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A SPLITTING PROPERTY OF ORIENTED HOMOTOPY EQUIVALENCE FOR A HYPERELEMENTARY GROUP

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1. Introduction

Let G be a finite group. In this paper a G-space means a complex G-representation space of finite dimension. For a G-space V we denote by $S(V)$ its unit sphere with respect to some G-invariant inner product. After tom Dieck [1] and [2] we call two G-spaces V and W oriented homotopy equivalent if there exists a G-map f: $S(V)\to S(W)$ such that for each subgroup H of G the induced map $f^H:S(V)^H\to S(W)^H$ on the H-fixed point sets has degree one with respect to the coherent orientations which are inherited from the complex structures on V^H and W^H. Let $R(G)$ be the complex G-representation ring, $R_h(G)$ the additive subgroup of $R(G)$ consisting of $x=V-W$ such that V and W are oriented homotopy equivalent, and $R_0(G)$ the additive subgroup of $R(G)$ consisting of $x=V-W$ such that dim $V^H=$dim W^H for all the subgroups H of G. We denote by $j(G)$ the quotient group $R_0(G)/R_h(G)$.

If G has a normal cyclic subgroup A and a Sylow p-subgroup H such that G is the semidirect product of H by A, we call G a hyperelementary group. Especially if G is the direct product of A and H, we call G an elementary group. tom Dieck showed that for an arbitrary finite group G the restriction homomorhpism from $j(G)$ to the direct sum of $j(K)$ is injective, where K runs over the hyperelementary subgroups of G ([1; Proposition 5.1]). Our purpose of this paper is to consider oriented homotopy equivalence for hyperelementary groups and to give a sufficient condition for a hyperelementary group to have a splitting property defined below.

Choose an integer m which is a multiple of the orders of the elements of G, and let $Q(m)$ be the field obtained by adjoining the m-*th* roots of unity to Q, where Q is the field of rational numbers. The Galois group $\Gamma=\Gamma(m)$ of $Q(m)$ over Q acts on $R(G)$ via its action on character value. Actually Γ acts on the set $Irr(G)$ of isomorphism classes of irreducible G-spaces. Let $Z[\Gamma]$ be the integral group ring of Γ, and $I(\Gamma)$ its augmentation ideal. Then we have $R_0(G)=I(\Gamma)R(G)$. We put $R_1(G)=I(\Gamma)R_0(G)$. According to [3] we have $R_1(G)\subset R_h(G)$. Let us say that G has *Property* 1 if $R_1(G)$ coincides with $R_h(G)$.

For example the abelian groups and the p-groups have Property 1, and some hyperelementary groups do not have Property 1 (see [1] and [6]). In section 3 we obtain other groups which have Property 1.

For each orbit $C\in X(G)=Irr(G)/\Gamma$, we let $F(C)$ be the free abelian group on elements of C. Then we have $R(G)=\oplus_{C\in X(G)}F(C)$. Let f_C be the canonical projection from $R(G)$ to $F(C)$. Let us say that G has *Property* 2 (we called this a splitting property) if for each element x of $R_h(G)$ and each element C of $X(G)$ $f_C(x)$ belongs to $R_h(G)$. This property is of our interest. If G has Property 1, then G has Property 2; the converse is not true. It is remarkable that $R_h(G)$ is determined by oriented homotopy equivalence between the irreducible G-spaces if G has Property 2.

Our main results are Theorems 6.11 and 6.12, and the latter indicates the importance of Property 2. Additionally we give a counter example to [1; Proposition 5.2] in section 7.

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2. Preparation

Let $S(G)$ the set of normal subgroups of G. If a G-space V is given we write $V=\oplus_{H\in S(G)}V(H)$, where $V(H)$ collects the faithful irreducible G/H-subspaces (see [2; p. 252]).

Lemma 2.1 ([2]). *If $x=V-W\in R_h(G)$, then for all $H\in S(G)$ we have $x(H)=V(H)-W(H)\in R_h(G)$.*

Let V and W be G-spaces. If f is an $N_G(H)$-map from $S(V)^H$ to $S(W)^H$ and g is an element of G, then there uniquely exists an $N_G(gHg^{-1})$-map h from $S(V)^{gHg^{-1}}$ to $S(W)^{gHg^{-1}}$ such that the following diagram is commutative:

$$\begin{array}{ccc} S(V)^H & \xrightarrow{f} & S(W)^H \\ \downarrow g_* & & \downarrow g_* \\ S(V)^{gHg^{-1}} & \xrightarrow{h} & S(W)^{gHg^{-1}} \end{array}$$

where g_* are the maps canonically given by the actions of g.

