline{\lim}_n P_n^\mu(\omega; (\exists) j \leq k_n, Y_j^n \notin K, \sigma_j^n < k'),$$

with  $k'$  chosen in Lemma 3. Because the terms under the limit are dominated by  $P_n^\mu(\omega; (\exists) t < k', X_t \notin K)$ , the proof of (1) is complete. (In what will follow we shall frequently use the same way of passing from  $X^n$  to  $X$ ).

To prove (4) we note first that

$$W_{[k-\delta, k]}(X) \leq 2 \sup_{k-\delta \leq t < k} d(X_t, X_{k-\delta}).$$

We have to show that for every  $k > 0$  and  $\varepsilon > 0$

$$(20) \quad \lim_{\delta \rightarrow 0} \overline{\lim}_n P_n^\mu(\sup_{k-\delta \leq t < k} d(X_t, X_{k-\delta}) > \varepsilon) = 0.$$

By the Markov property

$$P_n^\mu(\sup_{k-\delta \leq t < k} d(X_t^n, X_{k-\delta}^n) \geq \varepsilon) = \int P_n^\mu(\sup_{0 \leq t < \delta} d(X_t^n, X_t^n) > \varepsilon) P_n^\mu \circ (X_{k-\delta}^n)^{-1}(dx).$$

Using Lemma 4, we obtain in the same way like above

$$P_n^\mu(\sup_{0 \leq t < \delta} d(X_t^n, X_t^n) > \varepsilon) \leq P_n^\mu(\sup_{0 \leq t < \delta'} d(X_0^n, X_t^n) > \varepsilon) + P_n^\mu(\tau_{l_n} > \delta) + P_n^\mu(\sigma_{l_n}^n < \delta')$$

with  $\delta$  and  $l_n$  like in Lemma 3. Because the convergence in Lemma 3, (a), (b) is uniform with respect to  $x \in E$ , the two last terms vanish under  $\overline{\lim}_n \int \cdot \delta \rightarrow 0$  implies that  $\delta' \rightarrow 0$  and so we have to prove that

$$\lim_{\delta' \rightarrow 0} \overline{\lim}_n \int P^n(\sup_{0 \leq t < \delta'} d(X_t, X_0) > \varepsilon) P_n^\mu \circ (X_{k-\delta}^n)^{-1}(dx) = 0.$$

For a fixed  $\eta > 0$ , (1) assures that we may choose a compact set  $K_\eta$  such that

$$(21) \quad \overline{\lim}_n P_n^\mu(\omega; (\exists) t \leq k, X_t^\# \notin K_\eta) \leq \eta.$$

We dominate the above integral by

$$\sup_{x \in K_\eta} E^x(\sup_{0 \leq t \leq \delta'} d(X_t, X_0) > \varepsilon) + P_n^\mu \circ (X_{k-\delta}^n)^{-1}(CK_\eta).$$

Proposition 1, Appendix 2 assures that the first term vanishes under  $\lim_{\delta' \rightarrow 0}$ . By (21), for every  $\delta > 0$

$$\lim_n P_n^\mu \circ (X_{k-\delta}^n)^{-1}(CK_\eta) \leq \eta.$$

Because  $\eta$  is arbitrary small, the proof is complete.

An analogous proof goes for (3) and also for

$$(22) \quad \lim_{\delta \rightarrow 0} \overline{\lim}_n P_n^\mu(\sup_{t \leq s \leq t+\delta} d(X_t, X_s) > \varepsilon) = 0$$

for every  $t > 0, \varepsilon > 0$ .

The last relation will be used later.

To prove (2) we define for a fixed  $k > 0$  and  $\varepsilon > 0$  a discrete correspondent of  $W''_{k, \delta}$ , that is

$$T_n: E^{k_n} \rightarrow R$$

$$T_n(x_1, \dots, x_{k_n}) = \sup \min(d(x_i, x_j), d(x_i, x_p))$$

with  $i, j, p \in N$  such that

$$i - (l_n + 1) \leq j < i < p \leq k \wedge (i + (l_n + 1))$$

( $k_n$  and  $l_n$  are chosen with respect to  $k$  and  $\delta$  like in Lemma 3). By Lemma 3, (a), (c)

$$\overline{\lim}_n P_n^\mu(W''_{k, \delta}(X^n) > \varepsilon) = \overline{\lim}_n P^\mu(W''_{k, \delta} > \varepsilon, A_n, \tau_{k_n} > k).$$

We note that for  $\omega \in A_n \cap (\tau_{k_n} > k)$

$$W''_{k, \delta}(X^n(\omega)) > \varepsilon \Rightarrow T_n(Z_1^n(\omega), \dots, Z_{k_n}^n(\omega)) > \varepsilon.$$

