$\aleph_0$-CONTINUOUS MODULES

Mamoru KUTAMI

Generalizing the notion of right $\aleph_0$-continuous regular rings (see [2], [3]) we define that of quasi-$\aleph_0$- and $\aleph_0$-continuous modules and mainly study the directly finiteness of nonsingular $\aleph_0$-continuous modules over (von Neumann) regular rings.

Let R be a regular ring. By $\mathcal{F}$ we denote the family of all essentially $\aleph_0$-generated essential right ideals of R. It is shown that $\mathcal{F}$ becomes a right Gabriel topology on R (Proposition 5). From this fact the divisible hull $E_{\mathcal{F}}(M)$ of a given right R-module M is considered. Our main purpose of this note is to prove that a nonsingular $\aleph_0$-continuous R-module M is directly finite if and only if so is $E_{\mathcal{F}}(M)$. This is a generalization of a result due to Goodearl [3].

Throughout this paper R is a ring with identity and all R-modules considered are unitary right R-modules.

For a given R-module M, we denote its injective hull by $E(M)$ and the family of all submodules of M by $\mathcal{L}(M)$.

For $N \in \mathcal{L}(M)$ $N \leq_e M$ means that N is an essential submodule of M and $(N: x)$, for $x \in M$, denotes the right ideal $\{r \in R \,|\, xr \in N\}$.

Let M be an R-module. An S-closed submodule of M is a submodule B such that M/B is nonsingular. For any submodule A of M there exists the smallest S-closed submodule C of M containing A, which is called the S-closure of A in M (see [1]). We note that, when M is nonsingular, the S-closure C of A in M is uniquely determined as a submodule C such that $A \leq_e C$ and C is S-closed in M.

Lemma 1. *Let M be an R-module, and let A and B be submodules of M such that $A \leq_e B$. Then B is contained in the S-closure of A in M. In addition if M is nonsingular and B is a direct summand of M, then B coincides with the S-closure of A in M.*

Proof. Let C be the S-closure of A in M. Since $(B+C)/C$ is an epimorphic image of a singular module B/A, we see that $(B+C)/C$ is singular. On the other hand, $(B+C)/C$ is a submodule of a nonsingular module M/C,

whence $(B+C)/C$ is nonsingular. As a result, we have that $(B+C)/C=0$, and so $B\leq C$ as desired.

Let M be an R-module. We consider a subfamily $\mathcal{A}$ of $\mathcal{L}(M)$ which is closed under isomorphic images and essential extensions. For such an $\mathcal{A}$ the following conditions are studied in [5]:

(C_1)   For any $A\in\mathcal{A}$ there exists a direct summand A^* of M such that $A\leq_e A^*$.

(C_2)   If $A\in\mathcal{A}$ is a direct summand of M, then any exact sequence $0\to A \to M$ splits.

(C_3)   If $A\in\mathcal{A}$, $N\in\mathcal{L}(M)$ and both of them are direct summands of M with $A\cap N=0$, then $A\oplus N$ is also a direct summand of M.

Note that if M is nonsingular and $\mathcal{A}$ satisfies the condition (C_1), then, for each $A\in\mathcal{A}$, A^* coincides with the S-closure of A in M.

Following [5] we call M $\mathcal{A}$-continuous (resp. $\mathcal{A}$-quasi-continuous) if M satisfies the conditions (C_1) and (C_2) (resp. (C_1) and (C_3)). Especially if M is $\mathcal{L}(M)$-continuous (resp. $\mathcal{L}(M)$-quasi-continuous) we simply call M continuous (resp. quasi-continuous). It follows from [5] that $\mathcal{A}$-continuous modules are $\mathcal{A}$-quasi-continuous, quasi-injective modules are continuous and that M is $\mathcal{A}$-quasi-continuous if and only if M satisfies (C_1) and the condition:

(*)   For any $A\in\mathcal{A}$ and $N\in\mathcal{L}(M)$ such that N is a direct summand of M and $A\cap N=0$, every homomorphism from A to N can be extended to a homomorphism from M to N.

