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SOME STRUCTURE THEOREMS ON PSEUDO-SYMMETRIC SETS

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1. Introduction

In [2], a pseudo-symmetric set is defined as a binary system $(S, \circ)$ satisfying (1) $a\circ a=a$, (2) $(a\circ b)\circ c=(a\circ c)\circ(b\circ c)$ and (3) a mapping $\sigma_a: x\mapsto x\circ a$ is a permutation on S. The object of this paper is to develop a structuie theory for pseudo-symmetric sets. Denote σ_a by $\sigma(a)$. Then σ is considered as a mapping of S to the group of permutations on S satisfying the fundamental identity $\sigma(a^{\sigma(b)})=\sigma(b)^{-1}\sigma(a)\sigma(b)$, which results from (2). The mapping σ is called a pseudo-symmetric structure on S. The group of automorphisms generated by all σ_a is denoted by $G(S)$ or simply by G. The subgroup of G generated by all $\sigma_a^{-1}\sigma_b$ is denoted by $H(S)$. $H(S)$ is called the group of displacements of S according to the terminology in the theory of symmetric sets. It was found in previous works (See [1] and [2]) that there is a close connection between the structure of S and that of $H(S)$. In this paper, we shall investigate this connection more closely to find some structure theorems on S. To develop structure theory, we start with the concept of homomorphisms of our sets, which can be defined in a natural way. However, the concept of the kernel of a homomorphism is not available. To replace it, we introduce the concept of normal decompositions, which was already used in previous works (it was called coset-decompositions). Then, a more important concept is introduced. It is that of the group of displacements for a normal decomposition. As in the usual structure theory of groups, then, we proceed to consider sub- and factor-normal decompositions and the group of displacements for them. In the previous works, the structure of simple pseudo-symmetric sets was discussed. In this paper, we shall obtain a structure theorem on solvable (or nilpotent) pseudo-symmetric sets: S is solvable (or nilpotent) if and only if $H(S)$ is so. Lastly we remark that the theory developed here is more general. We do not need the conditions "effectiveness" and "transitivity". Also the condition (1) of the pseudo-symmetric set is not needed except for the main theorem of simple pseudo-symmetric sets. It should be also noted that the concept of normal decompositions can be applied for more general binary systems, for example, for groupoids.

2. Homomorphisms and normal decompositions

A homomorphism f of a pseudo-symmetric set S to a pseudo-symmetric set T is a mapping f of S to T such that $f(a\circ b)=f(a)\circ f(b)$. The pseudo-symmetric structure σ is a homomorphism of S to G, the latter being considered as a pseudo-symmetric set by definition: $x\circ y=y^{-1}xy$. $(x, y\in G.)$ This is clear from the fundamental identity given in **1**. Denote the image of S by σ by $I(S)$ or simply by I, i.e., $I=\{\sigma_a \mid a\in S\}$. Let K and L be groups, which are also considered as pseudo-symmetric sets as above. A group-homomorphism of K to L is naturally a pseudo-symmetric set-homomorphism. Especially, the natural homomorphism ν: $G\rightarrow G/N$ for a normal subgroup N is a (pseudo-symmetric set-)homomorphism. More generally, let T be a pseudo-symmetric subset of G. Then, the mapping $T\overset{\nu}{\rightarrow} TN/N$ is a homomorphism. Thus, we have a homomorphism of S to IN/N through $S\overset{\sigma}{\rightarrow} I\overset{\nu}{\rightarrow} IN/N$.

