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ON A GENERALIZATION OF SEMIPERFECT MODULES

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In this paper we shall generalize the notion of semiperfect modules in terms of preradicals, and show that almost all properties of semiperfect modules are preserved under this generalization. In particular we can derive immediately new characterizations of semiperfect rings and modules (Corollaries 1.4 and 1.8 below). Namely, when one deals with perfect or semiperfect rings, one can drop the smallness assumption from the definition of a projective cover.

Throughout this paper R will always denote a ring with identity and all modules will be assumed to be unital right R-modules, unless otherwise specified. For any module M we shall denote its Jacobson radical by $J(M)$. A submodule N of M is said to be *small* in M if $T+N=M$ implies $T=M$ for any submodule T of M. Dually, N is said to be *large* in M if $T\cap N=0$ implies $T=0$. By a *preradical* we shall mean a subfunctor of the identity functor on the category of modules. We refer to [5] for details concerning preradicals. Also we shall use freely the definitions and results of Bass [1] and Mares [2] in what follows.

1. A generalization of semiperfect modules

We start with some definitions: Let M be any module and ρ any preradical on modules. Then we shall say that an epimorphism $P\xrightarrow{\pi}M\rightarrow 0$ is a *ρ-semicover* of M if P is a projective module and if $\operatorname{Ker}\pi\subset\rho(P)$. A projective module will be said to be *ρ-semiperfect* (resp. *ρ-perfect*) if any factor module of it (resp. of any direct sum of its copies) has a ρ-semicover. If a ring R is ρ-semiperfect as a right module, then we call R a *ρ-semiperfect ring*. Similarly we define a *ρ-perfect ring*.

Clearly a projective module is ρ-perfect if and only if any direct sum of its copies is ρ-semiperfect. As we see later (Theorem 1.3 below), it turns out that ρ-semiperfect and semiperfect modules are closely related to each other.

Basic facts about semicovers which will be needed later are summarized in the following lemmas. Before proving those we note the useful fact that $\rho(P)=P\rho(R)$ for any preradical ρ and any projective module P.

Lemma 1.1. *Let ρ be any preradical on modules.*

(1) *If $\{P_i \xrightarrow{\pi_i} M_i \to 0\}_{i \in I}$ is any family of ρ-semicovers, then $\oplus_i P_i \xrightarrow{\oplus_i \pi_i} \oplus_i M_i \to 0$ is also a ρ-semicover.*

(2) *Let M be a right module over the factor ring $R/\rho(R)$. Then M has a ρ-semicover as R-module if and only if it is isomorphic to $P/\rho(P)$ for some R-projective module P. In that case M is projective as $R/\rho(R)$-module.*

(3) *If ρ is a radical (i.e. $\rho(L/\rho(L))=0$ for any module L) and M has a ρ-semicover, then $\rho(M)=M\rho(R)$.*

Proof. (1) This is immediate since preradicals commute with direct sums.

(2) The "if" part is trivial. Since $M\rho(R)=0$ by hypothesis, the "only if" part follows trivially from the remark before the lemma. The fact that $P/\rho(P)$ is $R/\rho(R)$-projective for any R-projective module P implies the last statement.

(3) This follows from [5, Lemma 1.2] and the above remark again.

The following definition is the only condition that we will impose on our preradicals in the sequel: A preradical ρ is said to be *normal* if $\rho(P) \neq P$ for every nonzero projective module P.

For example, any preradical ρ satisfying $\rho(R) \subset J(R)$ is normal by [1, Proposition 2.7]. Further observations on normality will be done in the next section. For the moment we need the following lemma:

Lemma 1.2. *Let ρ be a normal preradical on modules.*

(1) *Let M be a module and $P \xrightarrow{\pi} M \to 0$ a ρ-semicover. Suppose that M has also a projective cover $Q \xrightarrow{\xi} M \to 0$. Then there exists an isomorphism $f: P \to Q$ such that $\pi = \xi f$.*

(2) *If a module M has a ρ-semicover and $M\rho(R)=M$, then $M=0$.*

Proof. (1) The proof is similar to that of [2, Corollary 3.4]. (Cf. the proof of the "if" part of Theorem 1.3 below.)

(2) This is immediate since $\rho(P)=P\rho(R)$ for any projective module P.