We now introduce the notion of quasi-$\aleph_0$-continuous modules and $\aleph_0$-continuous modules. Let M be an R-module and consider the family $\mathcal{A}(M)$ of all submodules A of M such that A contains a countably generated essential submodule. Then $\mathcal{A}(M)$ is closed under isomorphic images and essential extensions. We say that M is $\aleph_0$-continuous (resp. quasi-$\aleph_0$-continuous) if M is $\mathcal{A}(M)$-continuous (resp. $\mathcal{A}(M)$-quasi-continuous).

An R-module M is directly finite provided that M is not isomorphic to any proper direct summand of itself. If M is not directly finite, then M is said to be directly infinite. It is well-known that M is directly finite if and only if for all $f, g\in End_R(M)$, $fg=1$ implies $gf=1$.

Theorem 2. *For a given nonsingular $\aleph_0$-continuous R-module M, the following conditions are equivalent:*

(a)   *M is directly finite.*

(b)   *M contains no infinite direct sums of nonzero pairwise isomorphic submodules.*

(c)   *Any submodule of M is directly finite.*

Proof. (a)$\Rightarrow$(b):   Assume that M is directly finite. It suffices to show

that if $\{A_1, A_2, \cdots\}$ is an independent sequence of pairwise isomorphic cyclic submodules of M, then $A_1=0$. Set $B_i=\bigoplus_{n=0}^{\infty}A_{3n+i}$ for $i=1, 2, 3$. Then $\bigoplus_{n=1}^{\infty}A_n$ $=B_1\oplus B_2\oplus B_3$ and $B_2\cong B_3$, and $B_1\oplus B_2\cong B_3$. By the condition (C_1), there exists a direct summand $B_i{}^*$ of M such that $B_i\leq_e B_i{}^*$ for each i. Using the condition (*) we have a homomorphism $f\colon B_2{}^*\to B_3{}^*$ which is an extension of the isomorphism $B_2\cong B_3$. Then f is a monomorphism, because $B_2\leq_e B_2{}^*$. Also, using the condition (C_2) we see that $f(B_2{}^*)$ is a direct summand of M containing B_3; hence by the uniqueness of the S-closure $f(B_2{}^*)=B_3{}^*$. Thus $B_2{}^*$ is isomorphic to $B_3{}^*$. Similarly, $(B_1\oplus B_2)^*$ is isomorphic to $B_3{}^*$. By the condition (C_3), $B_1{}^*\oplus B_2{}^*$ is the S-closure of $B_1\oplus B_2$ in M. Therefore $(B_1\oplus B_2)^*=B_1{}^*\oplus B_2{}^*$, and so $B_1{}^*\oplus B_2{}^*\cong B_2{}^*$. As M is directly finite, $B_1{}^*\oplus B_2{}^*$ is also directly finite, from which $B_1{}^*=0$. Thus we see that $A_1=0$ as desired.

(b)$\Rightarrow$(c): Let N be a submodule of M, and let $N=N_1\oplus N_2$ with an isomorphism $f\colon N\cong N_1$. Then,

$$
\begin{aligned}
N &= N_1\oplus N_2 \\
&= f(N_1)\oplus f(N_2)\oplus N_2 \\
&= f^2(N_1)\oplus f^2(N_2)\oplus f(N_2)\oplus N_2 \\
&= \cdots .
\end{aligned}
$$

It follows that $\{N_2, f(N_2), f^2(N_2), \cdots\}$ is an independent sequence of pairwise isomorphic submodules of M. By assumption $N_2=0$ and so N is directly finite.

(c)$\Rightarrow$(a) is clear.

The following lemma is well-known and, as is easily seen, the same conclusion is valid for sums of infinite many submodules.