Let f: $S\rightarrow T$ be a homomorphism. We define an equivalent relation on S: $a\sim b$ if and only if $f(a)=f(b)$. Then, S is decomposed into equivalent classes: $S=\cup S_i$, where $S_i=f^{-1}(t_i)$ for some $t_i\in T$. It satisfies that (i) G induces a group of permutations on $X=\{S_i\}_i$, the set of classes S_i, and (ii) σ_a and σ_b induce the same permutation on X if a and b belong to a same class S_i. Conversely, suppose that $S=\cup S_i$ with $S_i\cap S_j=\phi$ if $i\neq j$ and that the above conditions (i) and (ii) are satisfied. Then, we can define a pseudo-symmetric structure $\bar{\sigma}$ on X such that $\bar{\sigma}(S_i)$ is the permutation on X induced by $\sigma(a_i)$ where $a_i\in S_i$. In this case, the mapping $a_i\mapsto S_i$ is a homomorphism of S to X, and the decomposition $S=\cup S_i$ coincides with the decomposition of S relative to the homomorphism $S\rightarrow X$. We call a decomposition of S satisfying (i) and (ii) a *normal decomposition* of S and denote it by D: $S=\cup S_i$. The pseudo-symmetric set $X=\{S_i\}$ will be denoted by S/D. Thus, there is a homomorphism of S onto S/D and the decomposition associated with it is D. Two trivial examples: Let $\{e\}$ be a one-point trivial pseudo-symmetric set and consider the trivial homomorphism $S\rightarrow\{e\}$. The normal decomposition associated with it is $S=S$. We denote this decomposition by S, which will give no complication. Thus, S: $S=S$. The factor pseudo-symmetric set in this case is trivial; $S/S\cong\{e\}$. The second trivial example comes from the identity homomorphism $S\rightarrow S$. We denote the normal decomposition associated with it by E. Thus, E: $S=\cup\{a\}$, i.e., each class is a one-point set $\{a\}$ $(a\in S)$. We have $S/E\simeq S$.

Let N be a normal subgroup of G. As was given before, there is the homomorphism $S\rightarrow IN/N$. We denote the normal decomposition associated with it by D_N. We have

(1) $$S/D_N \cong IN/N.$$

Note that D_N is given by the equivalence: $a \sim b$ if and only if $\sigma_a \equiv \sigma_b \bmod N$. If $N=G$, then $D_G=S$ (the trivial decomposition). If $N=(1)$ (the identity), then the decomposition $D_{(1)}$ is given by $S=\cup S_i$ with $S_i=\{a \in S \mid \sigma_a=\sigma_{a_i}\}$ for fixed a_i. We have $S/D_{(1)} \cong I$.

Let f: $S \to T$ be a homomorphism of S onto T. Then $S/D \cong T$, where D is the normal decomposition associated with f. $G(S)$ induces a group of automorphisms on S/D in a natural way. So, we have a homomorphism f^*: $G(S) \to G(T)$. f^* is onto because $f^*(\sigma_a)=\sigma(f(a))$ (if $a \in S$) by definition and $G(T)$ is generated by $\sigma(f(a))$ (note that f is onto.).

Lemma 1. *Let N be a normal subgroup of G, and f the homomorphism $S \to S/D_N$. Then, the kernel K of f^*: $G \to G(S/D_N)$ is a normal subgroup of G containing N and $K/N \subseteq Z(G/N)$, the center of G/N.*

Proof. We have the homomorphism f': $S \to IN/N$. Naturally, K coincides with the kernel of $(f')^*$. If $\rho \in G$, then $(f')^*(\rho)$ is the conjugation by ρ on IN/N. (This is seen by taking $\rho=\sigma_a$.) It is clear that K contains N. Since IN/N generates G/N, the latter part of Lemma 1 is also clear.