Let be  $0 \leq t', t, t'' \leq k$  such that  $t - \delta \leq t' \leq t \leq t'' \leq t + \delta$  and  $d(X_{t'}^n, X_t^n) > \varepsilon, d(X_t^n, X_{t''}^n) < \varepsilon$ . We define  $i$  by  $\tau_i \leq t < \tau_{i+1}$  and  $j$  and  $p$  the corresponding integers with respect to  $t'$  and  $t''$

$$t - t' < \delta \Rightarrow \tau_i(\omega) - \tau_{j+1}(\omega) < \delta.$$

Because  $\omega \in A_n$ , it follows that  $i - (j+1) \leq l_n$ , that is  $i - j \leq l_n + 1$ . Because  $\omega \in (\tau_{k_n} > k)$ ,  $l' \leq k \Rightarrow \tau_p(\omega) \leq k \Rightarrow p \leq k_n$ .

$$X_i^n = Z_i^n, \quad X_{i'}^n = Z_j^n \quad \text{and} \quad X_{i''}^n = Z_p^n$$

(see (11)) implies that

$$\min(d(Z_i^n, Z_j^n), d(Z_i^n, Z_p^n)) < \varepsilon$$

and the above implication is proved. We may now conclude that

$$\overline{\lim}_n P_n^\mu(W_{k, \delta}''(X^n) > \varepsilon) \leq \overline{\lim}_n P_n^\mu(T_n(Z_1^n, \dots, Z_{k_n}^n) > \varepsilon).$$

By Lemma 4 first and then by Lemma 3, (b), (d) the last term is equal to

$$\overline{\lim}_n P_n^\mu(T_n(Y_1^n, \dots, Y_{k_n}^n) > \varepsilon, B_n, \sigma_{k_n}^n < k').$$

In the same way like above we may dominate this term by

$$P(W_{k', \delta'}''(X) > \varepsilon).$$

So we have proved that

$$\overline{\lim}_n P_n^\mu(W_{k, \delta}''(X) > \varepsilon) \leq P^\mu(W_{k', \delta'}''(X) > \varepsilon).$$

$$k < \infty \Rightarrow k' < \infty, \quad \delta \rightarrow 0 \Rightarrow \delta' \rightarrow 0,$$

and so we may refer to the tightness of  $X$ , and the proof of (2) is complete.

To prove the convergence of the marginal distributions we have to verify the hypotheses of Lemma 3, Appendix 3.

The first one is an immediate consequence of (i) in our Theorem. For (ii) we have to verify condition (c) enunciated in the beginning of the paper, i.e.

(c) for any compact set  $L \subseteq E$ ,  $\alpha > 0$  and  $\varepsilon > 0$ , there is some compact set  $K \subseteq E$  such that

$$\overline{\lim}_n \sup_{x \in L} U_\alpha^{(n)}(x, CK) \leq \varepsilon.$$

To do it we shall prove that for every compact set  $\tilde{K} \subseteq E$  and  $k > 0$ , we may choose another compact set  $K \subseteq E$  such that  $\tilde{K} \subseteq K$  and

$$(23) \quad \overline{\lim}_n \sup_{x \in L} U_\alpha^{(n)}(x, CK) \leq e^{\alpha k'} \sup_{x \in L} U_\alpha(x, C\tilde{K}) + e^{-\alpha k}$$

( $k'$  is defined with respect to  $k$  in Lemma 3). If (23) is true, we choose  $k$  such that

$$e^{-\alpha k} < \frac{\varepsilon}{2}$$

and  $\varepsilon'$  such that  $e^{\alpha k'} \varepsilon' < \frac{\varepsilon}{2}$ . Then, Lemma 2, Appendix 2 assures that there is some compact set  $K \subseteq E$  such that

$$\sup_{x \in K} U_{\alpha}(x, C\tilde{K}) < \varepsilon'.$$

So, the compact set  $K$  mentioned in (23) is that needed in (c).

If  $A^n = \cup U_i^n$  over all  $i$  such that  $U_i^n \cap \tilde{K} \neq \emptyset$ , then by the definition of  $U_i^n$  we may choose a compact set  $K$  such that  $A^n \subset K$  for every  $n \in N$ .

