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ON PATHWISE UNIQUENESS AND COMPARISON OF SOLUTIONS OF ONE-DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

SHINTARO NAKAO

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1. Introduction

In this paper we shall discuss the pathwise uniqueness and comparison problems for solutions of one-dimensional stochastic differential equations. Let $a(t, x)$ and $b(t, x)$ be bounded Borel functions defined on $[0, \infty)\times R$ with values in R. Consider the following one-dimensional stochastic differential equation;

$$(1)\quad \begin{cases} dx(t) = a(t, x(t))dB(t)+b(t, x(t))dt\,, \\ x(0) = x_0\,, \end{cases}$$

where $B(t)$ is a one-dimensional Brownian motion with $B(0)=0$ and $x_0 \in R$ is a non-random initial value. In [3], I showed that $a(t, x)=a(x)$ is uniformly positive and of bounded variation on any compact interval and $b(t, x)$ is time independent, then the pathwise uniqueness holds for the equation (1). A. Yu. Veretennikov [5] extended the above result to the case that the coefficients are time dependent. The purpose of this paper is to obtain another extension of the result of [3] different from that of A. Yu. Veretennikov.

$\mathrm{VI}([0, \infty)\times R)$ denotes the space of all functions defined on $[0, \infty)\times R$ such that for $t\geqq 0$ $f(t, x)$ is nondecreasing in x and for $x\in R$ $f(t, x)$ is of bounded variation in t on any compact interval. Throughout this paper we shall assume that $a(t, x)$ satisfies the following condition.

CONDITION A. $a(t, x)$ satisfies the following conditions;

(i) $a(t, x)$ is Borel measurable and there exist positive constants a_1 and a_2 such that $0<a_1\leqq a(t, x)\leqq a_2$ for $(t, x)\in[0, \infty)\times R$,

(ii) there exist $\alpha_1(t, x)\in \mathrm{VI}([0, \infty)\times R)$ and $\alpha_2(t, x)\in \mathrm{VI}([0, \infty)\times R)$ such that $\dfrac{1}{a(t, x)}=\alpha_1(t, x)-\alpha_2(t, x)$ for a.e. $(t, x)\in[0, \infty)\times R$,

(iii) for $t>0$ and $N>0$ there exists a positive constant $L(t, N)$ such that

$|||\alpha_i(\cdot, x)|||_t$ [*1)] $\leqq L(t, N)$ for $x \in [-N, N]$ and $i=1, 2$.

In this paper we adopt the definitions in [1] about the solution of (1) and the pathwise uniqueness of (1). We obtain the following theorem.

Theorem 1. *Suppose that $a(t, x)$ satisfies Condition A and $b(t, x)$ is bounded Borel measurable. Then the pathwise uniqueness holds for the stochastic differential equation* (1).

We now consider the following stochastic differential equations;

$$(2) \quad \begin{cases} dx(t) = a(t, x(t))dB(t)+b_1(t, x(t))dt\,, \\ x(0) = x_0 \in R \end{cases}$$

and

$$(3) \quad \begin{cases} dy(t) = a(t, y(t))dB(t)+b_2(t, y(t))dt\,, \\ y(0) = x_0\,. \end{cases}$$

The following comparison theorem is a generalization of a result of [4].

Theorem 2. *Suppose that $a(t, x)$, $b_1(t, x)$ and $b_2(t, x)$ satisfy the following conditions;*

(i) *$a(t, x)$ satisfies Condition A,*

(ii) *$b_1(t, x)$ and $b_2(t, x)$ are bounded Borel functions such that $b_1(t, x) \leqq b_2(t, x)$ for $(t, x) \in [0, \infty) \times R$ a.e.*

Let $(x(t), B(t))$ and $(y(t), B(t))$ be solutions of the stochastic differential equations (2) *and* (3) *respectively defined on a same probability space $(\Omega, \mathcal{F}, P)$ with a reference family $(\mathcal{F}_t)_{t\geqq 0}$ such that $x(0)=y(0)=x_0 \in R$. Then it holds that $x(t) \leqq y(t)$ a.s. for $t \geqq 0$.*

In section 2 we prove Theorem 1 and give an example of $a(t, x)$ which satisfies Condition A. In section 3 we prove Theorem 2 by a new method.