3. Subdecompositions, factor decompositions and groups of displacements of normal decompositions

Let D_1: $S=\cup S_i$ and D_2: $S=\cup T_j$ be two normal decompositions of S. When the decomposition D_2 is finer than D_1, i.e., $S_i \cap T_j=T_j$ or ϕ, we say that D_2 is a (normal) subdecomposition of D_1 and denote $D_2 \leq D_1$. So, we have $E \leq D \leq S$ for every normal decomposition D. Let M and N be normal subgroups of G. Then,

(2) $$D_M \leq D_N \quad \text{if } M \subseteq N.$$

Suppose that $D_2 \leq D_1$. Then, there exists a homomorphism $S/D_2 \to S/D_1$, where an element T_j of S/D_2 is mapped to an element S_i of S/D_1 if $T_j \subseteq S_i$. The normal decomposition of S/D_2 associated with the homomorphism is denoted by D_1/D_2 and is called a factor normal decomposition. Thus, D_1/D_2: $S/D_2=\cup \bar{S}_i$, where $\bar{S}_i=\{T_j \mid T_j \subseteq S_i\}$. We have

(3) $$(S/D_2)/(D_1/D_2) \cong S/D_1.$$

Now we define the group of displacements of a normal decomposition D: $S=\cup S_i$ by $H(D)=\langle \sigma_a^{-1}\sigma_b \mid a, b \in S_i, i=1, 2, \cdots \rangle$. $H(D)$ is a normal subgroup of G contained in $H(S)$. Note that $H(S)$ for the decomposition S coincides with the group of displacements $H(S)$ of the pseudo-symmetric set S. Let f: $S \to S/D$ and f^*: $G \to G(S/D)$ for a normal decomposition D. Then, $H(D)$

is contained in the kernel of f^*. Besides the trivial example $H(S)$, we have another trivial example: $H(E)=(1)$. There is another interesting example. Let S' denote the orbit-decomposition of S; S': $S=\cup(a^G)$. It is easy to see that it is a normal decomposition. S/S' is the set of G-orbits and is a trivial pseudo-symmetric set. In this case, we have $H(S')=G'$, the commutator subgroup of G. For, $\sigma_a^{-1}\sigma_b^{-1}\sigma_a\sigma_b=\sigma_a^{-1}\sigma(a^{\sigma(b)})\in H(S')$. Clearly we have

$$(4) \qquad H(D_2)\subseteq H(D_1) \quad \text{if } D_2\leq D_1 .$$

Especially, $G'=H(S')\subseteq H(S)$, which is also clear from definition. Let N be a normal subgroup and D a normal decomposition. Then,

$$(5) \qquad H(D_N)\subseteq N .$$

$$(6) \qquad D\leq D_{H(D)} .$$

Generally, D is not equal to $D_{H(D)}$. But, we have that $H(D_{H(D)}/D)=(1)$. For, $H(D_{H(D)}/D)$ is a subgroup of $G(S/D)$ generated by $f^*(\sigma_a^{-1})f^*(\sigma_b)$ with $\sigma_a\equiv\sigma_b$ mod $H(D)$, where f: $S\to S/D$, and hence $\sigma_a^{-1}\sigma_b\in H(D)$ and $f^*(\sigma_a^{-1})f^*(\sigma_b)=1$.

4. Some structure theorems

Theorem 1. *Let N be a normal subgroup of G and D a normal decomposition of S. Then, $D_{H(D_N)}=D_N$ and $H(D_{H(D)})=H(D)$.*

Proof. (5) and (2) imply $D_{H(D_N)}\leq D_N$. Conversely, $D_N\leq D_{H(D_N)}$ by (6). Therefore, the first identity follows. (6) and (4) imply $H(D)\subseteq H(D_{H(D)})$. Conversely, $H(D_{H(D)})\subseteq H(D)$ by (5). Therefore, the second identity follows.

Theorem 2. *Let K be the kernel of f^*: $G\to G(S/D)$. Then, K contains $H(D)$ and $K/H(D)\subseteq Z(G/H(D))$.*

Proof. Apply Lemma 1 for $N=H(D)$, and we have $K'/H(D)\subseteq Z(G/H(D))$, where K' is the kernel of the homomorphism $G\to G(S/D_N)$. But, $K\subseteq K'$ since $D\leq D_N$ by (6).