Proposition 2.2. *Let V and W be G-spaces. We have $V^H-W^H\in R_h(N_G(H))$ if and only if we have $V^{gHg^{-1}}-W^{gHg^{-1}}\in R_h(N_G(gHg^{-1}))$.*

Proof. This proposition follows from the fact that each g_* of the above diagram preserves the orientation of the sphere.

Let V and W be G-spaces such that $\dim V^H=\dim W^H$ for all subgroups H of G (i.e. $V-W\in R_0(G)$). We put $n=\dim V(=\dim W)$. If g is an element of

G, g has n eigenvalues $a_1(g), \cdots, a_n(g)$ (resp. $b_1(g), \cdots, b_n(g)$) with respect to its action on V (resp. W). We reorder $(a_j(g))$ and $(b_j(g))$ as follows: there is an integer k such that for each $j<k$ we have $a_j(g)=b_j(g)=1$ and for each $j\geqq k$ we have $a_j(g)\neq 1$ and $b_j(g)\neq 1$. We get an algebraic integer $z(g)$ defined by

$$z(g)=\prod_{j=1}^{n}(1-b_j(g))/(1-a_j(g)),$$

where we put $(1-b_j(g))/(1-a_j(g))=1$ for $j<k$. Summing up these algebraic integers $z(g)$ over the elements g of G we have an integer $P=P(G; W-V)$, that is,

$$P(G; W-V)=\sum_{g\in G} z(g).$$

Lemma 2.3 (due to T. Petrie). *Let V and W be G-spaces as above. V and W are oriented homotopy equivalent if and only if the following two conditions* (i) *and* (ii) *are satisfied.*

(i) *For each non-trivial subgroups H of G (i.e. $H\neq\{1\}$), we have $V^H-W^H\in R_h(N_G(H))$.*

(ii) *It holds that $P(G; W-V)\equiv 0 \bmod |G|$.*

Provided (i), then (ii) is equivalent to the condition: $P(G; V-W)\equiv 0 \bmod |G|$.

Let s be a positive integer, and V a G-space of dimension n. We are going to define an element $Q(s; V)$ of $R(G)$. Let $x(1), \cdots, x(n)$ be indeterminates, and $y(i)$ the elementary symmetric polynomial of degree i for each $1\leqq i\leqq n$. We define a polynomial Q of $y(1), \cdots, y(n)$ by

$$Q(y(1), \cdots, y(i), \cdots, y(n))=\prod_{j=1}^{n}(1+x(j)+\cdots+x(j)^{s-1}).$$

We define $Q(s; V)$ by

$$Q(s; V)=Q(V, \cdots, \Lambda^j V, \cdots, \Lambda^n V),$$

where $\Lambda^j V$ is the j-fold exterior power of V. By the usual identification we let $Q(s; V)(g)$ stands for trace $(g; Q(s; V))$. Then it holds that

$$Q(s; V)(g)=\prod_{j=1}^{n}(1+a_j(g)+\cdots+a_j(g)^{s-1}), \tag{2.4}$$

where $a_1(g), \cdots, a_n(g)$ are all the eigenvalues of g on V. Since $Q(s; V)\in R(G)$, we have

$$\sum_{h\in H} Q(s; V)(h)\equiv 0 \bmod |H| \tag{2.5}$$

for each subgroup H of G.

3. A few remasks about Property 1

Let L be a finite abelian group. We denote the integral group ring of L by $Z[L]$, the augmentation ideal of $Z[L]$ by $I(L)$, i.e.

$$I(L) = \{\sum_{x\in L} z(x)x\colon z(x)\in Z, \text{ and } \sum_{x\in L} z(x) = 0\}\ ,$$

where Z is the ring of integers.

Proposition 3.1. *We have the following.*
(i) *For $x, x'\in L$, it holds that $xx'-x\equiv x'-1 \bmod I(L)^2$.*
(ii) *For $x\in L$ and $z\in Z$, it holds that $zx-z1\equiv x^z-1 \bmod I(L)^2$.*
(iii) *$I(L)/I(L)^2$ is isomorphic to L.*

Since the proof is straightforward, we omit it.

Let G be a direct product $H\times K$ as finite group. We denote by ϕ the Euler function, that is, for a positive integer n $\phi(n)$ is the number of the units of $Z_n=Z/(n)$.