2. Proof of pathwise uniqueness theorem

First we shall prepare two lemmas for the proof of Theorem 1. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a reference family $(\mathcal{F}_t)_{t\geqq 0}$ and let $B(t)$ be a one-dimensional $(\mathcal{F}_t)$-Brownian motion defined on $(\Omega, \mathcal{F}, P)$ with $B(0)=0$. Consider the stochastic process defined by

$$x(t) = x_0 + \int_0^t \sigma(s)dB(s) + \int_0^t \gamma(s)ds\,,$$

where $\sigma(s)$ and $\gamma(s)$ are bounded measurable stochastic processes on $(\Omega, \mathcal{F}, P)$

1) Let f(s) be a real function defined on $[0, \infty)$. $|||f|||_t$ denotes the total variation of f(s) on [0,t].

adapted to $(\mathcal{F}_t)$ and x_0 is a real number. Set $\sigma=\sup_{(t,\omega)}|\sigma(t,\omega)|$ and $\gamma=\sup_{(t,\omega)}|\gamma(t,\omega)|$. For $N>0$, $\tau_N=\inf\{t;\ |x(t)|\geqq N\}$. Let $g(t,x)$ be a Lebesgue measurable function defined on $[0,\infty)\times R$. Setting

$$G(t,x)=\int_0^x g(t,y)dy \quad \text{for} \quad (t,x)\in[0,\infty)\times R$$

and

$$V(t)=G(t,x(t))-G(0,x_0)-\int_0^t g(s,x(s))\sigma(s)dB(s)\,,$$

we shall estimate the expectation of $|||V|||_{t\wedge\tau_N}$*2).

Lemma 1. *Suppose that $g(t,x)$ belongs to $VI([0,\infty)\times R)$ and is continuously differentiable in (t,x). Then it holds that for $t>0$ and $N>0$*

$$E[|||V|||_{t\wedge\tau_N}]\leqq 2(N+t\gamma)M(t,N)+4NK(t,N)\,,$$

where E denotes the expectation with respect to P,

$$M(t,N)=\sup\{|g(s,y)|\,;\,(s,y)\in[0,t]\times[-N,N]\}$$

and

$$K(t,N)=\sup\{|||g(\cdot,y)|||_t;\,y\in[-N,N]\}\,.$$

Proof. Itô's formula implies that

$$\begin{aligned}V(t)&=\int_0^t g(s,x(s))\gamma(s)ds+\int_0^t\frac{\partial}{\partial s}G(s,x(s))ds+\frac{1}{2}\int_0^t\frac{\partial}{\partial x}g(s,x(s))\sigma(s)^2ds\\&=I_1(t)+I_2(t)+I_3(t)\,.\end{aligned}$$

It is easy to see that $E[|||I_1|||_{t\wedge\tau_N}]\leqq t\gamma M(t,N)$ and $E[|||I_2|||_{t\wedge\tau_N}]\leqq 2NK(t,N)$. Since $|||I_3|||_{t\wedge\tau_N}=V(t\wedge\tau_N)-I_1(t\wedge\tau_N)-I_2(t\wedge\tau_N)$, we have $E[|||I_3|||_{t\wedge\tau_N}]\leqq E[V(t\wedge\tau_N)]+t\gamma M(t,N)+2NK(t,N)$. On the other hand it holds that $E[V(t\wedge\tau_N)]=E[G(t\wedge\tau_N,x(t\wedge\tau_N))-G(0,x_0)]\leqq 2NM(t,N)$. Combining the above estimates, we have $E[|||V|||_{t\wedge\tau_N}]\leqq 2(N+t\gamma)M(t,N)+4NK(t,N)$, which completes the proof.