Corollary 1. *Let N be a normal subgroup of G. Then, $N/H(D_N)\subseteq Z(G/H(D_N))$.*

When σ is one to one, we say that S is effective. This means $S\cong I$, or $D_{(1)}=E$. In this case, $Z(G)=(1)$. For, if $\rho\in Z(G)$, ρ induces the trivial automorphism on I (by conjugation). $I\cong S$ implies $\rho=1$. When $S=a^G$, we say that S is transitive. This means $S=S'$. So, in this case, $H(S)=G'$.

Corollary 2. (1) *Suppose that S is effective. If N is a normal subgroup of G such that $(1)\subset N\subset H(S)$, then $E<D_N<S$.* (2) *Suppose that S is transi-*

tive. If D is a normal decomposition of S such that $D_{(1)}<D<S$, then $(1)\subset H(D)\subset H(S)$.

Proof. (1) $D_N\neq S$, since $H(S)\supset N\supseteq H(D_N)$. Suppose $D_N=E$. Then, $N=N/H(D_N)$ is contained in $Z(G)=Z(G/H(D_N))$ by Corollary 1. But $Z(G)=(1)$, and hence $N=(1)$, which is a contradiction. (2) First, we remark that $S=a^G$ implies $S=a^{H(S)}$. For, by definition of $H(S)$, $G=\langle\sigma_b, H(S)\rangle$ for any element b. Let b be any element in S, and we have $b=a^\tau$ with $\tau\in G$. Then, we can replace τ by ρ (if necessary) such that $a^\tau=a^\rho$ and $\rho\in H(S)$, because $a^\tau=a^{\tau\sigma_b^i}$ (i is any integer) and $\tau\sigma_b^i$ $(=\rho)$ can be an element in $H(S)$ for a suitable i. Now we prove (2). $H(D)\neq(1)$, since $D_{H(D)}\geq D>D_{(1)}$. Suppose that $H(D)=H(S)$. Then, $S=a^{H(D)}$, which implies that D is the trivial decomposition $S=S$. This contradicts the assumption $D\neq S$.

From Corollary 2, we can derive the main theorem on a simple pseudo-symmetric set: Suppose that S is effective and transitive. Then, S has no normal decomposition D such that $E<D<S$ if and only if there is no normal subgroup N such that $(1)\subset N\subset H(S)$. We also note that in the above discussion we used the condition (1) $a\circ a=a$ (actually $b^{\sigma_b}=b$), and this is the only place we use (1).

When $D=D_N$, Theorem 2 can be strengthen:

Theorem 3. *Let K be the kernel of $f^*: G\to G(S/D_N)$ for a normal subgroup N. Then, $K/H(D_N)=Z(G/H(D_N))$.*

Proof. Let $\tau\in G$ be such that $\tau H(D_N)\in Z(G/H(D_N))$. τ induces the trivial automorphism on $IH(D_N)/H(D_N)$. Since $D_N=D_{H(D_N)}$ by Theorem 1, we have $S/D_N=S/D_{H(D_N)}$. But, the latter is isomorphic with $IH(D_N)/H(D_N)$. So, $f^*(\tau)=1$ and hence $\tau\in K$. This with Theorem 2 implies Theorem 3.

Theorem 4. *Let M and N be normal subgroups of G such that $N\subseteq M$. Then, there exist a subgroup M_1/N of M/N and an onto-homomorphism $M_1/N\to H(D_M/D_N)$.*

Proof. Let K be the kernel of $f^*: G\to G(S/D_N)$, and J the kernel of $h^*: H(D_M)\to H(D_M/D_N)$, where h^* is the restriction of f^* on $H(D_M)$. Then, $J=K\cap H(D_M)$. Since $K\supseteq N$, we have $J\supseteq N\cap H(D_M)$. On the other hand, $H(D_M)/H(D_M)\cap N\cong H(D_M)N/N$. Let $M_1=H(D_M)N$. Clearly, $M_1\subseteq M$. We have the canonical homomorphism $M_1/N\to H(D_M)/J\cong H(D_M/D_N)$.