Next we prove the following inequality:

$$(24) \quad E_n^x \int_{\tau_i}^{\tau_{i+1}} e^{-\alpha t} I_{CK} \circ X_i^n dt \leq E^x \int_{\sigma_i^n}^{\sigma_{i+1}^n} I_{C\tilde{K}} \circ X_i dt.$$

Because  $X_i^n = Z_i^n$  for  $\tau_i \leq t < \tau_{i+1}$  and  $\tau_{i+1} - \tau_i = \tau_1 \circ \theta_{\tau_i}$ , we have

$$\int_{\tau_i}^{\tau_{i+1}} e^{-\alpha t} I_{CK} \circ X_i^n dt = I_{CK} Z_i^n \int_{\tau_i}^{\tau_{i+1}} e^{-\alpha t} dt \leq I_{CK} \circ Z_i^n \tau_1 \circ \theta_{\tau_i}.$$

We dominate the term in the left of (24) by  $E_n^x(I_{CK} \circ Z_i^n \tau_1 \circ \theta_{\tau_i})$  which by the strong Markov property is

$$E_n^x(I_{CK} \circ Z_i^n E^{Z_i^n}(\tau_1)) = \prod_n^i (I_{CK} q_n^{-1})(x).$$

By the definition of  $\sigma_i^n$  and  $K$  we have

$$E^x \int_{\sigma_i^n}^{\sigma_{i+1}^n} I_{C\tilde{K}} \circ X_i dt \geq E^x(I_{CK} \circ Y_i^n \int_{\sigma_i^n}^{\sigma_{i+1}^n} 1 dt)$$

In the same way like above, the last term is equal to

$$\prod_n^i (I_{CK} q_n^{-1})(x)$$

and (24) is proved.

Next, to prove (23), we shall change  $X^n$  by  $X$  in the same way as above:

$$(25) \quad U_{\alpha}^{(n)}(x, CK) = E_n^x \int_0^k e^{-\alpha t} I_{CK} \circ X_t^n dt + E_n^x \int_k^{\infty} e^{-\alpha t} I_{CK} \circ X_t dt.$$

The second term in the sum is dominated by  $e^{-\alpha k}$ . The first one is dominated by

$$E_n^x \left( \int_0^k e^{-\alpha t} I_{CK} \circ X_t^n dt; \tau_{k_n} > k \right) + \frac{1}{\alpha} P_n^x(\tau_{k_n} \leq k).$$

By Lemma 3.a we may ignore the second term in the above sum. The first one is dominated by

$$E^n_x \left( \int_0^{\tau_{k_n}} e^{-\alpha t} I_{C\bar{K}} \circ X_t^n dt \right)$$

which by (22) is dominated by

$$\begin{aligned} E^x \left( \int_0^{\sigma_{k_n}^n} I_{C\bar{K}} \circ X_t dt \right) &\leq E^x \left( \int_0^{\sigma_{k_n}^n} I_{C\bar{K}} \circ X_t dt; \sigma_{k_n}^n < k' \right) \\ &\quad + E^x \left( \int_0^{\sigma_{k_n}^n} I_{C\bar{K}} \circ X_t dt; \sigma_{k_n}^n \geq k' \right). \\ \int_0^{\sigma_{k_n}^n} I_{C\bar{K}} \circ X_t dt &\leq \sigma_{k_n}^n \end{aligned}$$

and therefore we may dominate the second term in the sum by  $E^x(\sigma_{k_n}^n; \sigma_{k_n}^n \geq k')$  which we may ignore (see Lemma 3.e). To dominate the first term we note that on  $\sigma_{k_n}^n < k'$

$$\int_0^{\sigma_{k_n}^n} I_{C\bar{K}} \circ X_t dt \leq \int_0^{k'} I_{C\bar{K}} \circ X_t dt \leq e^{\alpha k'} \int_0^{k'} e^{-\alpha t} I_{C\bar{K}} \circ X_t dt$$

and therefore

$$E^x \left( \int_0^{\sigma_{k_n}^n} I_{C\bar{K}} \circ X_t dt; \sigma_{k_n}^n < k' \right) \leq e^{\alpha k'} U_\omega(x, C\bar{K}).$$

The proof of (23) is complete and also that of (c).

We verify now the last condition in Lemma 3, Appendix 3. For every  $t \geq 0$ ,  $f \in U_b(E)$  (uniformly continuous and bounded) and  $\varepsilon > 0$ , there is some  $\delta_\varepsilon > 0$  such that

$$(26) \quad \overline{\lim}_n \sup E_n^\mu (|f(X_t^n) - f(X_s^n)|) \leq \varepsilon$$

where the supremum is over all  $s$  such that  $t \leq s \leq t + \delta_\varepsilon$ . We choose  $\eta_\varepsilon > 0$  such that

$$\begin{aligned} d(x, y) \leq \eta_\varepsilon &\Rightarrow |f(x) - f(y)| \leq \frac{\varepsilon}{2} \\ E_n^\mu (|f(X_t^n) - f(X_s^n)|) &\leq E_n^\mu (|f(X_t^n) - f(X_s^n)|; d(X_t^n, X_s^n) < \eta_\varepsilon) \\ &\quad + 2\|f\| E_n^\mu (d(X_s^n, X_t^n) > \eta_\varepsilon). \end{aligned}$$

The first term is less as  $\frac{\varepsilon}{2}$  and therefore

$$\overline{\lim}_n \sup E_n^\mu (|f(X_t^n) - f(X_s^n)|) \leq \frac{\varepsilon}{2} + 2\|f\| \overline{\lim}_n E_n^\mu (\sup d(X_t^n, X_s^n) > \eta_\varepsilon)$$

(the supremum like above). (22) assures that we may choose  $\delta_\varepsilon$  needed in (26) and the proof is complete.