Let $\rho(s,y)$ be a non-negative C^∞-function defined on R^2 such that its support is contained in the closed unit ball and $\int_{R^2}\rho(s,y)dsdy=1$. For $\delta>0$ set

$$\rho_\delta(s,y)=\frac{1}{\delta^2}\rho\left(\frac{s}{\delta},\frac{y}{\delta}\right). \tag{4}$$

We now consider

$$V_\delta(t)=G_\delta(t,x(t))-G_\delta(0,x_0)-\int_0^t g_\delta(s,x(s))\sigma(s)dB(s)\,,$$

2) Let a and b be real numbers. $a\wedge b=\min\{a,b\}$.

where

$$g_\delta = \bar{g}*\rho_\delta{}^{*3)} \quad \text{and} \quad G_\delta(t, x) = \int_0^x g_\delta(t, y)dy\,.$$

Lemma 2. *Suppose that* $g(t, x) \in VI([0, \infty)\times R)$ *satisfies that for* $t>0$ *and* $N>0$ *there exists a positive constant* $K(t, N)$ *such that* $|||g(\cdot, x)|||_t \leqq K(t, N)$ *for* $x \in [-N, N]$. *Then it holds that for* $0<\delta\leqq 1$, $t>0$ *and* $N>0$

$$E[|||V|||_{t\wedge\tau_N}] \leqq 2(N+t\gamma)M(t+1, N+1)+4NK(t+1, N+1)\,,$$

where

$$M(t, N) = \sup\{|g(s, y)|\,;\,(s, y)\in [0, t]\times[-N, N]\}\,.$$

Proof. It is easy to see that $|||g_\delta(\cdot, x)|||_t \leqq K(t+\delta, N+\delta)$ for $x\in[-N, N]$ and $\sup\{|g_\delta(s, y)|\,;\,(s, y)\in[0, t]\times[-N, N]\} \leqq M(t+\delta, N+\delta)$. Hence Lemma 2 is an easy consequence of Lemma 1.

Proof of Theorem 1. Let $a_0=1>a_1>a_2>\cdots>a_k>\cdots\to 0$ be a sequence such that $\int_{a_k}^{a_{k-1}} \frac{1}{u}du=k$ for $k=1, 2, \cdots$. Then there exists a twice continuously differentiable and odd function $\psi_k(u)$ on R such that $0\leqq\psi_k(u)\leqq 1$ for $u\in[0, \infty)$,

$$\psi_k(u) = \begin{cases} 0 & \text{for} \quad 0\leqq u\leqq a_k \\ 1 & \text{for} \quad a_{k-1}\leqq u\,, \end{cases}$$

and

$$(5) \qquad 0\leqq \psi_k^{(1)}(u)^{*4)} \leqq \frac{2}{ku} \quad \text{for} \quad a_k<u<a_{k-1}\,.$$

Set $\alpha(t, x)=\alpha_1(t, x)-\alpha_2(t, x)$, $\alpha_\delta=\bar{\alpha}*\rho_\delta$ and $h_\delta(t, x)=\int_0^x \alpha_\delta(t, y)dy$, where ρ_δ is the function defined by (4).

Let $(x(t), B(t))$ and $(y(t), B(t))$ be solutions of (1) defined on a same quadruplet $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$. Set $\eta_N=\{t;\,|x(t)|\geqq N \text{ or } |y(t)|\geqq N\}$. Theorem 2 of N. V. Krylov [2] assures that for $k=1, 2, \cdots$ there exists a positive constant $\delta_k=\delta_k(t, N) \leqq \frac{1}{k}$ such that

$$(6) \qquad \max_{u\in R}|\psi_k^{(i)}(u)|\,E\left[\int_0^{t\wedge\eta_N} |a\cdot\alpha_{\delta_k}(s, x(s))-1|^i ds\right] \leqq \frac{1}{k} \qquad \text{for} \quad i=1, 2.$$

Obviously the same estimates as (6) hold for $(y(t))$. For simplicity we set $h_k=h_{\delta_k}$, $\tilde{\alpha}_k=\alpha_{\delta_k}$, $z_k(t)=h_k(t, x(t))-h_k(t, y(t))$ and $J(k, t)=(x(t)-y(t))\psi_k(z_k(t))$.

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3) For a function g(t,x) defined on $[0,\infty)\times R$, $\bar{g}(t,x)$ denotes the function on $R\times R$ such that $\bar{g}(t,x) = \begin{cases} g(t,x) & t\geqq 0 \\ g(0,x) & t<0. \end{cases}$ $\bar{g}*\rho_\delta$ denotes the convolution of $\bar{g}$ and ρ_δ.