Corollary 3. *In Theorem 4, if M/N is abelian, so is $H(D_M/D_N)$.*

Theorem 5. *Let M and N be as in Theorem 4, and assume that $N\subseteq H(S)$. If $M/N\subseteq Z(H(S)/N)$, then $H(D_M/D_N)\subseteq Z(H(S/D_N))$.*

Proof. Let $\tau\in H(D_M)\subseteq M$, and $\rho\in H(S)$. We want to show $f^*(\tau)f^*(\rho)$

$=f^*(\rho)f^*(\tau)$, where $f^*: G\to G(S/D_N)$. Let $\sigma=\tau^{-1}\rho^{-1}\tau\rho$. Then, $\sigma\in N$ because $M/N\subseteq Z(H(S)/N)$. Hence, $f^*(\sigma)=1$, proving the above identity.

Theorem 6. *If $D_2\leq D_1$ and $H(D_2)=H(D_1)$, then $H(D_1/D_2)=(1)$. If $N\subseteq M$ and $D_N=D_M$, then $M/N\subseteq Z(G/N)$.*

Proof. Let $D=D_{H(D_1)}\,(=D_{H(D_2)})$. Then, $D_2\leq D_1\leq D$ by (6) and Theorem 1. By the remark given after (6), $H(D/D_2)=(1)$. Hence, $H(D_1/D_2)=(1)$. Let $T=H(D_M)\,(=H(D_N))$. Then, $T\subseteq N\subseteq M$ by (2) and Theorem 1. By Corollary 1, $M/T\subseteq Z(G/T)$. Considering the canonical homomorphism $G/T\to G/N$, we can conclude that $M/N\subseteq Z(G/N)$.

Although the correspondences $D\mapsto H(D)$ and $N\mapsto D_N$ are not one to one, Theorem 6 tells that their discrepancies are within trivial (or, abelian).

5. Solvable and nilpotent pseudo-symmetric sets

A normal decomposition D is called abelian if $H(D)$ is abelian. A pseudo-symmetric set S is called solvable if there exists a sequence of normal decompositions

(7) $$S=D_0\geq D_1\geq \cdots \geq D_{n-1}\geq D_n=E$$

such that D_i/D_{i+1} are abelian ($i=0, 1, \cdots, n-1$).

Theorem 7. *S is solvable if and only if $H(S)$ is solvable.*

Proof. Suppose that S is solvable as above. Then, we have a sequence of normal subgroups of G: $H(S)=H(D_0)\supseteq H(D_1)\supseteq \cdots \supseteq H(D_{n-1})\supseteq H(E)=(1)$. Let J_i be the kernel of the homomorphism $h_i^*: H(D_i)\to H(D_i/D_{i+1})$. J_i contains $H(D_{i+1})$ and $J_i/H(D_{i+1})$ is abelian by Theorem 2. $H(D_i)/J_i$ $(\cong H(D_i/D_{i+1}))$ is also abelian by assumption. Thus, $H(S)$ is solvable. Conversely, suppose that $H(S)$ is solvable. Let $H(S)=N_0\supseteq N_1\supseteq \cdots \supseteq N_{n-1}\supseteq N_n=(1)$ be a sequence of normal subgroups of $H(S)$ such that N_i/N_{i+1} are all abelian. We may also assume that N_i are G-normal. Let $D_i=D_{N_i}$, and we have $S=D_0\geq D_1\geq \cdots \geq D_n=D_{(1)}\geq D_{n+1}=E$. By Corollary 3, D_i/D_{i+1} are abelian ($i=0, 1, \cdots, n-1$). Also, $H(D_n/D_{n+1})=(1)$ implies D_n/D_{n+1} is abelian. Thus, S is solvable.

A normal decomposition D is called central if $H(D)\subseteq Z(H(S))$. S is called nilpotent if there exists (7) such that D_i/D_{i+1} are all central, i.e., $H(D_i/D_{i+1})\subseteq Z(H(S/D_{i+1}))$.

