ON M-RINGS AND GENERAL ZPI-RINGS

Dedicated to Professor Kentaro Murata on his 60th birthday

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In the preceding paper [10], we have proved that a left Noetherian M-ring is a so called “general ZPI-ring” in the commutative case. Also we know that in an M-ring the multiplication of prime ideals is commutative [8]. In the present paper we define general ZPI-rings in section 1 and we study general properties of them, and as an important example of such rings we can give a left Noetherian semi-prime Asano left order. In section 2 we research the condition for a left Noetherian general ZPI-ring to be an M-ring, using minimal prime divisors of an ideal. The notation “<” means a proper inclusion as the preceding papers [8], [9], [10].

1. M-rings and general ZPI-rings

DEFINITION. If the multiplication of any two prime ideals of a ring \( R \) is commutative, and any ideal of \( R \) can be written as a produkt of powers of prime (considering \( R \) as a prime ideal) ideals of \( R \), then we call \( R \) a general ZPI-ring. Therefore the multiplication of ideals is commutative.

In the commutative case a general ZPI-ring is necessarily Noetherian no matter whether the ring has an identity or not. But in our case the general ZPI-ring is not necessarily Noetherian as the example in [9] shows.

Proposition 1. Let \( R \) be a left Noetherian general ZPI-ring, let \( P \) be any prime ideal of \( R \), and let \( q \) be maximal in the set of prime ideals such that \( q<P \). Then for any ideal \( a \) with \( q<a<P \), there is an ideal \( b \) such that \( a=Pb=bP \).

Proof. Let \( a=p_1\cdots p_r<P \), since \( R \) is a general ZPI-ring. Then \( p_i\subseteq P \) for some \( p_i \). Since \( q<a\subseteq p_i \), \( q<p_i\subseteq P \), so \( p_i=P \). Therefore \( a=PP_1\cdots P_{i-1}P_{i+1}\cdots P_r=bP \), where \( b=p_1\cdots p_{i-1}P_{i+1}\cdots p_r \).

As in the commutative case we have

Proposition 2. Let \( R \) be be a left Noetherian general ZPI-ring, and let \( P \) be a maximal ideal of \( R \). Then there are no ideals between \( P \) and \( P^n \) (including the case that \( P=P^n \)), more generally for any positive integer \( n \), the only ideals
between \( P \) and \( P^a \) are \( P, P^2, \ldots, P^n \) (including the case that \( P^i = P^{i+1} \) for some \( i \), \( 1 \leq i < n \)).

**Remark.** Let \( R \) be as above. If every proper ideal \( \alpha \) of \( R \) can be written as a product of minimal prime divisors of \( \alpha \), then for any proper prime ideal \( \mathfrak{p} \) of \( R \) and for any positive integer \( n \), the only ideals between \( \mathfrak{p} \) and \( \mathfrak{p}^n \) are \( \mathfrak{p}, \mathfrak{p}^2, \ldots, \mathfrak{p}^n \).

**Proposition 3.** Let \( R \) be a left Noetherian general ZPI-ring, and let \( \mathfrak{m} = \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_r \} \) be the set of minimal prime ideals of \( R \). Then for any subset \( \{ \mathfrak{p}_{i_1}, \ldots, \mathfrak{p}_{i_k} \} \) of \( \mathfrak{m} \), \( \mathfrak{p}_{i_1} \cap \cdots \cap \mathfrak{p}_{i_k} = \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_k} \). Especially for the prime radical \( \mathfrak{N} \) of \( R \), \( \mathfrak{N}_{i_1} = \mathfrak{p}_{i_1} \cap \cdots \cap \mathfrak{p}_{i_k} = \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_k} \).

**Proof.** Since \( R \) is a general ZPI-ring, \( \mathfrak{p}_{i_1} \cap \cdots \cap \mathfrak{p}_{i_k} = \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_k} \) for some prime ideals \( \mathfrak{p}_{i_1}, \ldots, \mathfrak{p}_{i_k} \) of \( R \). Then for any \( \mathfrak{p}_j \), \( 1 \leq j \leq k \) we have \( \mathfrak{p}_j = 0 \pmod{\mathfrak{p}_i} \) for some \( \mathfrak{p}_i \), and so \( \mathfrak{p}_j = \mathfrak{p}_i \), therefore \( \mathfrak{p}_{i_1} \cap \cdots \cap \mathfrak{p}_{i_k} = \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_k} \). Now \( \mathfrak{p}_{i_1} \cap \cdots \cap \mathfrak{p}_{i_k} = \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_k} \), hence \( \mathfrak{p}_{i_1} \cap \cdots \cap \mathfrak{p}_{i_k} = \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_k} \).