Lemma 3. *Let A, B, C and D be submodules of a nonsingular module M such that $A\leq_e B$ and $C\leq_e D$. Then $(A+C)\leq_e(B+D)$.*

Lemma 4 ([2, Lemma 14.10]). *Let M be a projective module over a regular ring R, and let $\mathcal{L}$ denote the collection of all countably generated submodules of M.*
   (a) *If $J, K\in\mathcal{L}$, then $J\cap K\in\mathcal{L}$.*
   (b) *If J, $K\in\mathcal{L}$ and $f\in Hom_R(J, M)$, then $f^{-1}(K)=\{x\in J\mid f(x)\in K\}\in\mathcal{L}$.*

Now, let $\mathcal{E}$ be the collection of all countably generated essential right ideals of a ring R and let $\mathcal{F}$ be the collection of all right ideals which contain a member of $\mathcal{E}$. Then we have the following proposition.

Proposition 5. *If R is a regular ring, then $\mathcal{F}$ is a right Gabriel topology, i.e., $\mathcal{F}$ is not empty and satisfies the following conditions:*

(T$_1$) *If $I \in \mathcal{F}$ and $a \in R$, then $(I: a) \in \mathcal{F}$.*

(T$_2$) *If I is a right ideal and there exists $J \in \mathcal{F}$ such that $(I: a) \in \mathcal{F}$ for every $a \in J$, then $I \in \mathcal{F}$.*

Proof. Suppose that $I \in \mathcal{F}$ and $a \in R$. Then there exists $J \in \mathcal{E}$ such that $J \leq I$. Noting that $(I: a) \geq (J: a)$, we may show that $(J: a) \in \mathcal{E}$. $J \leq_e R$ implies that $(J: a) \leq_e R$ and by Lemma 4 $(J: a)$ is countably generated. Therefore $(J: a) \in \mathcal{E}$. Now suppose that I is a right ideal and that there exists $J \in \mathcal{F}$ such that $(I: \mathrm{a}) \in \mathcal{F}$ for each $a \in J$. Then there exists $K \in \mathcal{E}$ with $K \leq J$. Put $K = \sum_{n=1}^{\infty} a_n R$. Then by the assumption, there exists $I_n \in \mathcal{E}$ with $(I: a_n) \geq I_n$ for each n. The mapping $f: \bigoplus_{n=1}^{\infty} R \to K$ given by $f((r_n)) = \sum a_n r_n$ is an epimorphism. Note that $f(\bigoplus_{n=1}^{\infty} I_n) \leq K \cap I$ and $f(\bigoplus_{n=1}^{\infty} I_n)$ is countably generated. Since f induces an epimorphism $\bigoplus_{n=1}^{\infty} R / \bigoplus_{n=1}^{\infty} I_n \to K / f(\bigoplus_{n=1}^{\infty} I_n)$ and $\bigoplus_{n=1}^{\infty} R / \bigoplus_{n=1}^{\infty} I_n$ is singular by Lemma 3, $K / f(\bigoplus_{n=1}^{\infty} I_n)$ is also singular. Hence $f(\bigoplus_{n=1}^{\infty} I_n) \leq_e K \leq_e R$ and so $f(\bigoplus_{n=1}^{\infty} I_n) \in \mathcal{E}$. Thus $I \in \mathcal{F}$ as desired.

For a given module M over a regular ring R, we put

$$E_{\mathcal{F}}(M) = \{x \in E(M) \mid (M: x) \in \mathcal{F}\}$$
$$= \{x \in E(M) \mid xI \leq M \text{ for some } I \in \mathcal{E}\} .$$

$E_{\mathcal{F}}(M)$ is called the $\mathcal{F}$-injective hull or $\mathcal{F}$-divisible hull of M (cf. [7, p. 30]).