Theorem 8. *S is nilpotent if and only if $H(S)$ is nilpotent.*

Proof. Suppose S is nilpotent as above. We have the sequence: $H(S)$

$\supseteq H(D_1)\supseteq\cdots$ Again, let J_i be the kernel of h_i^*: $H(D_i)\to H(D_i/D_{i+1})$. Then, $J_i/H(D_{i+1})\subseteq Z(G/H(D_{i+1}))$ by Theorem 2. We must show that $H(D_i)/J_i\subseteq Z(H(S)/J_i)$. For it, recall the assumption $H(D_i/D_{i+1})\subseteq Z(H(S/D_{i+1}))$ and the two isomorphisms $H(D_i/D_{i+1})\cong H(D_i)/J_i$ and $H(S/D_{i+1})\cong H(S)/J_i'$, where $J_i=K\cap H(D_i)$ and $J_i'=K\cap H(S)$ (K is the kernel of the homomorphism $G\to G(S/D_{i+1})$). Hence, $J_i=J_i'\cap H(D_i)$. Let $\tau\in H(D_i)$ and $\rho\in H(S)$. We must show that $\tau\rho\equiv\rho\tau$ mod J_i. Let $\sigma=\tau^{-1}\rho^{-1}\tau\rho$. Then, $\sigma\in J_i'$. σ is naturally contained in $H(D_i)$. Therefore, $\sigma\in J_i$, and hence $\tau\rho\equiv\rho\tau$ mod J_i. We have proven that $H(D_i)/J_i\subseteq Z(H(S)/J_i)$, which finishes the proof that $H(S)$ is nilpotent. Conversely, suppose that $H(S)$ is nilpotent and let $H(S)=N_0\supseteq N_1\supseteq\cdots\supseteq N_n=(1)$ be such that $N_i/N_{i+1}\subseteq Z(N_0/N_{i+1})$. We may assume that N_i are all G-normal. Then, let $D_i=D_{N_i}$. By Theorem 5, $H(D_i/D_{i+1})\subseteq Z(H(S/D_{i+1}))$. Thus, S is nilpotent.

EXAMPLE 1. Let $G=S_4$, the symmetric group of degree 4. Let S be the symmetric set consisting of transpositions in S_4: $S=\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$. We have a normal decomposition D: $S=S_1\cup S_2\cup S_3$, where $S_1=\{(1, 2), (3, 4)\}$, $S_2=\{(1, 3), (2, 4)\}$, and $S_3=\{(1, 4), (2, 3)\}$. Then, $H(D)=\{$the identity, $(1, 2)(3, 4)$, $(1, 3)(2, 4)$, $(1, 4)(2, 3)\}$, which is an abelian group. It is also easy to see that $H(S/D)$ is an abelian group of order 3. Thus, $S\geq D\geq E$ is a solvable sequence of normal decompositions. So, S is solvable. $H(S)=A_4$ is naturally solvable.

EXAMPLE 2. Let L be a nilpotent Lie algebra of finite dimension. Define $\sigma(a)=\exp(ad\ a)$ for $a\in L$, and L is considered as a pseudo-symmetric set. We can show that L is a nilpotent pseudo-symmetric set as follows. Let C be the center of L. The (Lie algebra) homomorphism $L\to L/C$ is seen to be a pseudo-symmetric set homomorphism. Let D be the normal decomposition associated with the homomorphism. D is the usual cosets decomposition $L=\cup S_i$, where $S_i=\{a_i+C\}$. If $z\in C$, then we see that $\sigma(a+z)=\sigma(a)$. Therefore, $H(D)=(1)$. Next, we consider L/C in place of L and apply the above argument. We obtain a normal decomposition D' such that $D\leq D'$ and $H(D'/D)=(1)$. Repeating this, we obtain a nilpotent sequence of normal decompositions $L\geq\cdots\geq D'\geq D\geq E$. Thus, L is nilpotent.

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