We are now in a position to characterize ρ-semiperfect modules:

Theorem 1.3. *Let ρ be a normal preradical and P any projective module. Then P is ρ-semiperfect (resp. ρ-perfect) if and only if P is semiperfect (resp. perfect) and $\rho(P)=J(P)$.*

Proof. Since the perfect case follows immediately from the semiperfect case, we need only to prove the latter case.

"IF" part: Assume that P is semiperfect and $\rho(P)=J(P)$. Let N be any factor module of P and $Q \xrightarrow{\pi} N \to 0$ a projective cover. Then by [1. Lemma 2.3] we can write $P = Q \oplus P'$ for some P', identifying Q with a direct summand of P. Hence $\rho(P) = \rho(Q) \oplus \rho(P')$ and so $\rho(Q) = J(Q)$ by assumption. Thus Q is also a ρ-semicover of N since $\mathrm{Ker}\, \pi \subset J(Q)$. (Note: We have not yet used the normality of ρ.)

"ONLY IF" part: Assume that P is ρ-semiperfect. First of all we show the smallness of $\rho(P)$. Let $P = \rho(P) + T$ for some T and put $U = P/T$. Since then $U\rho(R) = U$, it follows from (1.2) (2) that $U = 0$. Hence $\rho(P)$ is small in P. Next we prove the complete reducibility of $P/\rho(P)$. To prove this, it will suffice to show that, for any submodule $N \supset \rho(P)$, there exists a submodule C of P such that $N + C = P$ and $N \cap C \subset \rho(P)$ (cf. the proof of [2, Theorem 3.5]). Now let $N \supset \rho(P)$ and let $Q \xrightarrow{\pi} P/N \to 0$ be a ρ-semicover. Then the projectivity of Q gives a homomorphism f making the following diagram commutative:

$$\begin{array}{ccccc} & & Q & & \\ & \overset{f}{\swarrow} & \downarrow \pi & & \\ P & \xrightarrow{\text{nat.}} & P/N & \longrightarrow & 0\,. \end{array}$$

Putting $C = \mathrm{Im}\, f$, it is readily seen that C satisfies the above two conditions, which shows the complete reducibility of $P/\rho(P)$. Finally we show that every direct decomposition of $P/\rho(P)$ can be lifted to a direct decomposition of P. Let $P/\rho(P) = \oplus_i L_i$ and $Q_i \to L_i \to 0$ a ρ-semicover for each i (Q_i exists by assumption). Then $\oplus_i Q_i \to P/\rho(P) \to 0$ is a ρ-semicover by (1.1) (1). Since the natural epimorphism $P \to P/\rho(P) \to 0$ is a projective cover, we get an isomorphism $\oplus_i Q_i \to P$ by (1.2) (1), which yields a desired direct decomposition of P. Now we can easily deduce, from these results, that P is semiperfect and $\rho(P) = J(P)$. In fact, we first have $\rho(P) = J(P)$ since $\rho(P) \subset J(P)$ and $P/\rho(P)$ is completely reducible. Then it follows from the above results and [2, Theorem 5.1] that P is semiperfect. This completes the proof of the theorem.

REMARK. We note that the requirement of normality on preradicals is needed in this theorem even if ρ is an exact radical (e.g. ρ = the identity functor).

In particular, since the Jacobson radical "J" is normal [1, Proposition 2.7], we obtain the following:

Corollary 1.4. *A projective module P is semiperfect (resp. perfect) if and only if P is J-semiperfect (resp. J-perfect).*

If a ring R is a ρ-semiperfect ring for a normal preradical ρ, then we have $\rho(R) = J(R)$ by (1.3) and hence $\rho(P) = J(P)$ for any projective module P. But this need not guarantee that $\rho = J$ even if R is ρ-perfect, as Example 2 at the end of this section will show. It is thus natural to ask when this equality holds. The next result presents an answer to this question:

Theorem 1.5. *Let ρ be a normal "radical". Then R is ρ-perfect if and only if $\rho=J$ and R is right perfect, where "J" denotes the Jacobson radical.*

Proof. By virtue of (1.3) it remains only to show that if R is ρ-perfect then $\rho(M)=J(M)$ for any module M. Indeed, R is then right perfect and $\rho(R)=J(R)$ by (1.3), so in particular every module M has both a ρ-semicover and a projective cover. But any projective cover becomes a J-semicover; hence by (1.1) (3) we have $\rho(M)=J(M)$, as desired.