**Lemma 4.** Let \( R \) be a left Noetherian semi-prime general ZPI-ring, and let \( \mathfrak{m} = \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_r \} \) be the set of minimal prime ideals of \( R \). Then for any \( 1 \leq i < r \) and any positive integers \( m_1, \ldots, m_r \), \( \mathfrak{p}_1^{m_1} \cdots \mathfrak{p}_r^{m_r} = 0 \).

**Theorem 1.** Let \( R \) be a left Noetherian semi-prime general ZPI-ring, and let \( \mathfrak{m} = \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_r \} \) be the set of minimal prime ideals of \( R \). If a proper ideal \( \alpha \) of \( R \) has the form \( \alpha = \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_s} \mathfrak{P}_{1} \cdots \mathfrak{P}_t \), where \( \mathfrak{p}_{i_j} \in \mathfrak{m} \) for \( i_1, \ldots, s \) and \( \mathfrak{P}_j \in \mathfrak{m} \) for \( j=1, \ldots, t \), then \( \mathfrak{P}_1 \cdots \mathfrak{P}_t \subseteq R \), i.e. essential as a left \( R \)-module, and the set \( \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_r \} \) is uniquely determined by \( \alpha \).

**Proof.** Let \( \mathfrak{P} \) be a prime ideal of \( R \). By proposition 2.11 [5] and Lemma 4, \( \mathfrak{P} \) is not essential as a left \( R \)-module if and only if \( \mathfrak{P} \in \mathfrak{m} \). Hence \( \mathfrak{P}_1 \cdots \mathfrak{P}_t \subseteq R \) as a left \( R \)-module. Let \( \alpha = \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_s} \mathfrak{Q}_1 \cdots \mathfrak{Q}_w \) where \( \mathfrak{p}_{i_j} \in \mathfrak{m} \) for \( 1 \leq j \leq k \), \( \mathfrak{Q}_j \in \mathfrak{m} \) for \( 1 \leq i \leq w \) be another form of \( \alpha \). Assume that two set \( \mathcal{M}_1 = \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_s \} \), \( \mathcal{M}_2 = \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_s \} \) are distinct. If \( \mathcal{M}_1 > \mathcal{M}_2 \), then \( 0 = \alpha \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_s} \mathfrak{Q}_1 \cdots \mathfrak{Q}_w \mathfrak{P}_1 \cdots \mathfrak{P}_t \), and \( \mathfrak{Q}_1 \cdots \mathfrak{Q}_w \) contains some regular element, hence \( 0 = \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_s} \mathfrak{P}_1 \cdots \mathfrak{P}_t \), contradicting Lemma 4. Next we consider the case that \( \mathcal{M}_1 \not\supset \mathcal{M}_2 \) and also \( \mathcal{M}_1 \not\subset \mathcal{M}_2 \). We denote the product of minimal prime ideals belonging to the set \( \mathcal{M}_1 \) by \( [\mathcal{M}_1] \), for example. Then \( 0 = \alpha [\mathfrak{m} - \mathcal{M}_1] \) since \( \alpha = \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_s} \mathfrak{P}_1 \cdots \mathfrak{P}_t \). On the other hand, \( \mathfrak{m} - \mathcal{M}_1 \not\subset \mathcal{M}_2 \) and \( \alpha = \mathfrak{p}_{i_1} \cdots \mathfrak{p}_{i_s} \mathfrak{Q}_1 \cdots \mathfrak{Q}_w \), hence \( 0 = \alpha [\mathfrak{m} - \mathcal{M}_1] \) which is a contradiction. So we have \( \mathcal{M}_1 = \mathcal{M}_2 \).

As a result of Theorem 1 we have

**Proposition 5.** Let \( R \) be as above, and let \( \mathfrak{m} = \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_r \} \). Then
(p^i_1...p^i_r) is a regular^1 ideal of R, where p_1,...,p_r are distinct minimal prime ideal of R, 1\leq i \leq r and \alpha_1,...,\alpha_r are any positive integers.