4) $f^{(i)}(u)$ denotes the i-th derivative of f(u).

The martingale part $m_k(t)$ of $z_k(t)$ is $\int_0^t (a\cdot\tilde{\alpha}_k(s, x(s)) - a\cdot\tilde{\alpha}_k(s, y(s)))dB(s)$. Setting $v_k(t)=z_k(t)-m_k(t)$, we have by Itô's formula

$$\begin{aligned}
&J(k, t)\\
&= \int_0^t \psi_k(z_k(s))d(x-y)(s)+\int_0^t (x(s)-y(s))\psi_k^{(1)}(z_k(s))dm_k(s)\\
&\quad+\int_0^t (x(s)-y(s))\psi_k^{(1)}(z_k(s))dv_k(s)\\
&\quad+\int_0^t \psi_k^{(1)}(z_k(s))(a(s, x(s))-a(s, y(s)))(a\cdot\tilde{\alpha}_k(s, x(s))-a\cdot\tilde{\alpha}_k(s, y(s)))ds\\
&\quad+\frac{1}{2}\int_0^t (x(s)-y(s))\psi_k^{(2)}(z_k(s))(a\cdot\tilde{\alpha}_k(s, x(s))-a\cdot\tilde{\alpha}_k(s, y(s)))^2 ds\\
&= J_1(k, t)+J_2(k, t)+J_3(k, t)+J_4(k, t)+J_5(k, t)\,.
\end{aligned}$$

Using that

$$(7)\qquad 0<\frac{x-y}{h_\delta(t, x)-h_\delta(t, y)}\leqq a_2 \quad\text{for}\quad t\geqq 0,\ x\neq y \quad\text{and}\quad \delta>0$$

and

$$(8)\qquad \lim_{k\to\infty}\psi_k(u)=\chi(u)=\begin{cases} -1 & \text{for}\quad u<0\\ 0 & \text{for}\quad u=0\\ 1 & \text{for}\quad u>0\,,\end{cases}$$

it is easy to see that

$$J(k, t\wedge\eta_N)\xrightarrow[k\to\infty]{} |x(t\wedge\eta_N)-y(t\wedge\eta_N)| \quad\text{in}\quad L^1(P)$$

and

$$J_1(k, t\wedge\eta_N)\xrightarrow[k\to\infty]{}\int_0^{t\wedge\eta_N}\chi(x(s)-y(s))d(x-y)(s) \quad\text{in}\quad L^1(P)\,.$$

By (5) and (7) we obtain

$$E[J_2(k, t\wedge\eta_N)^2]\leqq\left(\frac{2a_2}{k}\right)^2 E\left[\int_0^{t\wedge\eta_N}(a\cdot\tilde{\alpha}_k(s, x(s))-a\cdot\tilde{\alpha}_k(s, y(s)))^2 ds\right]\leqq 8\left(\frac{a_2}{a_1 k}\right)^2$$

and

$$E[|J_4(k, t\wedge\eta_N)|]\leqq\frac{2a_2}{k}E[|||v_k|||_{t\wedge\eta_N}]\,.$$

Since $\sup_k E[|||v_k|||_{t\wedge\eta_N}]$ is finite by Lemma 2, we have $\lim_{k\to\infty} E[|J_2(k, t\wedge\eta_N)+J_3(k, t\wedge\eta_N)|]=0$. (6) implies that $\lim_{k\to\infty} E[|J_4(k, t\wedge\eta_N)+J_5(k,t\wedge\eta_N)|]=0$. Consequently we have

$$|x(t\wedge\eta_N)-y(t\wedge\eta_N)|=\int_0^{t\wedge\eta_N}\chi(x(s)-y(s))d(x-y)(s)\,.$$