Lemma 6. *Let R be a regular ring and let M be an R-module.*

(a) *If $M = A \oplus B$ for some submodules A and B, then $E_{\mathcal{F}}(M) = E_{\mathcal{F}}(A) \oplus E_{\mathcal{F}}(B)$.*

(b) *Any R-homomorphism from M to an R-module N can be extended to an R-homomorphism from $E_{\mathcal{F}}(M)$ to $E_{\mathcal{F}}(N)$.*

Proof. (a) It is clear that $E(M) = E(A) \oplus E(B)$. Let $m \in E_{\mathcal{F}}(M)$. Then $m = a + b$ for some $a \in E(A)$ and $b \in E(B)$. Since $(M: m) \in \mathcal{F}$ and $(M: m) = (A: a) \cap (B: b)$, $(A: a) \in \mathcal{F}$ and $a \in E_{\mathcal{F}}(A)$. Likewise we have $b \in E_{\mathcal{F}}(B)$. Therefore $m \in E_{\mathcal{F}}(A) \oplus E_{\mathcal{F}}(B)$ and hence $E_{\mathcal{F}}(M) = E_{\mathcal{F}}(A) \oplus E_{\mathcal{F}}(B)$.

(b) Let $f: M \to N$ be a homomorphism. Then f can be extended to an R-homomorphism $\bar{f}: E(M) \to E(N)$. Let $m \in E_{\mathcal{F}}(M)$. Then $(M: m) \leq (N: \bar{f}(m)) \in \mathcal{F}$. Hence $\bar{f}(m)$ lies in $E_{\mathcal{F}}(N)$. Thus the restriction map $\bar{f}|_{E_{\mathcal{F}}(M)}$ of $\bar{f}$ is the desired one.

Proposition 7. *Let R be a regular ring. If M is a non-singular quasi-$\aleph_0$-continuous R-module, then so is $E_{\mathcal{F}}(M)$.*

Proof. First we show that the condition (C$_1$) holds for $\mathcal{A}(E_{\mathcal{F}}(M))$. Let

$L \in \mathcal{A}(E_{\mathcal{F}}(M))$. There exists a countably generated essential submodule N of L; say $N = \sum_{n=1}^{\infty} x_n R$. We can take $I_n \in \mathcal{E}$ such that $x_n I_n \leq M$ for each n. Since $K = \sum_{n=1}^{\infty} x_n I_n$ is countably generated, there exists a direct summand K^* of M with $K \leq_e K^*$. It follows that K is an essential submodule of $E_{\mathcal{F}}(K^*)$ and $E_{\mathcal{F}}(K^*)$ is a direct summand of $E_{\mathcal{F}}(M)$ by Lemma 6 (a). On the other hand, K is an essential submodule of N by Lemma 3 and hence is that of L. Therefore from Lemma 1 we see that $L \leq_e E_{\mathcal{F}}(K^*)$.

Next, we prove that the condition (*) for $\mathcal{A}(E_{\mathcal{F}}(M))$ holds. Let $A \in \mathcal{A}(E_{\mathcal{F}}(M))$ and $N \in \mathcal{L}(E_{\mathcal{F}}(M))$ such that N is a direct summand of $E_{\mathcal{F}}(M)$ and $A \cap N = 0$, and let $f: A \to N$ be a homomorphism. Now there exists a countably generated essential submodule B of A; say $B = \sum_{n=1}^{\infty} x_n R$. Then $f(B) = \sum_{n=1}^{\infty} f(x_n) R$ $\leq_e f(A) \leq N$. Since both x_n and $f(x_n)$ are in $E_{\mathcal{F}}(M)$ for each n, there exist I_n' and I_n'' in $\mathcal{E}$ such that $x_n I_n' \leq M$ and $f(x_n) I_n'' \leq M$. Then $I_n = I_n' \cap I_n''$ lies in $\mathcal{E}$ by Lemma 4 (a), and $x_n I_n \leq M$ and $f(x_n) I_n \leq M$. Putting $C = \sum_{n=1}^{\infty} x_n I_n$ and $D = \sum_{n=1}^{\infty} f(x_n) I_n$, we see that C and D are countably generated submodules of M with $C \leq_e B$ and $D \leq_e f(B)$. There exists a direct summand D^* of M such that $D \leq_e D^*$. Using the condition (*) for $\mathcal{A}(M)$, the restriction $f|_C: C \to D$ of f can be extended to a homomorphism $M \to D^*$. This also can be extended to a homomorphism $h: E_{\mathcal{F}}(M) \to E_{\mathcal{F}}(D^*)$ by Lemma 6 (b). Since $E_{\mathcal{F}}(D^*)$ is the $\mathcal{S}$-closure of D in $E_{\mathcal{F}}(M)$, $E_{\mathcal{F}}(D^*)$ is contained in N. Therefore h is a homomorphism from $E_{\mathcal{F}}(M)$ to N. Since $f|_C = h|_C$ and $C \leq_e B$, $h = f$.

