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ON THE STRICT CLASS NUMBER OF $Q(\sqrt{2p})$ MODULO 16, $p\equiv 1$ (mod 8) PRIME

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Let $p\equiv 1 \pmod 8$ be prime so that there are integers a, b, c, d, e, f with

$$(1)\qquad \begin{cases} p = a^2+b^2 = c^2+2d^2 = e^2-2f^2 \\ a\equiv 1 \pmod 4,\ b\equiv 0 \pmod 4,\ c\equiv 1 \pmod 4,\ d\equiv 0 \pmod 2, \\ \qquad\qquad e\equiv 1 \pmod 4,\ f\equiv 0 \pmod 4. \end{cases}$$

Throughout this note we consider only those primes p for which the strict class number $h^+(8p)$ of the real quadratic field $\boldsymbol{Q}(\sqrt{2p})$ (of discrimanant $8p$) satisfies

$$(2)\qquad h^+(8p)\equiv 0 \pmod 8.$$

These primes have been characterized by Kaplan [4]. Indeed such primes must satisfy [5]

$$(3)\qquad \begin{cases} p\equiv 1 \pmod{16},\ a\equiv 1 \pmod 8,\ b\equiv 0 \pmod 8,\ c\equiv 1 \pmod 8,\ \left(\dfrac{c}{p}\right) = 1, \\ d\equiv 0 \pmod 4,\ e\equiv 1 \pmod 8,\ \left(\dfrac{e}{p}\right) = +1. \end{cases}$$

In this note we give a new determination of $h^+(8p)$ modulo 16, and compare it with the determination given by Yamamoto in [15].

We begin by introducing some notation. We denote the fundamental unit (>1) of $\boldsymbol{Q}(\sqrt{2p})$ by η_{2p}. As one and only one of the equations $V^2-2pW^2 = -1$, -2, or $+2$ is solvable in integers V, W, we define

$$E_p = \begin{cases} -1, & \text{if } V^2-2pW^2 = -1 \text{ solvable,} \\ -2, & \text{if } V^2-2pW^2 = -2 \text{ solvable,} \\ +2, & \text{if } V^2-2pW^2 = +2 \text{ solvable.} \end{cases}$$

Clearly the norm $N(\eta_{2p})$ of η_{2p} satisfies

$$N(\eta_{2p}) = \begin{cases} +1, & \text{if } E_p = \pm 2, \\ -1, & \text{if } E_p = -1. \end{cases}$$

Further we let

$$\varepsilon_2 = 1+\sqrt{2}, \quad \varepsilon_p = T+U\sqrt{p}$$

denote the fundamental units (>1) of $\boldsymbol{Q}(\sqrt{2})$ and $Q(\sqrt{p})$ respectively, and set

$$(4) \qquad e_2 = -\sqrt{2}\ \varepsilon_2' = -\sqrt{2}\ (1-\sqrt{2}) = 2-\sqrt{2}\ ,$$

$$(5) \qquad e_p = -\sqrt{p}\ \varepsilon_p' = -\sqrt{p}\ (T-U\sqrt{p}) = pU-T\sqrt{p}\ .$$

Finally the fundamental unit of $\boldsymbol{Q}(\sqrt{2p})$ of norm $+1$ is denoted by $R+S\sqrt{2p}$ so that

$$R+S\sqrt{2p} = \begin{cases} \eta_{2p}, & \text{if } N(\eta_{2p}) = +1\ , \\ \eta_{2p}^2, & \text{if } N(\eta_{2p}) = -1\ . \end{cases}$$

Our starting point is the following result of Bucher [1: p. 8].

Lemma 1. *If $p\equiv 1$ (mod 8) is a prime such that $h^+(8p)\equiv 0$ (mod 8) then*

$$(6) \qquad (-1)^{\lambda(p)}\left(\frac{e_2}{p}\right)_4 \equiv R^{h^+(8p)/8} \pmod{p}\ ,$$

$$(7) \qquad (-1)^{\lambda(p)}\left(\frac{e_p}{2}\right)_4 \equiv R^{h^+(8p)/8} \pmod{4}\ ,$$

where

$$(8) \qquad \lambda(p) = \textit{number of quadratic residues of } p \textit{ less than } p/8\ .$$

[In the biquadratic residue symbols e_2 and e_p are to be taken modulo p and 16 respectively.]

It is convenient to set

$$(9) \qquad \alpha = (-1)^{\lambda(p)}\left(\frac{e_p}{2}\right)_4, \quad \beta = (-1)^{\lambda(p)}\left(\frac{e_2}{p}\right)_4 .$$

As (see for example [1: p. 4] or [8])

$$(10) \qquad \begin{cases} E_p = -1 \Rightarrow R\equiv -1 \pmod{p},\ R\equiv -1 \pmod{4}\ , \\ E_p = -2 \Rightarrow R\equiv -1 \pmod{p},\ R\equiv 1 \pmod{4}\ , \\ E_p = +2 \Rightarrow R\equiv 1 \pmod{p},\ R\equiv -1 \pmod{4}\ , \end{cases}$$

we note that Lemma 1 together with (10) gives immediately the following supplement to the biquadratic reciprocity law of Scholz type proved in [2].