Now we shall next give further characterizations of ρ-semiperfect rings. Before proceeding to prove the theorem we shall need a lemma due to Sandomierski [3, Lemma 3]. Its proof is the same as the original one and so will be omitted:

Lemma 1.6. *R is a semisimple Artinian if and only if every irreducible module is projective.*

Theorem 1.7. *Let ρ be a normal preradical. Then the following statements are equivalent:*

(1) *R is a ρ-semiperfect ring.*

(2) *Every irreducible module has a ρ-semicover.*

(3) *Every completely reducible module has a ρ-semicover.*

(4) *Every right $R/\rho(R)$-module has a ρ-semicover as an R-module and $\rho(R)$ is small in R as a right ideal.*

Proof. (1)$\Rightarrow$(4): Assume (1) holds. Then by (1.3), $\rho(R)$ is small in R as a right ideal. Let M be any right $R/\rho(R)$-module. Since $R/\rho(R)$ is semisimple Artinian by (1.3) again, we may write $M=\oplus_\alpha I_\alpha$ for suitable right $R/\rho(R)$-(hence R-) irreducible modules I_α. But each I_α has a ρ-semicover P_α by assumption, so M has a ρ-semicover $\oplus_\alpha P_\alpha$ by (1.1) (1).

(4)$\Rightarrow$(3): This follows immediately from the fact that a right ideal is small in R if and only if it annihilates all irreducible (hence all completely reducible) modules.

(3)$\Rightarrow$(2): Trivial.

(2)$\Rightarrow$(1): We first prove that $\rho(R)$ is small in R as a right ideal. In fact, let I be any irreducible module. Since I has a ρ-semicover, it then follows from (1.2) (2) that $I\rho(R)=0$. Hence $\rho(R)$ is small in R by the above-mentioned fact. Next we prove the complete reducibility of $R/\rho(R)$. Let I be any right $R/\rho(R)$-irreducible module. Then since we may regard I as an R-irreducible module, it has a ρ-semicover and hence is projective as $R/\rho(R)$-module by (1.1) (2). Hence (1.6) shows the complete reducibility of $R/\rho(R)$. Since then any direct summand of $R/\rho(R)$ has a ρ-semicover by assumption and by (1.1) (1), it will follow from the same argument as in the proof of (1.3) that R is ρ-semiperfect. This completes the proof of the theorem.

Since $J(R)$ is always small in R, we can readily deduce the following characterizations of semiperfect rings, from this theorem and (1.4). In particular, Statement (2) below is a generalization of a result given by Sandomierski [3, Theorem 4] and Mueller [4, p. 465], because projective covers become J-semicovers.

Corollary 1.8. *The following statements are equivalent for any ring R:*
(1) *R is semiperfect.*
(2) *Every irreducible module has a J-semicover.*
(3) *Every completely reducible module has a J-semicover.*
(4) *Every right $R/J(R)$-module has a J-semicover as an R-module.*

REMARK. Comparing (1.8) (3) with [3, Theorem 5] will indirectly show the existence of a module having a J-semicover but no projective covers. (Cf. Example 4 below.)

Now we conclude this section with several examples:

EXAMPLE 1. The *singular preradical* "Z" (see [5, 7] for definition) is normal for any ring: This is an immediate consequence of the next general result.

Lemma 1.9. *Let P be any projective module and L any submodule. Then the following conditions are equivalent:*
(1) *P/L is singular (i.e. $Z(P/L)=P/L$) as an R-module.*
(2) *L is large in P.*

Proof. Since (2)$\Rightarrow$(1) always holds, we need only to prove (1)$\Rightarrow$(2). First we will prove this for free modules, and in this case a proof is implicit in the proof of [7, Proposition 1.20 (b)]. But, for completeness, we reproduce it here. Let F be a free module with a basis $(x_i)_{i\in I}$ and N any submodule for which F/N is singular. Then for each i there exists a large right ideal A_i such that $x_iA_i\subset N$. Since N contains $\oplus_i x_iA_i$ and the latter is large in F, it follows that N is large in F. Now, let P be a projective module and suppose that P/L is singular for a submodule L of P. Let $F=P\oplus Q$, F free. Then, by the above, $L\oplus Q$ is large in F and hence L is large in P, as was to be shown.