Letting $N\to\infty$ it holds that

$$(9)\qquad |x(t)-y(t)|=\int_0^t \chi(x(s)-y(s))d(x-y)(s)\,.$$

(9) implies that

$$\begin{aligned}
&x(t)\wedge y(t)\\
&=\frac{1}{2}\{x(t)+y(t)-|x(t)-y(t)|\}\\
&=x_0+\int_0^t \frac{1}{2}\{a(s,\,x(s))+a(s,\,y(s))-\chi(x(s)-y(s))(a(s,\,x(s))-a(s,\,y(s)))\}\,dB(s)\\
&\qquad+\int_0^t \frac{1}{2}\{b(s,\,x(s))+b(s,\,y(s))-\chi(x(s)-y(s))(b(s,\,x(s))-b(s,\,y(s)))\}\,ds\\
&=x_0+\int_0^t a(s,x(s)\wedge y(s))dB(s)+\int_0^t b(s,\,x(s)\wedge y(s))ds\,.
\end{aligned}$$

In the same way max $\{x(t), y(t)\}$ is a solution of (1). Since the uniqueness in law holds for (1), we conclude $x(t)=y(t)$ a.s. The proof is completed.

REMARK. Let $a(t, x)$ be a uniformly positive and bounded Borel function and let $b(t, x)$ be a bounded Borel function. Set $h(t, x)=\int_0^x \frac{1}{a(t, y)}dy$. Suppose that there exists a solution $(\tilde{x}(t), \tilde{B}(t))$ with $\tilde{x}(t)=x_0+\int_0^x a(s, \tilde{x}(s))d\tilde{B}(s)$ such that $h(t, \tilde{x}(t))-h(0, x_0)$ is a continuous quasimartingale and the martingale part of $h(t, \tilde{x}(t))-h(0, x_0)$ is the one-dimensional Brownian motion $\tilde{B}(t)$. Let $(x_1(t), B(t))$ and $(x_2(t), B(t))$ be solutions defined on a same quadruplet $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ such that

$$x_i(t)=x_0+\int_0^t a(s,\,x_i(s))dB(s)+\int_0^t b(s,\,x_i(s))ds \qquad i=1,\,2\,.$$

Then it holds that $x_1(t)=x_2(t)$ a.s. for $t\geqq 0$.

Proof. By the assumption the sample paths of $h(t, x_1(t))-h(t, x_2(t))$ are continuous and of bounded variation on any compact interval with probability one. Let $\psi_k(u)$ $(k=1, 2, \cdots)$ be the function defined in the proof of Theorem 1. Itô's formula implies

$$\begin{aligned}
&(x_1(t)-x_2(t))\psi_k(h(t,\,x_1(t))-h(t,\,x_2(t)))\\
&\quad=\int_0^t \psi_k(h(s,\,x_1(s))-h(s,\,x_2(s)))d(x_1-x_2)(s)\\
&\qquad+\int_0^t (x_1(s)-x_2(s))\psi_k^{(1)}(h(s,\,x_1(s))-h(s,\,x_2(s)))d(h(s,\,x_1(s))-h(s,\,x_2(s)))\,.
\end{aligned}$$

Letting $k\to\infty$ we have

$$|x_1(t)-x_2(t)|=\int_0^t \chi(x_1(s)-x_2(s))d(x_1-x_2)(s),$$

which implies the conclusion of Remark.

Finally we state an example of $a(t, x)$ which satisfies Condition A.

EXAMPLE. Let $f(t)$ be a continuous function defined on $[0, \infty)$. For $t>0$ and $c\in R$, $n(t, c)$ denotes the number of the connected components of $\{s\in(0, t); f(s)<c\}$. Define

$$a(t, x)=\begin{cases} 2 & \text{for} \quad x\leqq f(t) \\ 1 & \text{for} \quad x>f(t). \end{cases}$$

If $\sup_{c\in[-N,N]} n(t, c)$ is finite for $t>0$ and $N>0$, then $a(t, x)$ satisfies Condition A. But this example does not satisfy those sufficient conditions in the preceding papers [1], [3], [5].

3. Proof of comparison theorem

Let W_x be the space of all continuous functions w defined on $[0, \infty)$ with values in R such that $w(0)=x\in R$. $\mathcal{B}_t(W_x)$ denotes the σ-field generated by $w(s)$ $0\leqq s\leqq t$ and P^W denotes the Wiener measure on W_0. Let $\overline{\mathcal{B}}_t(W_0)$ be the completion of $\mathcal{B}_t(W_0)$ with respect to P^W.