#### 4. Appendices

Appendix 1.

**Lemma 1.** Let  $(F_n)_{n \in \mathbb{N}}$  be a sequence of distributions on  $R_+$ ,  $(k_n)_{n \in \mathbb{N}}$  a

sequence of positive integers such that  $\lim_n k_n a_n = l$  with  $a_n = \int_0^\infty z F_n(dz)$ . If

(a)  $\lim_n a_n = 0$ ,

(b)  $\lim_n \frac{1}{a_n} \int_a^\infty z F_n(dz) = 0$  for every  $a > 0$ ,

then

$$\lim_n F_n^{*k_n} = \varepsilon_l.$$

Proof. It will suffice to show that  $\lim_n \varphi_n(t)^{k_n} = e^{-ilt}$  with

$$\varphi_n(t) = \int_0^\infty e^{-itz} F_n(dz).$$

By (a),  $\lim_n F_n = \varepsilon_0$  and so  $\lim_n \varphi_n(t) = 1$ . We may organize the above limit in an exponential form and it remains to show that  $\lim_n k_n (\varphi_n(t) - 1) = -ilt$ . By the choose of  $k_n$ , that is  $\lim_n \frac{1}{a_n} (1 - \varphi_n(t)) = it$ . We write  $1 - \varphi_n(t)$  in the following form:

$$it \int_0^\infty z F_n(dz) + \int_0^\infty z \left( \frac{1 - \cos tz}{z} \right) F_n(dz) + i \int_0^\infty z \left( \frac{\sin z}{z} - t \right) F_n(dz).$$

Both  $z \rightarrow \frac{1 - \cos tz}{z}$  and  $z \rightarrow \frac{\sin z}{z} - t$  are bounded continuous functions which vanishes when  $z \rightarrow 0$ , therefore it will suffice to show that for such a function  $\alpha$ ,  $\lim_n \int_0^\infty z \alpha(z) F_n(dz) = 0$ .

Let  $M$  be such that  $|\alpha(z)| < M$  for  $z > 0$ , and for a fixed  $\varepsilon > 0$ ,  $a_\varepsilon > 0$  such that  $z < a_\varepsilon \Rightarrow |\alpha(z)| < \varepsilon$ .

$$\frac{1}{a_n} \left| \int_0^\infty z \alpha(z) F_n(dz) \right| \leq \frac{1}{a_n} \int_0^{a_\varepsilon} z |\alpha(z)| F_n(dz) + \frac{M}{a_n} \int_{a_\varepsilon}^\infty z F_n(dz).$$

By (b) the second term of the sum vanishes when  $n \rightarrow \infty$ . The first one is dominated by  $\varepsilon$ , which is arbitrary small, and so the proof is complete.

**Corollary 2.**  $\overline{\lim}_n k_n a_n < l \Rightarrow \lim_n F_n^{*k_n}(l + \varepsilon, \infty) = 0$ .

**Lemma 3.** If  $a_n \uparrow \infty$  and  $\lim_n \frac{k_n}{a_n} = l + \varepsilon$  with  $k_n \in \mathbb{N}$ ,  $\varepsilon > 0$ , then  $\lim_n k_n e_{a_n}^{*k_n}(0, l) = 0$ .

Proof. Let  $(E, \mathcal{H}, P)$  be a probability space and  $f_n: E \rightarrow \mathbb{R}$ ,  $n \in \mathbb{N}$ , a sequence of independent variables,  $e_1$  distributed ( $e_a$  is the exponential distribu-

tion with parameter  $a$ ).  $\frac{1}{a_n} f_i, i \in N$  are independent and  $e_{a_n}$  distributed and so

$$e_{a_n}^{*k_n}(0, l) = P\left(\frac{1}{a_n} f_1 + \dots + \frac{1}{a_n} f_{k_n} < l\right) = P\left(\frac{f_1 + \dots + f_{k_n}}{k_n} < l\right).$$