EXAMPLE 2. An example of a normal "left exact" preradical ρ which does not coincide with the Jacobson radical over a ρ-perfect ring R: Let R be any quasi-Frobenius ring which is not semisimple Artinian, and let $\rho=Z$ (the singular preradical). Then we have $\rho(R)=J(R)$ by the right self-injectivity of R, so R is ρ-perfect by (1.3). But ρ is clearly different from the Jacobson radical since R is not semisimple Artinian. (Note: we may assume further, in the above, that R is also a commutative local ring. E.g., take $R=\boldsymbol{Z}/p^n\boldsymbol{Z}$ where $\boldsymbol{Z}$ is the ring of integers, p a prime and $n>1$.)

Exampl 3. The smallness assumption in (1.7) (4) cannot be dropped in general even when ρ is a normal "radical": Let $R=\boldsymbol{Z}$ (the integers) and p any prime number. Set $\rho(M)=Mp$ for every module M. Then ρ is a radical which is normal by (2.2) or (2.3) below. On the other hand, since $R/\rho(R)$ is a field, each $R/\rho(R)$-module is a direct sum of copies of $R/\rho(R)$. Hence it has a ρ-semicover as an R-module by (1.1) (1), since $R/\rho(R)$ has the ρ-semicover R as an R-module. However, R is not ρ-semiperfect because $\rho(R)$ is not small in R.

Example 4. An example of a module which has a ρ-semicover for a normal radical ρ, but has no projective covers: Let ρ be any normal radical for which $\rho(R)$ is not right T-nilpotent (e.g. R=the localization of $\boldsymbol{Z}$ at any prime number and ρ=the Jacobson radical). Let F be a free module with countably infinite rank. Put $M=F/\rho(F)$. Then M has a ρ-semicover, namely F itself, but has no projective covers. For otherwise $\rho(F)$ would be small in F by (1.2) (1), which would imply by the same argument as in [3, proof of Theorem 5] that $\rho(R)$ is right T-nilpotent; a contradiction.

2. Criteria for normality

In this section we will examine the normality of preradicals. First we have the following simple remark:

Proposition 2.1. *Let R be any ring and ρ any preradical on right R-modules. If ρ is normal, then the ideal $\rho(R)$ contains no nonzero idempotent elements.*

Proof. This is clear since we have $\rho(eR)=eR$ for any idempotent element e in $\rho(R)$.

It is readily seen by examples that the converse is not true in general. There are, however, two special but important cases where this converse does work (Proposition 2.2 below). To show this, we recall two definitions: First, the *trace ideal* of a module M, denoted $\tau(M)$, is defined to be $\Sigma\ \mathrm{Im}\, f$, where the sum is taken over all $f\in \mathrm{Hom}_R(M, R)$. It is well known that for a projective module P, $\tau(P)$ is an idempotent two-sided ideal of R and $P\tau(P)=P$. Next, a ring R is called an *I-ring* if any nonnil right ideal contains a nonzero idempotent element (see [6] for details). Note that any right or left perfect ring is an I-ring.

Proposition 2.2. *The converse to* (2.1) *holds if R is either* (i) *a commutative Noetherian ring or* (ii) *an I-ring.*

Proof. (i) Let R be a commutative Noetherian ring and ρ any preradical on R-modules such that $\rho(R)$ contains no nonzero idempotents. Suppose $\rho(P)=P$ for a projective module P. It then follows clearly that $\rho(R)\supset\tau(P)$. But $\tau(P)$ is a finitely generated (idempotent) ideal by assumption, so we easily see that it is generated by an idempotent element. Hence, by the assumption of $\rho(R)$, we have $\tau(P)=0$ and so $P=0$, which shows the normality of ρ.

(ii) This follows immediately from the fact that in any I-ring R a right ideal T contains no nonzero idempotents if and only if $T\subset J(R)$.

Next we shall characterize those rings over which every preradical different from the identity functor is always normal. First, following the usual terminology, we say that a module M is a *generator* for the category of modules if every module is an epimorphic image of a direct sum of copies of M. As is well known, M is a generator if and only if $\tau(M)=R$.

Proposition 2.3. *The following conditions are equivalent for any ring R:*
(1) *Any preradical different from the identity functor is always normal.*
(2) *Any nonzero projective module is a generator.*

Proof. (1)$\Rightarrow$(2): For any nonzero projective module P, define a (non-normal) radical ρ such that $\rho(M)=M\tau(P)$ for all M, and use the remark above.