Proof of Theorem 2. Fix a initial value $x_0\in R$. If the pathwise uniqueness holds for the stochastic differential equation (1), then there exists a unique function $F(w)$ defined on W_0 with values in W_{x_0} such that

(i) $F(w)$ is $\overline{\mathcal{B}}_t(W_0)/\mathcal{B}_t(W_{x_0})$-measurable for each $t\geqq 0$,

(ii) any solution $(x(t), B(t))$ of (1) with $x(0)=x_0$ can be represented in the form $x(\cdot)=F(B(\cdot))$ a.s. (cf. [1]).

Let $F_1(w)$ and $F_2(w)$ be the above functions for the stochastic differential equations (2) and (3) respectively. It is sufficient to prove that $F_1(w)$ *5) $\leqq F_2(w)$ a.s. (P^W).

Set $a^k=\bar{a}*\rho_{1/k}$ and $b_i^k=\bar{b}_i*\rho_{1/k}$ $(i=1, 2)$, where ρ_δ is the mollifier defined by (4). Let $(\Omega, \mathcal{F}, P)$ be a probability space with a reference family $(\mathcal{F}_t)$ such that there exists a one-dimensional $(\mathcal{F}_t)$-Brownian motion $B(t)$ with $B(0)=0$. Obviously there exist solutions $(x_k(t), B(t))$ and $(y_k(t), B(t))$ defined on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ such that for $k=1, 2, \cdots$

$$x_k(t)=x_0+\int_0^t a^k(s, x_k(s))dB(s)+\int_0^t b_1^k(s, x_k(s))ds$$

and

5) For $w_1, w_2\in W$, $w_1\leqq w_2$ means that $w_1(t)\leqq w_2(t)$ for each $t\geqq 0$.

$$y_k(t) = x_0 + \int_0^t a^k(s, y_k(s))dB(s) + \int_0^t b_2^k(s, y_k(s))ds\,.$$

Since the family of the laws P^{Z_k} of $Z_k(t)=(x_k(t), y_k(t), B(t))$ $(k=1, 2, \cdots)$ is tight, there exist a subsequence (k_n) and a sequence of stochastic process $(\bar{x}_{k_n}(t), \bar{y}_{k_n}(t), \bar{B}_{k_n}(t))$ defined on a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ satisfying the following conditions;

(i) for each k_n the law of $(\bar{x}_{k_n}(t), \bar{y}_{k_n}(t), \bar{B}_{k_n}(t))$ is $P^{Z_{k_n}}$,

(ii) there exists a stochastic process $(\bar{x}(t), \bar{y}(t), \bar{B}(t))$ defined on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ such that $(\bar{x}_{k_n}(t), \bar{y}_{k_n}(t), \bar{B}_{k_n}(t))$ converges to $(\bar{x}(t), \bar{y}(t), \bar{B}(t))$ uniformly on each compact interval a.s.

Since $b_1^k(t, x) \leqq b_2^k(t, x)$, it holds that $\bar{x}_k(t) \leqq \bar{y}_k(t)$ a.s. for $t \geqq 0$ and $k=k_1, k_2, \cdots$ (cf. [1]). Noting that $(\bar{x}(t), \bar{B}(t))$ and $(\bar{y}(t), \bar{B}(t))$ are solutions of (2) and (3) respectively, we have $F_1(\bar{B}(\cdot))=\bar{x}(\cdot) \leqq \bar{y}(\cdot)=F_2(\bar{B}(\cdot))$ a.s. $(\bar{P})$. Therefore we conclude $F_1(w) \leqq F_2(w)$ a.s. (P^W). The proof is completed.

The above method can be applicable for the following general case.

Remark. Let $a(t, x)$ be a uniformly positive bounded Borel function on $[0, \infty) \times R$. Let $b_1(t, x)$ and $b_2(t, x)$ be bounded Borel functions such that $b_1(t, x) \leqq b_2(t, x)$ for $(t, x) \in [0, \infty) \times R$ a.e. If the pathwise uniqueness holds for the equations (2) and (3), then the conclusion of Theorem 2 holds.

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Department of Mathematics
Osaka University
Toyonaka Osaka 560, Japan