Proposition 6. Let R be a left Noetherian general ZPI-ring, and let P be a maximal ideal of R such that P^i > P^{i+1} for any positive integer i. Then \bigcap_{n=1}^\infty P^n is a prime ideal of R.

Proof. Set \bigcap_{n=1}^\infty P^n = a. Let A, B be ideals of R such that AB \equiv 0 (mod a) and \exists B \equiv 0 (mod a). Therefore there is a maximal i \geq 0 such that A \subseteq P^i and so A \not\subseteq P^{i+1}. Similarly there is a maximal j \geq 0 such that B \subseteq P^j and so B \not\subseteq P^{j+1}, where P^0 = R. Then P^{i+1} \subseteq (A, P^{i+1}) \subseteq P^i, therefore (A, P^{i+1}) = P^i by Proposition 2, and similarly (B, P^{i+1}) = P^j. Hence P^{i+j} = (A, P^{i+1}) (B, P^{i+1}) \subseteq P^{i+j+1}, thus P^{i+j} = P^{i+j+1} contradicting the assumptions.

Remark. Let R be as above. Let p be any proper prime ideal such that for any positive integer \alpha p^\alpha > p^{\alpha+1}. If every proper ideal of R can be written as a product of minimal prime divisors, then \bigcap_{n=1}^\infty p^n is a prime ideal of R.

Theorem 2. Let R be a Noetherian (left and right) prime ring with an identity. If R satisfies the following
1) R is a general ZPI-ring;
2) every non-zero proper prime ideal of R is maximal;
3) every ideal of R is projective both as a left and as a right R-module,
the R is an M-ring.

Proof. We shall prove the existence of an ideal C with A = BC = CB for ideals A, B such that O < A < B < R. Let A = P^i_1...P^i_r < B = Q^\beta_1...Q^\beta_\alpha where P_1,...,P_\alpha, Q_1,...,Q_\beta are prime ideals of R and e_k > 0 for k = 1,...,\alpha, f_j > 0 for j = 1,...,\beta, so for every \bar{Q}_k there is some P_k with P_k = Q_k for k = 1,...,\beta. Hence A = Q^\beta_1...Q^\beta_\beta P^\beta_{\beta+1}...P^\beta_{\alpha+1} < B = Q^\alpha_1...Q^\alpha_\alpha. Now by Proposition 2.2 [3], each maximal ideal of R is either idempotent or invertible. Let Q_1,...,Q_j be the set of idempotent maximal ideals in the set of maximal ideals Q_1,...,Q_\beta (including the case that \{Q_1,...,Q_j\} is empty). Then A = Q_1...Q_j Q^j_1...Q^j_\beta P^\beta_{\beta+1}...P^\beta_{\alpha+1}...P^\beta_{\alpha+1} < B = Q_1...Q_j Q^\alpha_1...Q^\alpha_\alpha, where Q_1,...,Q_\beta are invertible ideals of R. If e_{j+1} < f_{j+1} for example, multiplying (Q^\alpha_j)^{j+1} on each side, we have Q_1...Q_j Q^{j_1+1}_1...Q^\alpha_\alpha P^\beta_{\beta+1}...P^\beta_{\alpha+1}...P^\beta_{\alpha+1} < Q_1...Q_j Q^{j_1+1}_1...Q^\alpha_\alpha P^\beta_{\beta+1}...P^\beta_{\alpha+1}...P^\beta_{\alpha+1} (mod Q_1,...,Q_j), which is a contradiction. Therefore e_{j+1} \geq f_{j+1}, e_{\beta} \geq f_{\beta}. Thus A = B Q^{j_1+1}_1...Q^\alpha_\alpha P^\beta_{\beta+1}...P^\beta_{\alpha+1}...P^\beta_{\alpha+1}, hence R is an M-ring.

^1) We call an R-ideal a regular ideal.
Remark. If $R$ is a Noetherian semi-prime ring with an identity, then we may replace the condition 2) by the following:

2') the proper prime ideals of $R$ are either comaximal minimal prime ideals or maximal prime ideals of $R$.

The theorem is valid also in this case, because $R = R_1 \oplus \cdots \oplus R_i \oplus \cdots \oplus R_n$ where $R_i = R/p_i$ for every $i$ and $\{p_1, \ldots, p_n\} = \text{min-}\mathcal{P}$, so every $R_i$ is a Noetherian general ZPI-ring satisfying the condition 2).