By the choice of  $k_n$ , for a sufficiently large  $n$ , we may dominate this term by

$$P\left(\frac{f_1 + \dots + f_{k_n}}{k_n} < \left(1 + \frac{\varepsilon}{l}\right)^{-1}\right) \leq P\left(\left|\frac{f_1 + \dots + f_{k_n}}{k_n} - 1\right| > c\right) = P\left(\sum_{i=1}^{k_n} (f_i - 1)^4 > k_n^4 c^4\right)$$

with

$$c = \frac{\varepsilon}{l} \left(1 + \frac{\varepsilon}{l}\right)^{-1}.$$

By Chebyshev's inequality we may dominate it by  $\frac{1}{k_n^4 c^4} \int \left(\sum_{i=1}^{k_n} (f_i - 1)\right)^4 dP$ . Because  $f_i$  are independent with mean value 0, this is

$$\frac{1}{k_n^4 c^4} \left( \sum_{i=1}^n \int (f_i - 1)^4 dP + \sum_{i \neq j} \int (f_i - 1)^2 dP \int (f_j - 1)^2 dP \right).$$

This sum is dominated by  $(Mk_n^2) (k_n^4 c)^{-1}$  for a sufficiently large  $M$ . So,

$$\lim_n k_n e_{a_n}^{*k_n}(0, l) \leq \lim_n k_n M (k_n^2 c)^{-1} = 0.$$

Appendix 2. The first proposition follows from an idea exposed in [4].

**Proposition 1.** *Let  $E$  be a L.C.C.B. space with a metric  $d$  such that  $B_h(x)$  is relatively compact, and  $X$  a standard process with semigroup  $(P_t)_{t>0}$ . If for every  $f \in C_c(E)$ ,  $\lim_{t \rightarrow 0} P_t f = f$  uniformly on  $E$ , then for every compact set  $K \subseteq E$  and  $\varepsilon > 0$*

$$(1) \quad \lim_{h \rightarrow 0} \sup_{x \in K} P^x(\sup_{0 \leq t \leq h} d(X_0, X_t) > \varepsilon) = 0.$$

Proof. We note first that to prove (1) will suffice to show that for every  $L \subseteq G \subseteq E$ ,  $L$  compact set and  $G$  relatively compact open set,

$$(2) \quad \lim_{h \rightarrow 0} \sup_{x \in L} P^x(D_{CG} \leq h) = 0$$

$(D_A(\omega) = \inf\{t; X_t(\omega) \in A\})$  for any measurable set  $A$ ).

If (2) is true, the proof of (1) goes like this: for every  $x \in K$  we choose an open set  $V_x$  and a compact one  $K_x$  such that

$$x \in V_x \subseteq K_x \subseteq B_{\varepsilon/2}(x).$$

We consider  $V_{x_i}$ ,  $i \in N$ , a finite covering for  $K$ . Then  $K_{x_i}$ ,  $i \in N$ , will also be a covering of  $K$ . For any  $x \in K$  there is some  $i \leq n$  such that  $x \in K_{x_i} \subseteq B_{\varepsilon/2}(x_i)$ , therefore  $B_{\varepsilon/2}(x_i) \subseteq B_\varepsilon(x)$  and so  $D_{CB_\varepsilon(x)} \geq D_{CB_{\varepsilon/2}(x_i)}$ . Because  $(\sup_{t \leq h} d(X_0, X_t) > \varepsilon) = (D_{CB_\varepsilon(x)} \leq h)$   $P^x$  a.s., we may conclude that

$$\sup_{x \in K} P^x(\sup_{t \leq h} d(X_0, X_t) > \varepsilon) \leq \max_{j \leq n} \sup_{x \in K_j} P^x(D_{CB_{\varepsilon/2}(x_i)} \leq h).$$

Now we take  $L_i = K_i$  and  $G_i = B_{\varepsilon/2}(x_i)$  (which is relatively compact), and (2)  $\Rightarrow$  (1) is proved.

To prove (2) we choose a relatively compact open set  $U$  such that  $L \subseteq U \subseteq \bar{U} \subseteq G$  and note that

$$(3) \quad P^x(D_{CG} \leq h) \leq P^x(D_{CG} \leq h, X_h \in U) + P^x(X_h \in CU).$$

Let be  $f \in C(E)$  such that  $I_{CU} \leq f \leq I_{CL}$ . Then  $P^x(X_h \in CU) = (P_h I_{CU})(x) \leq P_h f(x)$  and for  $x \in L$ ,  $f(x) = 0$ . So,

$$\begin{aligned} \sup_{x \in L} P^x(X_h \in CU) &\leq \sup_{x \in L} P_h f(x) \\ &= \sup_{x \in L} |P_h f(x) - f(x)| = \sup_{x \in L} |P_h g(x) - g(x)| \end{aligned}$$

with  $g = 1 - f$ . Because  $\bar{U}$  is compact,  $g \in C_c(E)$  and by our hypothesis this term vanishes when  $h \rightarrow 0$ .