Corollary 1. *If $p\equiv 1$ (mod 8) is a prime such that $h^+(8p)\equiv 0$ (mod 8) then*

$$\left(\frac{e_2}{p}\right)_4\left(\frac{e_p}{2}\right)_4 = \begin{cases} +1, & \textit{if } N(\eta_{2p}) = -1\ , \\ (-1)^{h^+(8p)/8}, & \textit{if } N(\eta_{2p}) = +1\ . \end{cases}$$

Next we examine each of the three quantities $\lambda(p)$, $\left(\frac{e_2}{p}\right)_4$, $\left(\frac{e_p}{2}\right)_4$, which appear in α and β.

First, from (8), we have

$$\lambda(p) = \frac{1}{2} \sum_{0<x<p/8} \left\{1+\left(\frac{x}{p}\right)\right\},$$

that is

$$\lambda(p) = \frac{1}{16}(p-1)+\frac{1}{2} \sum_{0<x<p/8} \left(\frac{x}{p}\right). \tag{11}$$

Now it is well-known that for primes $p\equiv 1 \pmod 8$ (see for example [3: p. 694])

$$\sum_{0<x<p/8} \left(\frac{x}{p}\right) = \frac{1}{4}(h(-4p)+h(-8p)), \tag{12}$$

where $h(-4p)$ and $h(-8p)$ are the class numbers of $\boldsymbol{Q}(\sqrt{-p})$ and $\boldsymbol{Q}(\sqrt{-2p})$ respectively. Hence, from (11) and (12), we obtain

$$\lambda(p) = \frac{1}{16}(p-1+2h(-4p)+2h(-8p)).$$

Then appealing to the easily proved result

$$\frac{p-1}{16} \equiv \frac{a-1}{8} \pmod 2 \tag{13}$$

we have

$$(-1)^{\lambda(p)} = (-1)^{(a-1+h(-4p)+h(-8p))/8}. \tag{14}$$

Secondly, by a theorem of Emma Lehmer [9], we have

$$\left(\frac{\varepsilon_2}{p}\right)_4 = (-1)^{d/4},$$

and so by (4) we obtain

$$\left(\frac{e_2}{p}\right)_4 = \left(\frac{2}{p}\right)_8 (-1)^{d/4}.$$

Now by the Reuschle [11]–Western [12] criterion for 2 to be an eighth power (see also [13]), we haae

$$\left(\frac{2}{p}\right)_8 = (-1)^{b/8},$$

so

$$(15) \qquad \left(\frac{e_2}{p}\right)_4 = (-1)^{(b+2d)/8}\,.$$

Thirdly, as $h^+(8p)\equiv 0 \pmod 8$, we have $h(-4p)\equiv 0 \pmod 8$ [4], and so $T\equiv 0 \pmod 8$ [6]. Moreover, as $p\equiv 1 \pmod 8$, $\sqrt{p}$ is defined modulo 16 and is odd, so that $T\sqrt{p}\equiv T \pmod{16}$, and we have from (5), as $p\equiv 1 \pmod{16}$,

$$\left(\frac{e_p}{2}\right)_4 = (-1)^{(pU+T-1)/8} = (-1)^{(T+U-1)/8}\,.$$

Appealing to (13) and the easily-proved result

$$U\equiv \frac{1}{2}(p+1) \pmod{16}\,,$$

as well as a theorem of Williams [14]

$$h(-4p)\equiv T \pmod{16}\,,$$

we obtain

$$(16) \qquad \left(\frac{e_p}{2}\right)_4 = (-1)^{(a-1+h(-4p))/8}\,.$$

From (9), (14), (15), (16), we see that

$$(17) \qquad \alpha = (-1)^{h(-8p)/8},\quad \beta = (-1)^{(a-1+b+2d+h(-4p)+h(-8p))/8}\,.$$

Then by Lemma 1 we obtain the following theorem.

Theorem. *If $p\equiv 1$ (mod 8) is a prime such that $h^+(8p)\equiv 0$ (mod 8) and α and β are as given in* (17), *then*

$$\begin{aligned}
\alpha=\beta=1 \quad &\Rightarrow h^+(8p)\equiv 0 \pmod{16},\\
\alpha=1,\ \beta=-1 &\Rightarrow h^+(8p)\equiv 8 \pmod{16},\ E_p=-2\,,\\
\alpha=-1,\ \beta=1 &\Rightarrow h^+(8p)\equiv 8 \pmod{16},\ E_p=+2\,,\\
\alpha=\beta=-1 \quad &\Rightarrow h^+(8p)\equiv 8 \pmod{16},\ E_p=-1\,.
\end{aligned}$$

As an immediate consequence of our Theorem we have the following corollary.