At the end of this note we provide an example to show that the converse of Proposition 7 is not true in general.

Now we are in position to prove our main theorem.

Theorem 8. *Let R be a regular ring, and let M be a non-singular $\aleph_0$-continuous R-module. Then M is directly finite if and only if so is $E_{\mathcal{F}}(M)$.*

Proof. The "only if" part. Assume that M is directly finite. If $E_{\mathcal{F}}(M)$ is directly infinite, then there exists an independent sequence of nonzero pairwise isomorphic cyclic submodules of $E_{\mathcal{F}}(M)$; say $\{x_n R\}_{n=1}^{\infty}$. Let $f_n: x_n R \to x_{n+1} R$ be an isomorphism with $f_n(x_n) = x_{n+1}$, $n = 1, 2, \cdots$. For each n, there exists $I_n \in \mathcal{E}$ such that $x_n I_n \leq M$. Then $A_n = x_n(I_1 \cap \cdots \cap I_n)$ and $B_n = x_n(I_1 \cap \cdots \cap I_{n+1})$ are countably generated submodules of M such that $B_n \leq_e A_n \leq_e x_n R$ and the restriction $f_n|_{B_n}$ of f_n to B_n is an isomorphism between B_n and A_{n+1}. By the assumption, there exist direct summands A_n^* and B_n^* of M with $A_n \leq_e A_n^*$ and $B_n \leq_e B_n^*$ and further $f_n|_{B_n}$ can be extended to an isomorphism g_n between

$B_n{}^*$ and $A_{n+1}{}^*$. Hence we have $\bigoplus_{n=1}^{\infty}B_n{}^*\cong\bigoplus_{n=1}^{\infty}A_{n+1}{}^*$. On the other hand, by Lemma 1, for each n $A_n\leq B_n{}^*$. So $A_n\leq_e B_n{}^*$ which implies that $B_n{}^*=A_n{}^*$. Therefore $\bigoplus_{n=1}^{\infty}B_n{}^*=B_1{}^*\oplus(\bigoplus_{n=1}^{\infty}A_{n+1}{}^*)$. However, this is a contradiction, because $\bigoplus_{n=1}^{\infty}B_n{}^*$ is directly finite by Theorem 2. Therefore $E\mathcal{F}(M)$ is directly finite.

The "if" part. Assume that $E\mathcal{F}(M)$ is directly finite, and consider f and g in $End_R(M)$ such that $fg=1_{End_R(M)}$. By Lemma 6 (b), there exist $\bar{f}$ and $\bar{g}$ in $End_R(E\mathcal{F}(M))$ such that $\bar{f}$ and $\bar{g}$ are extensions of f and g respectively. Noting that $M\leq_e E\mathcal{F}(M)$ and $E\mathcal{F}(M)$ is nonsingular, we obtain that $\bar{f}\bar{g}=1_{End_R(E\mathcal{F}(M))}$. By the assumption $\bar{g}\bar{f}=1_{End_R(E\mathcal{F}(M))}$, from which $gf=1_{End_R(M)}$. Therefore M is directly finite.

Proposition 9. *Assume that M is a nonsingular $\aleph_0$-continuous R-module with the following condition*:

($\sharp$) *For any submodules A and B of M with $A\cap B=0$, any isomorphism from A to B can be extended to a homomorphism of $\bar{A}$ to $\bar{B}$, where $\bar{A}$ and $\bar{B}$ are the S-closures of A and B in M respectively. Then, M is directly finite if and only if so is $E(M)$.*

Proof. Noting that $E(M)$ is nonsingular $\aleph_0$-continuous, the "if" part is clear by Theorem 2.