To dominate the first term in (3) we note that  $X_{D_{CG}} \in CG$  and so if we denote  $X_{D_{CG}} = Y$  and  $D_{CG} = T$  by applying the strong Markov property for two variables functions, we get

$$\begin{aligned} P^x(T \leq h, X_h \in U) &= \int_{(T \leq h, Y \in CG)} P^y(X_{h-T(\omega)} \in U) P^x(d\omega) \\ &\leq \sup_{t \leq h} \sup_{y \in CG} P^y(X_t \in U). \end{aligned}$$

For  $f \in C(E)$  such that  $I_{\bar{U}} \leq f \leq I_G$ ,

$$P^y(X_t \in U) = (P_t I_U)(y) \leq P_t f(y) = |P_t f(y) - f(y)|$$

for every  $y \in CG$ .  $\text{supp. } f \subseteq \bar{G}$  which is compact and the proof ends.

**Lemma 2.** *Let  $Q$  be a kernel on  $E$ , L.C.C.B. space. If  $Q(C_b(E)) \subseteq C_b(E)$ , then, for every compact  $L \subseteq E$  and  $\varepsilon > 0$ , there is some compact set  $K_\varepsilon \subseteq E$  such that*

$$\sup_{x \in L} Q(x, CK_\varepsilon) \leq \varepsilon.$$

Proof. For every  $x \in L$  we choose  $K_x, K'_x$  compact sets and  $f_x \in C_b(E)$

such that  $Q(x, CK_x) < \varepsilon$ ,  $K_x \subseteq \text{Int } K'_x$  and  $I_{K_x} \leq f_x \leq I_{K'_x}$ . It is obvious that  $Q(I-f_x)(x) < \varepsilon$ .  $y \rightarrow Q(I-f_x)(y)$  is a continuous function, so we may choose  $V_x$  such that  $Q(I-f_x)(y) < \varepsilon$  for every  $y \in V_x$ . Let  $V_{x_i}$ ,  $i \in N$  be a finite covering of  $L$ . Then, the compact  $K_\varepsilon$  will be  $\bigcup_{i \leq n_i} V_{x_i}$ . Indeed, for an  $x \in L$  there exists  $i$  such that  $x \in V_{x_i}$

$$Q(x, CK_\varepsilon) \leq Q(x, CK_{x_i}) \leq Q(x, 1-f_{x_i}) < \varepsilon.$$

Appendix 3. We introduce first some notations:

$$\begin{aligned} R_+^k &= ((t_1, \dots, t_k); t_i \geq 0), \quad s^k = (s_1, \dots, s_k), \\ ds^k &= ds_1 \dots ds_k \\ A(\alpha^k, s^k) &= \exp\left(-\sum_{i=1}^k \alpha_i s_i\right). \end{aligned}$$

For a permutation  $\sigma$  on  $(1, 2, \dots, k)$  we denote

$$\Lambda_\sigma^k = \{(s_1, \dots, s_k); 0 \leq s_{\sigma(1)} \leq \dots \leq s_{\sigma(k)}\}.$$

If  $\sigma$  is the identic permutation we ignore it and write  $\Lambda^k$ . We consider a standard process and for  $0 \leq t_1 < \dots < t_k$  and  $f_i \in C_b(E)$   $i \leq n$  we define

$$U_{\alpha^k} f_1 \dots f_k = E^\mu \int_{R_+^k} A(\alpha^k, s^k) \prod_{i \leq k} f_i(X_{s_i}) ds^k.$$

We note that  $U_{\alpha^k} f_1 \dots f_k$  is the Laplace transform for the distribution

$$F(dt^k) = h(t^k) dt^k \quad \text{with} \quad h(t^k) = E^\mu\left(\prod_{i \leq k} f_i(X_{t_i})\right).$$

We define also

$$H_{\sigma, \alpha^k} f_1 \dots f_k = E^\mu \int_{\Lambda_\sigma^k} A(\alpha^k, s^k) \prod_{i \leq k} f_i(X_{s_i}) ds^k.$$

If  $\sigma$  is the identic permutation, we ignore it in our notation. Because  $m(\partial \Lambda_\sigma^k) = 0$  ( $m$  is the Lebesgue measure),

$$(1) \quad U_{\alpha^k} f_1 \dots f_k = \sum_{\sigma} H_{\sigma, \alpha^k} f_1 \dots f_k.$$

**Lemma 1.** Let  $X^n$   $n \in N$  and  $X$  be standard processes with state space  $E$ , L.C.C.B., such that:

- (i)  $U_\omega(C_b(E)) \subseteq C_b(E)$ ,
- (ii) for every  $f \in C_b(E)$ ,  $\lim_n U_\omega^{(n)} f = U_\omega f$  uniformly on compacts.