(2)$\Rightarrow$(1): This follows immediately from the fact that if M is a generator and ρ a preradical, then $\rho=$the identity functor if and only if $\rho(M)=M$.

REMARK. In Condition (1) above, the word "preradical" can be replaced by the word "idempotent preradical". In fact, for any preradical ρ, there exists an idempotent preradical $\hat{\rho}$ such that $\rho(M)=M$ if and only if $\hat{\rho}(M)=M$ for any module M. (See [5, Proposition 1.1].)

As an important example of such rings, we get the following by (2.2) (i):

EXAMPLE. Every indecomposable commutative Noetherian ring satisfies (2.3) (1). (Here, "*indecomposable*" means "contains no nontrivial idempotents".)

Finally, we shall examine conditions for an ideal to be of the form $\rho(R)$ for some normal preradical ρ. For this purpose, we consider the following property for a module M:

(*) "$\mathrm{Hom}_R(P, M)\neq 0$ *for any nonzero projective module P*."[1]

For example, let $C=\oplus_\alpha I_\alpha$ where $\{I_\alpha\}$ is the family of all distinct isomorphism types of irreducible modules. Then C has this property since any nonzero projective module has a maximal submodule [1, Remark, p. 474]. On the other hand, it is well known that $J(R)=\mathrm{Ann}_R(C)$, where the notation "Ann_R" denotes the annihilator. The next proposition will generalize this situation:

1) (*) is equivalent to $M\tau(P)\neq 0$ for any nonzero projective module P. See Sandomierski: Proc. Amer. Math. Soc. **31** (1972), 27–31.

Proposition 2.4. *Let $\mathfrak{a}$ be any two-sided ideal of any ring R. Then the following conditions are equivalent:*

(1) *$\mathfrak{a}=\rho(R)$ for some normal preradical ρ.*

(2) *$\mathfrak{a}=\mathrm{Ann}_R(M)$ for some module M with property (*).*

(3) *$R/\mathfrak{a}$ has property (*) as a right R-module.*

Proof. (1)$\Rightarrow$(3): Assume (1) holds. Let P be any nonzero projective module and let $P\oplus Q=F$, F free. Write $F=\oplus_\alpha R_\alpha$, where $R_\alpha=R_R$ for each α. Since ρ is normal, $P/\rho(P)\neq 0$ and hence there exists a projection of $F/\rho(F)$ onto some $R_\alpha/\rho(R_\alpha)$ $(=R/\mathfrak{a})$ which is not zero on $P/\rho(P)$. Thus, composing this projection with the natural epimorphism $P\to P/\rho(P)$, we have $\mathrm{Hom}_R(P, R/\mathfrak{a})\neq 0$

(3)$\Rightarrow$(2): Take $M=R/\mathfrak{a}$.

(2)$\Rightarrow$(1): Assume (2) holds. For any module L, set $\rho_M(L)=\bigcap_f \mathrm{Ker} f$, where f runs through all elements of $\mathrm{Hom}_R(L, M)$. Then $\rho_M(R)=\mathrm{Ann}_R(M)$ and ρ_M is a normal radical since M satisfies (*). This completes the proof of the proposition.

Corollary 2.5. *The following conditions are equivalent for any ring R:*

(1) *$\rho(R)=0$ for any normal preradical ρ.*

(2) *Any module which has property (*) is faithful.*

EXAMPLE. Every I-ring R with $J(R)=0$ (in particular, every von Neumann regular ring and every primitive ring with a minimal one-sided ideal) satisfies (2.5) (1).

References

[1] H. Bass: *Finitistic dimension and a homological generalization of semi-primary rings,* Trans. Amer. Math. Soc. **95** (1960), 466–488.

[2] E.A. Mares: *Semi-perfect modules,* Math. Z. **82** (1963), 347–360.

[3] F.L. Sandomierski: *On semi-perfect and perfect rings,* Proc. Amer. Math. Soc. **21** (1969), 205–207.

[4] B.J. Mueller: *On semi-perfect rings,* Illinois J. Math. **14** (1970), 464–467.

[5] B. Stenström: Rings and modules of quotients, Springer Lecture Notes 237, Berlin, 1971.

[6] N. Jacobson: Structure of rings, Amer. Math. Soc. Colloquium Publ., vol. 37, 1964.

[7] K.R. Goodearl: Ring theory, Marcel Dekker, New York, 1976.

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