Corollary 2. *If $p\equiv 1$ (mod 8) is a prime such that $h^+(8p)\equiv 0$ (mod 8) then*

$$(18) \qquad \begin{cases} h^+(8p)\equiv T+a+b+2d-1 \pmod{16}, \text{ if } N(\eta_{2p})=+1\,,\\ 0 \quad\ \equiv T+a+b+2d-1 \pmod{16}, \text{ if } N(\eta_{2p})=-1\,; \end{cases}$$

and

$$(19) \qquad \begin{cases} h(-8p)\equiv 0 \pmod{16}, & \textit{if } E_p=-2\,,\\ h(-8p)\equiv h^+(8p) \pmod{16}, & \textit{if } E_p=-1,\ +2\,. \end{cases}$$

We remark that the congruences in (18) appear to be new but that those of (19) are contained in [7], [8].

Finally we compare our Theorem with the following result of Yamamoto [15].

Lemma 2. *If* $p\equiv 1$ (*mod* 8) *is a prime such that* $h^+(8p)\equiv 0$ (*mod* 8) *then*

$$\left(\frac{e}{p}\right)_4 = \left(\frac{z-2^{h(p)}}{2}\right)_4 = 1 \Rightarrow h^+(8p)\equiv 0 \ (\textit{mod } 16),$$

$$\left(\frac{e}{p}\right)_4 = 1,\ \left(\frac{z-2^{h(p)}}{2}\right)_4 = -1 \Rightarrow h^+(8p)\equiv 8 \ (\text{mod } 16),\ E_p = -2\,,$$

$$\left(\frac{e}{p}\right)_4 = -1,\ \left(\frac{z-2^{h(p)}}{2}\right)_4 = 1 \Rightarrow h^+(8p)\equiv 8 \ (\text{mod } 16),\ E_p = +2\,,$$

$$\left(\frac{e}{p}\right)_4 = -1,\ \left(\frac{z-2^{h(p)}}{2}\right)_4 = -1 \Rightarrow h^+(8p)\equiv 8 \ (\text{mod } 16),\ E_p = -1\,,$$

where $h(p)$ is the class number of $\boldsymbol{Q}(\sqrt{p})$ and (z, w) is a solution of

$$z^2-pw^2 = 2^{h(p)+2}, \quad z\equiv 2^{h(p)}+1 \ (\text{mod } 4)\,.$$

Clearly from our Theorem and Lemma 2 we have the following corollary.

Corollary 3. *If* $p\equiv 1$ (*mod* 8) *is a prime such that* $h^+(8p)\equiv 0$ (*mod* 8) *then*

$$(-1)^{h(-8p)/8} = \left(\frac{e}{p}\right)_4 .$$

However corollary 3 is not quite as general as the following result of Leonard and Williams [10: Theorem 2] (since it is possible to have $h(-8p)\equiv 0$ (mod 8) but $h^+(8p)\not\equiv 0$ (mod 8), for example $p=73$):

$$(-1)^{h(-8p)/8} = \left(\frac{e}{p}\right)_4 ,$$

if p is a prime such that $h(-8p)\equiv 0$ (mod 8) and e is chosen so that $e\equiv 1$ (mod 8). We remark that Yamamoto [15] has shown that $(-1)^{h(-8p)/8}=\left(\frac{2c}{p}\right)_4$, if $p\equiv 1$ (mod 8) is a prime such that $h(-8p)\equiv 0$ (mod 8).

We conclude with a few examples.

EXAMPLE 1. $p=113$

Here $a=-7$, $b=8$, $c=9$, $d=4$, $e=25$, $f=16$,

$$h(-4p) = 8, \quad h(-8p) = 8\,,$$

so

$$\alpha = -1,\ \beta = -1\,.$$

Hence, by Theorem, $h^+(8p)\equiv 8 \pmod{16}$ and $E_p=-1$.
Indeed $h^+(8p)=8$ and $15^2-226\cdot 1^2=-1$.

EXAMPLE 2. $p=353$
Here $a=17$, $b=8$, $c=-15$, $d=8$, $e=49$, $f=32$,

$$h(-4p)=16, \quad h(-8p)=24,$$

so

$$\alpha=-1, \beta=+1.$$

Hence, by Theorem, $h^+(8p)\equiv 8 \pmod{16}$ and $E_p=+2$.
Indeed $h^+(8p)=8$ and $186^2-706\cdot 7^2=+2$.

EXAMPLE 3. $p=1217$
Here $a=-31$, $b=16$, $c=33$, $d=8$, $e=97$, $f=64$,

$$h(-4p)=32, \quad h(-8p)=32,$$

so

$$\alpha=+1, \beta=+1.$$

Hence, by Theorem, $h^+(8p)\equiv 0 \pmod{16}$. Indeed $h^+(8p)=16$.

EXAMPLE 4. $p=257$
Here $a=1$, $b=16$, $c=-15$, $d=4$, $e=17$, $f=4$,

$$h(-4p)=16, \quad h(-8p)=16,$$

so

$$\alpha=+1, \beta=-1.$$

Hence, by Theorem 1, $h^+(8p)\equiv 8 \pmod{16}$ and $E_p=-2$.
Indeed $h^+(8p)=8$ and $68^2-514\cdot 3^2=-2$.

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