Theorem 3. Let $R$ be a left Noetherian semi-prime Asano left order. Then $R$ is a general ZPI-ring and also an $M$-ring, and the proper prime ideals of $R$ are either comaximal idempotent minimal prime ideals or maximal prime ideals of $R$. Every proper ideal $a$ of $R$ has the form $a = p_1 \cdots p_i \cdots p_n$ where $p_k \in \text{min-}\mathcal{P}$ for $1 \leq k \leq i$ and $p_1, \ldots, p_n$ are maximal prime ideals of $R$ which are regular.

Proof. Let $Q = Q_1 \oplus \cdots \oplus Q_i \oplus \cdots \oplus Q_n$ be the left quotient ring of $R$ which is semisimple Artinian, where $Q_1, \ldots, Q_n$ are simple Artinian rings. Now we can deduce that $R = R_1 \oplus \cdots \oplus R_i \oplus \cdots \oplus R_n$ where $R_i$ is a left Noetherian Asano left order of $Q_i$ for $1 \leq i \leq n$. Each proper prime ideal of $R$ has either the form $p_1 \cdots p_i \cdots p_n$ or the form $p_i \oplus p_{i+1} \oplus \cdots \oplus p_n$ where $p_{i-1}$ is a maximal prime ideal of $R_i$ for $1 \leq i \leq n$. Every proper ideal $a$ of $R$ has the form $a = a_1 \cdots a_i \cdots a_n$ where $a_i$ is an ideal of $R_i$ for $1 \leq i \leq n$. In order to make the proof concise we assume that $a_i = \cdots = a_{i-1} = 0$ (including the case that $\{a_1, \ldots, a_{i-1}\}$ is empty) and $a_i = p_{i(1)} \cdots p_{i(m)} \cdots a_n = p_{(a_1)} \cdots p_{(a_n)}$. Then $a = p_1 \cdots p_{i-1} p_{i(1)} \cdots p_{i(m)} \cdots p_{n(1)} \cdots p_{n(\lambda)}$ where $p_{i(j)} = R_i \oplus \cdots \oplus R_{i-1} \oplus p_{i(j)} \oplus R_{i+1} \oplus \cdots \oplus R_n$, thus $R$ is a general ZPI-ring. Then it is easy to see that $R$ is an $M$-ring.

By Proposition 6 we have

Corollary 4. Let $R$ be a left Noetherian semi-prime Asano left order and let $P$ be a regular prime ideal of $R$, then $\prod_{n=1} P^n = \mathcal{P}$ is a minimal prime ideal of $R$.

2. Minimal prime divisors of ideals

Let $a$ be a proper ideal of $R$. A minimal prime divisor of $a$ is a prime ideal $p$ with $a \subseteq p$ such that there are no prime ideals $p'$ with $a \subseteq p' \subsetneq p$. We denote the set of minimal prime divisors of $a$ by $\text{min-}\mathcal{P}_a$. The set $\text{min-}\mathcal{P}$ of minimal prime ideals of $R$ is $\text{min-}\mathcal{P}_a$. As a consequence of Theorem 3 [10] and Proposition 1 [8], we have

Proposition 7. Let $R$ be a left Noetherian general ZPI-ring. Moreover if $R$ is an $M$-ring, then
For any prime ideal \( p, q \) with \( p < q, p = q = q \).

Let \( \alpha \) be any proper ideal of \( R \), and let \( \min-\mathcal{P}_\alpha = \{ p_1, \ldots, p_s \} \). Then \( \alpha = \prod p_i^{f_i} \) for some positive integers \( f_i, \ldots, f_s \).

**Remark.** Let \( R \) be a left Noetherian general ZPI-ring. Then i) of the above condition (*) is equivalent to the following:

i') For any prime ideal \( p \) and any ideal \( b \) properly containing \( p, p = b \neq p \).

Next we consider the converse of this apparent proposition.

**Proposition 8.** Let \( R \) be a left Noetherian general ZPI-ring which satisfies the condition (*) in Proposition 7 and let \( \alpha \) be a proper ideal of \( R \). Then for any minimal prime divisor \( p \) of \( \alpha \), either \( p^i = p^{i+1} \) for some positive integer \( i \) or else there is some positive integer \( j \) such that \( p^j \not\supset \alpha \).