Then, for every  $\alpha^k \in R_+^k$  and  $f_i \in C_b(E)$   $i \leq k$

$$\lim_n U_{\alpha^k}^{(n)} f_1 \dots f_k = U_{\alpha^k} f_1 \dots f_k$$

$(U_{\alpha}^{(n)})$  is defined with respect to  $X^n$  in the same way as  $U_{\alpha^k}$  with respect to  $X$ . For  $k=1$  we shall write  $\alpha$  instead of  $\alpha^1$ , so  $U_{\alpha}(U_{\alpha}^{(n)})$  is the resolvent of  $X(X^n)$ .

Proof. (1) assures that it will suffice to prove

$$\lim_n H_{\sigma, \alpha^k}^{(n)} f_1 \cdots f_k = H_{\sigma, \alpha^k} f_1 \cdots f_k.$$

It is no loss of generality to do it only when  $\sigma$  is the identic permutation. We shall proceed by induction on  $k$ . For  $k=1$ , that is (ii). By the Markov property we obtain

$$E^{\mu} \left( \prod_{i \leq k} f_i(X_{s_i}) \right) = E^{\mu} \left( \prod_{i \leq k-1} f_i(X_{s_i}) E^{X_{s_{k-1}}} (f_k(X_{s_k - s_{k-1}})) \right).$$

Then, applying twice Fubini's theorem we get

$$H_{\alpha^k} f_1 \cdots f_k = \int_{\Delta^{k-1}} ds^{k-1} A(\alpha^{k-1}, s^{k-1}) E^{\mu} \left\{ \prod_{i \leq k-1} f_i(X_{s_i}) E^{X_{s_{k-1}}} \left( \int_{s_{k-1}}^{\infty} e^{-\alpha_k s_k} f_k(X_{s_k - s_{k-1}}) ds_k \right) \right\}.$$

By the changement  $s = s_k - s_{k-1}$  we get

$$\int_{s_{k-1}}^{\infty} e^{-\alpha_k s_k} f_k(X_{s_k - s_{k-1}}) ds_k = e^{-\alpha_k s_{k-1}} \int_0^{\infty} e^{-\alpha_k s} f_k(X_s) ds$$

and therefore

$$E^{X_{s_{k-1}}} \left( \int_{s_{k-1}}^{\infty} e^{-\alpha_k s_k} f_k(X_{s_k - s_{k-1}}) ds_k \right) = e^{-\alpha_k s_{k-1}} U_{\alpha_k} f_k(X_{s_{k-1}}).$$

Condition (i) assures that  $U_{\alpha_k} f_k \in C_b(E)$  and we may write

$$H_{\alpha^k} f_1 \cdots f_k = H_{\beta^{k-1}} f_1 \cdots f_{k-2} g_{k-1}$$

with

$$\beta^{k-1} = (\beta_1, \dots, \beta_{k-1}), \quad \beta_i = \alpha_i \quad \text{for } i \leq k-2, \\ \beta_{k-1} = \alpha_{k-1} + \alpha_k \quad \text{and} \quad g_{k-1} = f_{k-1} U_{\alpha_k} f_k.$$

We may establish an analogous relation for every  $n \in \mathbb{N}$ . In this case  $\beta^{k-1}$  will be the same, but

$$g_{k-1}^n = f_{k-1} U_{\alpha_k}^{(n)} f_k$$

which is no more continuous. Nevertheless the definition of  $H_{\beta^{k-1}}^{(n)}$  makes sense, and we write

$$H_{\beta^{k-1}}^{(n)} f_1 \cdots f_{k-2} g_{k-1}^n - H_{\beta^{k-1}} f_1 \cdots f_{k-2} g_{k-1} = d_1^n + d_2^n$$

with

$$d_1^n = H_{\beta^{k-1}}^{(n)} f_1 \cdots f_{k-2} g_{k-1}^n - H_{\beta^{k-1}} f_1 \cdots f_{k-2} g_{k-1}$$

$$d_2^n = H_{\beta^{k-1}}^{(n)} f_1 \cdots f_{k-2} g_{\beta-1} - H_{\beta^{k-1}} f_1 \cdots f_{k-2} g_{k-1}$$

$g_{k-1} \in C_b(E)$  and by the induction hypothesis we get  $\lim_n d_2^n = 0$ . For a fixed  $\varepsilon > 0$ , let  $K_\varepsilon \subseteq E$  be a compact such that

$$H_{\beta^{k-1}} f_1 \cdots f_{k-2} \mathbb{I}_{C_{K_\varepsilon}} \leq \varepsilon.$$