The "only if" part. Assume that M is directly finite. If $E(M)$ is directly infinite, there exists an infinite and independent sequence of nonzero pairwise isomorphic submodules of $E(M)$; say $\{A_n\}_{n=1}^{\infty}$. Let f_n be the isomorphism between A_n and A_{n+1} for each n. Set $B_1=A_1\cap M$ and define inductively $B_{n+1}=f_n(B_n)\cap M$ and $C_n=f_n^{-1}(B_{n+1})$ for $n=1, 2, \cdots$. Then, for each n, B_n and C_n are submodules of $A_n\cap M$ with $C_n\leq_e B_n$ and the S-closure $\bar{B}_n$ of B_n in M coincides with $\bar{C}_n$, the S-closure of C_n in M, by the similar way in the proof of Theorem 8. The restriction map $f_n|_{C_n}$ is an isomorphism from C_n to B_{n+1}. Using the condition ($\sharp$), $f_n|_{C_n}$ can be extended to a monomorphism $\bar{f}_n\colon \bar{C}_n\to \bar{B}_{n+1}=\bar{C}_{n+1}$. Consequently $\{\bar{C}_1, \bar{f}_1(\bar{C}_1), \bar{f}_2(\bar{f}_1(\bar{C}_1)), \cdots\}$ is an independent sequence of nonzero pairwise isomorphic submodules of M, which is a contradiction by Theorem 2. Thus the proof is completed.

Corollary 10. *Let M be a nonsingular continuous R-module. Then M is directly finite if and only if so is $E(M)$.*

Finally we show the following result.

Theorem 11. *Let R be a regular ring and M a finitely generated projective $\aleph_0$-continuous R-module. If A is a projective maximal submodule of M, then A is a direct summand of M.*

Proof. A can be written as a direct sum of cyclic submodules; say $A= \bigoplus_{\alpha \in I} x_\alpha R$ [4]. We claim that I is a finite set. If I is an infinite set, then we have a countable subset J of I such that $I-J$ is an infinite set and so $A=(\bigoplus_{\alpha \in J} x_\alpha R) \oplus (\bigoplus_{\beta \in I-J} x_\beta R)$. Since M is $\aleph_0$-continuous, we have a direct summand B^* of M such that $\bigoplus_{\alpha \in J} x_\alpha R \leq_e B^*$ and $A \leq B^* \oplus (\bigoplus_{\beta \in I-J} x_\beta R) \leq M$. If $A=B^* \oplus (\bigoplus_{\beta \in I-J} x_\beta R)$, then $\bigoplus_{\alpha \in J} x_\alpha R$ coincides with B^* and is finitely generated. If $B^* \oplus (\bigoplus_{\beta \in I-J} x_\beta R)= M$, then $\bigoplus_{\beta \in I-J} x_\beta R$ is finitely generated. In any case we have a contradiction. Therefore A is finitely generated and so it is a direct summand of M.

As a consequence of Theorem 11, we obtain the following which is a slight generalization of [6, Corollary].

Corollary 12. *If R is a right hereditary, right $\aleph_0$-continuous, regular ring, then R is a semi-simple artinian ring.*

REMARK. In general, the converse of Proposition 7 is not true. For example, take a field F and set $R_n=M_{2^n}(F)$ for all $n=1, 2, \cdots$. Map each $R_n \rightarrow R_{n+1}$ along the diagonal, i.e., map $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$, and set $R=\varinjlim R_n$. Then R is a simple, right hereditary, not artinian, regular ring with a unique dimension function (see [2]). Note that for a regular ring R, R_R is quasi-$\aleph_0$-continuous if and only if R_R is $\aleph_0$-continuous. Therefore we see that $E_{\mathscr{F}}(R)=E_{\mathscr{L}(R)}(R) =E(R)$ is a nonsingular quasi-$\aleph_0$-continuous R-module, but R is not quasi-$\aleph_0$-continuous by Corollary 12.

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Department of Mathematics
Yamaguchi University
Yoshida, Yamaguchi 753
Japan