**Proof.** We assume that for any positive integer \( i \), \( p^i > p^{i+1} \), and we shall show that \( p^j \not\supset \alpha \) for some positive integer \( j \). If \( p^i > p^{i+1} \) for any positive integer \( i \) and moreover \( p^k \supset \alpha \) for any positive integer \( k \), then \( \alpha \subseteq \prod_{k=1}^n p^k = \wp \) where \( \wp \) is a prime ideal by the remark of Proposition 6, a contradiction.

**Proposition 9.** Let \( R \) be a left Noetherian general ZPI-ring which satisfies the condition (*), let \( \alpha \) be a proper ideal of \( R \), and let \( \min-\mathcal{P}_\alpha = \{ p_1, \ldots, p_s \} \). Then for any \( i \neq j \) and any positive integer \( e_i, e_j \), \( (\psi^{(i)}, (\psi_i)_{i,j}) \) is an idempotent ideal of \( R \).

**Proof.** First we prove that \( \psi_i((\psi^{(i)}, (\psi_i)_{i,j}) = \psi_i \). Since \( p_i \equiv 0 \pmod{p^{(i)}} \), \( p_i \equiv (\psi_i)_{i,j} = \psi_i \). Similarly \( (\psi_i)_{i,j} = \psi_i \). Now we know that \( p_i \equiv 0 \pmod{P_i} \) for every \( P_i \), \( 1 \leq k \leq s \). If \( p_i = P_i \) for some \( p_i \), then \( (\psi_i, (\psi_i)_{i,j}) = P_i \). Hence \( p_i \equiv 0 \pmod{p_i} \), a contradiction. Therefore \( p_i < P_i \) for \( 1 \leq k \leq s \), hence \( (\psi_i, (\psi_i)_{i,j}) = P_i \). Then \( (\psi_i, (\psi_i)_{i,j}) = (\psi_i, \psi_i) \).

**Lemma.** Under the same assumptions as above, for any \( i \neq j \) and any positive integer \( e_i, e_j \), \( (\psi_i \cap \psi_j) = (\psi_i, \psi_j) \).

**Proof.** First we prove that \( (\psi_i \cap \psi_j) = (\psi_i, \psi_j) \). For some positive integer \( \rho \), \( \alpha \subseteq (\psi_i \cap \psi_j) = (\psi_i, \psi_j) \). Let \( (\psi_i, \psi_j) = (Q_1 \cdots Q_t) \), where \( Q_i, \cdots, Q_t \) are minimal prime divisors of \( (\psi_i, \psi_j) \). For every \( Q_i, \alpha \equiv 0 \pmod{Q_i} \) and \( \psi_j \equiv 0 \pmod{Q_i} \), hence \( p_i < Q_i \) and \( p_j < Q_i \). From the above arguments \( (\psi_i, \psi_j) = (\psi_i, \psi_j) \).
\[ \bigcap_{i=1}^{r} p_i = \bigcap_{i=1}^{r} p_i \cap p_i \] by the condition (*). Hence 
\[ \bigcap_{i=1}^{r} (p_i \cap p_i) = \left( \bigcap_{i=1}^{r} p_i \right) \cap \left( \bigcap_{i=1}^{r} p_i \right) \subseteq \left( \bigcap_{i=1}^{r} p_i \right) \cap \left( \bigcap_{i=1}^{r} p_i \right) = \bigcap_{i=1}^{r} p_i. \]
The other inclusion is obvious, so \( \bigcap_{i=1}^{r} p_i = \bigcap_{i=1}^{r} p_i. \)

Now by the induction we have

**Theorem 5.** Let \( R \) be a left Noetherian general ZPI-ring which satisfies the condition (*), let \( \alpha \) be a proper ideal of \( R \), and let \( \min-\mathcal{O}_\alpha = \{p_1, \ldots, p_r\} \). Then for any subset \( \{p_i, \ldots, p_k\} \) of \( \min-\mathcal{O}_\alpha \) and for any positive integers \( i, \ldots, k, p_i \cap \cdots \cap p_k = p_i \cdots p_k. \)