We choose another compact set  $K'$  such that  $K \subseteq \text{Int } K'$  and a function  $\varphi \in C_b(E)$  such that

$$\begin{aligned} \mathbb{I}_{K_\varepsilon} &\leq \varphi \leq \mathbb{I}_{K'} \\ d_1^n &= H_{\beta^{k-1}}^{(n)} f_1 \cdots f_{k-2} ((g_{k-1}^n - g_{k-1}) \varphi) + H_{\beta^{k-1}}^n f_1 \cdots f_{k-2} ((g_{k-1}^n - g_{k-1}) (\mathbb{I} - \varphi)). \end{aligned}$$

We dominate the first term of the sum by

$$\prod_{i \leq k-2} \|f_i\| \sup_{s \in K'} |g_{k-1}^n - g_{k-1}| \leq \prod_{i \leq k-1} \|f_i\| \sup_{s \in K'} |U_{\alpha_k}^{(n)} f_k - U_{\alpha_k} f_k|$$

which by (ii) vanishes when  $n \rightarrow \infty$ . The second term is dominated by

$$2 \|f_k\| \cdot \|f_{k-1}\| H_{\beta^{k-1}}^{(n)} f_1 \cdots f_{k-2} (1 - \varphi).$$

which by the induction hypothesis converges to

$$2 \|f_k\| \|f_{k-1}\| H_{\beta^{k-1}} f_1 \cdots f_{k-2} (1 - \varphi).$$

By the choice of  $K_\varepsilon$  and  $\varphi$  this term is dominated by  $2 \|f_k\| \|f_{k-1}\| \varepsilon$ .  $\varepsilon$  is arbitrary small and so the proof is complete.

**Lemma 2.** *Let  $F_n$ ,  $n \in \mathbb{N}$ , and  $F$  be distributions on  $R_+^k$  of the form*

$$F_n(ds^k) = h_n(s^k) ds^k \quad \text{and} \quad F(ds^k) = h(s^k) ds^k.$$

If

- (i)  $\lim_n F_n = F$ ,
- (ii)  $h_n$ ,  $n \in \mathbb{N}$ , are equal right continuous,

then

$$\lim_n h_n(t^k) = h(t^k) \quad \text{for every } t^k \in R_+^k.$$

*Proof.* Let us suppose that there is some  $t^k \in R_+^k$  such that  $\lim_n h_n(t^k) \neq h(t^k)$ . By passing to a subsequence we may assume that for some  $\varepsilon > 0$ ,  $h_n(t^k) > h(t^k) + \varepsilon$  for every  $n \in \mathbb{N}$ . Let  $\delta_\varepsilon$  be a constant such that for every  $s^k = (s_1, \dots, s_k)$  with  $t_i \leq s_i \leq t_i + \delta_\varepsilon$  for every  $i \leq k$ , holds that  $|h(t^k) - h(s^k)| \leq \frac{\varepsilon}{3}$  and  $|h_n(t^k) - h_n(s^k)| \leq \frac{\varepsilon}{3}$  for every  $n \in \mathbb{N}$ .

We define  $A = \prod_{i \leq k} [t_i, t_i + \delta_\varepsilon]$ . Then  $F(\partial A) = 0$  and therefore  $\lim_n F_n(A) = F(A)$ , that is

$$(2) \quad \lim_n \int_A (h_n(s^k) - h(s^k)) ds^k = 0,$$

$$h_n(s^k) - h(s^k) = (h_n(s^k) - h_n(t^k)) + (h_n(t^k) - h(t^k)) + (h(t^k) - h(s^k)).$$

For  $s^k \in A$  the first and the last term of the above sum are dominated by  $\frac{\varepsilon}{3}$ , and the middle term is greater than  $\varepsilon$ . So  $h_n(s^k) - h(s^k) \geq \frac{\varepsilon}{3}$  for  $s^k \in A$ , which is in contradictory with (2).

**Lemma 3.** *Let  $X_n$ ,  $n \in N$  and  $X$  be standard processes. If*

- (i)  $U_\alpha(C_b(E)) \subseteq C_b(E)$ ,
- (ii)  $\lim_n U_\alpha^{(n)} f = U_\alpha f$  uniformly on compacts for every  $\alpha > 0$  and  $f \in C_b(E)$ ,
- (iii)  $\lim_{\delta \rightarrow 0} \lim_n \sup_{t \leq s \leq t + \delta} E_n(|f(X_t^n) - f(X_s^n)|) = 0$  for every  $f \in C_b(E)$  and  $t \geq 0$ ,

then, for every  $t_1 < t_2 \dots < t_k$  and  $f_i \in U_b(E)$   $i \leq k$ ,

$$(3) \quad \lim_n E_n^{\mu} \left( \prod_{i \leq k} f_i(X_{t_i}^n) \right) = E^{\mu} \left( \prod_{i \leq k} f_i(X_{t_i}) \right).$$

Proof. Lemma 3 is a simple consequence of Lemma 1 and Lemma 2.

REMARK. (3) is sufficient to assure the convergence of the marginal distributions.

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