**Theorem 6.** Let \( R \) be a left Noetherian general ZPI-ring which satisfies the condition (*), let \( \alpha \) be a proper ideal of \( R \), and let \( \alpha = \alpha_1 \cdots \alpha_r \) where \( \min-\mathcal{O}_\alpha = \{p_1, \ldots, p_r\} \) and \( x_i > 0 \) for \( 1 \leq i \leq r \). Let \( \{p_1, \ldots, p_r\} \) be the subset of \( \min-\mathcal{O}_\alpha \) every \( p_i \) of which has a maximal index \( \alpha_i \) such that \( p_i \supseteq \alpha \) and so \( p_i \cdots p_i \supseteq \alpha \), and assume that for \( p_{k+1}, \ldots, p_r \), there are no maximal \( \beta_j \) among indices \( \beta_j \) such that \( p_j \supseteq \alpha \) (including the case that one of the sets \( \{1, \ldots, k\}, \{k+1, \ldots, r\} \) is empty). Then \( \alpha \) has the form \( \alpha = p_1 \cdots p_k \cdots p_{k+1} \cdots p_r \), where \( \beta_i \) is any positive integer such that \( x_i \leq \beta_i \leq \alpha_i \) for \( 1 \leq i \leq r \) and \( y_j \) is any positive integer with \( x_j \leq y_j \) for \( k < j \leq r \).

**Proof.** By Theorem 5 \( \alpha = \bigcap_{i=1}^{k} p_i \cap \cdots \cap p_k \supseteq \bigcap_{i=1}^{k} p_i \cap \cdots \cap p_{k+1} \cap \cdots \cap p_r \), since \( x_i \leq \beta_i \leq \alpha_i \) for \( 1 \leq i \leq k \) and \( x_j \leq y_j \) for \( k < j \leq r \). Conversely \( \alpha \supseteq \bigcap_{i=1}^{k} p_i \) for \( 1 \leq i \leq k \) since \( \beta_i \leq \alpha_i \), and also \( \alpha \supseteq \bigcap_{i=1}^{k} p_i \cap \cdots \cap p_{k+1} \cap \cdots \cap p_r \) for any \( y_j \geq x_j, k < j \leq r \); hence \( \alpha \supseteq \bigcap_{i=1}^{k} \cdots \cap \bigcup_{i=1}^{k} p_i \cap \cdots \cap p_{k+1} \cap \cdots \cap p_r \). Thus \( \alpha = \bigcap_{i=1}^{k} \cdots \cap \bigcup_{i=1}^{k} p_i \cap \cdots \cap p_{k+1} \cap \cdots \cap p_r \) by Theorem 5.

The following definition of primary ideal is defined in [2]. Let \( \alpha \) be an ideal of \( R \). If for ideals \( A, B \) \( A B \equiv 0 \) (mod \( \alpha \)) implies \( A \equiv 0 \) (mod \( \alpha \)) or \( B \equiv 0 \) (mod \( \alpha \)) for some positive integer \( \rho \), then \( \alpha \) is called \( r \)-primary. And a 1-primary ideal is defined similarly. A 1- and \( r \)-primary ideal is called a \( \alpha \) primary ideal.

**Theorem 7.** Let \( R \) be as above. Then for every proper prime ideal \( \mathfrak{p} \) of \( R \) \( \mathfrak{p}^e \) is a primary ideal for any positive integer \( e \).

**Proof.** Let \( A B \equiv 0 \) (mod \( \mathfrak{p}^e \)) for ideals \( A, B \). We may assume that \( A \equiv a \) and \( B \equiv a \) where we set \( \mathfrak{p}^e = a \). We set anew \( A = (A, \alpha), B = (B, \alpha) \). Then \( A, B \equiv 0 \) (mod \( \mathfrak{p}^e \)); and \( A \equiv 0 \) (mod \( \mathfrak{p} \)) if and only if \( A \equiv 0 \) (mod \( \mathfrak{p}^e \)), etc. Therefore it is sufficient to prove that for ideals \( A > a, B > a \), if \( A B \equiv 0 \) (mod \( \mathfrak{p}^e \)), then \( A \equiv 0 \) (mod \( \mathfrak{p}^e \)) or \( B \equiv 0 \) (mod \( \mathfrak{p}^e \)) for some positive integer \( n \). Hence we prove that for ideals \( A, B \) such that \( A < A, B, A, B \equiv 0 \) (mod \( \mathfrak{p}^e \)) and for any positive integer \( m \) \( B^m \equiv 0 \) (mod \( \mathfrak{p}^e \)), then \( A \equiv 0 \) (mod \( \mathfrak{p}^e \)). Let \( \min-\mathcal{O}_\mathfrak{p} = \{P_1, \ldots, P_r\} \), and let \( A = P_1 \cdots P_r \) for some positive integers \( \delta_1, \ldots, \delta_r \). Since \( A B \equiv 0 \) (mod \( \mathfrak{p}^e \)), however \( B \equiv 0 \) (mod \( \mathfrak{p} \)), hence \( A \equiv 0 \) (mod \( \mathfrak{p} \)). Therefore \( A < A \subseteq P_i \equiv 0 \) (mod \( \mathfrak{p} \)) for some \( P_i \), hence \( P_i = \mathfrak{p} \) since \( \mathfrak{p} \) is a
minimal prime divisor of $\alpha$; so $A = \mathfrak{p}_1^a \mathfrak{p}_2^b \cdots \mathfrak{p}_t^h$, i.e. $\mathfrak{p}$ is a minimal prime divisor of $A$. Let $\min-\mathfrak{p}_A = \{q_1, \ldots, q_i\}$. Since $a = \mathfrak{p}^t < B = q_1^{\nu_1} \cdots q_i^{\nu_i}$ for some positive integers $\nu_1, \ldots, \nu_i$, $\mathfrak{p} < q_i$ for every $q_i$ and since $\mathfrak{p}$ is a factor of $A$ $AB = A$ by the condition (\#), i.e. $A \equiv 0 \pmod{\mathfrak{p}^t}$.

**Theorem 8.** Let $R$ be a left Noetherian general ZPI-ring which satisfies the condition (\#). Then $R$ is an $M$-ring.

Proof. Let $0 < A < B < R$ be ideals of $R$, let $\min-\mathfrak{p}_A = \{P_1, \ldots, P_s\}$, $\min-\mathfrak{p}_B = \{Q_1, \ldots, Q_t\}$, and let $A = P_1^{\alpha_1} \cdots P_s^{\alpha_s}, B = Q_1^{\beta_1} \cdots Q_t^{\beta_t}$ where $\beta_1, \ldots, \beta_t$ are positive integers and as for $\alpha_1, \ldots, \alpha_s$ by Theorem 6 we can choose them as large as possible. Then for every $Q_i$, there is some $P_j$ such that $P_j \subseteq Q_i$. If $P_j \subset Q_i$ for every $Q_i$ then $A = A B = B A$, so there is nothing to prove. If there are some $Q_i$ such that $P_j \supseteq Q_i$ for $1 \leq i \leq m$ and for every $Q_j$ ($m < j < b$) there are some $P_k$ with $P_k \subset Q_j$. Furthermore, as to $P_1, \ldots, P_m$, let $P_1, \ldots, P_m$ be minimal prime divisors of $A$ which have maximal indices such that $P_j^t \supseteq A$ for $1 \leq j \leq s$, and let $P_{s+1}, \ldots, P_m$ be those which do not have such indices as above. On prime ideals $P_j, 1 \leq j \leq s, A \subseteq P_j^\beta_j$, and $A \subset B \subseteq \mathfrak{q}_j^\alpha_j = P_j^\alpha_j$, so $A \subseteq \mathfrak{q}_j^\alpha_j$, hence $\beta_j \leq \alpha_j$ for $1 \leq j \leq s$ by Theorem 6. On prime ideals $P_{s+1}, \ldots, P_m$ we may assume that $\beta_i \leq \alpha_i$ for $s < i \leq m$, by Theorem 6. Therefore $A = P_1^{\alpha_1 - \beta_1} \cdots P_m^{\alpha_m - \beta_m} P_{s+1}^{\beta_{s+1}} \cdots P_m^{\beta_m}$, $P_s^{\alpha_s} = P_1^{\alpha_1 - \beta_1} \cdots P_m^{\alpha_m - \beta_m} P_{s+1}^{\beta_{s+1}} \cdots P_m^{\beta_m}$ ($Q_{s+1}^{\alpha_{s+1}} \cdots Q_t^{\alpha_t}$) $P_1^{\alpha_1} \cdots P_m^{\alpha_m} = B C$, say. Hence $R$ is an $M$-ring.

We summarize

**Theorem 9.** Let $R$ be a left Noetherian general ZPI-ring. Then $R$ is an $M$-ring if, and only if,

1) For any prime ideals $\mathfrak{p}, \mathfrak{q}$ of $R$ such that $\mathfrak{p} \subset \mathfrak{q}$, $\mathfrak{p} = \mathfrak{q}$, and

2) Any proper ideal $\alpha$ of $R$ can be written as a product of powers of minimal prime divisors of $\alpha$.

References


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