Proposition 3.2. *Let V be an irreducible H-space, and W an irreducible K-space. Assume $(\phi(|H|), \dim W)=(\phi(|K|), \dim V)=1$. Then for an element*

$$x = \sum_{\gamma\in\Gamma} z(\gamma)\gamma(V\otimes W)\in R_0(G)\ ,$$

x belongs to $R_1(G)$ if and only if $\mathrm{Res}_H^G x\in R_1(H)$ and $\mathrm{Res}_K^G x\in R_1(K)$, where $z(\gamma)$ are integers.

Proof. The only if part is clear. We are going to prove the if part. Γ acts on the orbits $\Gamma(V\otimes W)$, ΓV and ΓW which are subsets of $Irr(G)$, $Irr(H)$ and $Irr(K)$ respectively. Let $\Gamma_{V\otimes W}$, Γ_V and Γ_W be the isotropy subgroups of $V\otimes W$, V and W respectively. We have $\Gamma_{V\otimes W}=\Gamma_V\cap\Gamma_W$. Put $M=\Gamma/\Gamma_V$ and $N=\Gamma/\Gamma_W$. The order of M (resp. N) divides $\phi(|H|)$ (resp. $\phi(|K|)$). Since $x\in R_0(G)$, there exists $\mu\in\Gamma$ such that

$$x\equiv(\mu-1)(V\otimes W) \bmod R_1(G)\ .$$

We put $y=(\mu-1)\ (V\otimes W)$. $\mathrm{Res}_H^G x\in R_1(H)$ and $\mathrm{Res}_K^G x\in R_1(K)$ are equivalent to $\mathrm{Res}_H^G y\in R_1(H)$ and $\mathrm{Rcs}_K^G y\in R_1(K)$ respectively. We have $\mathrm{Res}_H^G y=(\dim W)(\mu-1)V$. By Proposition 3.1 (ii) it holds that

$$\mathrm{Res}_H^G y\equiv(\mu^{\dim W}-1)V \bmod R_1(H)\ .$$

$\mathrm{Res}_H^G y\in R_1(H)$ implies $\mu^{\dim W}\in\Gamma_V$. Since $(|M|, \dim W)=1$, we have $\mu\in\Gamma_V$. In the same way we obtain $\mu\in\Gamma_W$. Therefore we have $\mu\in\Gamma_{V\otimes W}$; this means $y=0$ in $R(G)$. Consequently x belongs to $R_1(G)$.

For a group G we denote by $C(G)$ its center. Since the dimensions of the irreducible G-spaces divide $|G/C(G)|$, we have the following proposition.

Proposition 3.3. *If both H and K have Property 1 and if it holds that $(|H/C(H)|, \phi(|K|))=(|K/C(K)|, \phi(|H|))=1$, then $G=H\times K$ has Property 1.*

As the abelian groups and the p-groups have Property 1, we have the following.

Corollary 3.4. *Let H be an abelian group, and K a p-group. Provided $(\phi(|H|), p)=1$, then $G=H\times K$ has Property 1.*

Corollary 3.5. *Let H be a p-group and K a q-group. Provided $(p, q)=(p, q-1)=(q, p-1)=1$, then $G=H\times K$ has Property 1.*

4. The irreducible spaces of the hyperelementary group

Let G have a normal cyclic subgroup A and a Sylow p-subgroup H such that G is the semidirect product of H by A, that is, G is a hyperelementary group. The irreducible representations of G can be constructed by the method of little group of Wigner and Mackey (see [7; 8.2]).

Since A is cyclic, its irreducible representations form a group Y. The group G acts on Y by

$$(g\chi)(a) = \chi(g^{-1}ag)$$

for $g\in G$, $\chi\in Y$, $a\in A$. This action induces the action of G on the set $Irr(A)$ of irreducible A-spaces. For $V\in Irr(A)$ and $g\in G$, we have an irreducible A-space g_*V by this action. Let $\{V(i): i\in Y/H\}$ be a system of representatives for the orbits of H. For each $i\in Y/H$, let $H(i)$ be the subgroup of H consisting of those elements h such that $h_*V(i)=V(i)$, and let $G(i)=AH(i)$ be the corresponding subgroup of G. We can canonically extend $V(i)$ to the $G(i)$-space, that is, $h\in H(i)$ acts trivially on $V(i)$. Let W be an irreducible $H(i)$-space; W can be extended to $G(i)$-space, too. By taking the tensor product of $V(i)$ and W we obtain an irreducible $G(i)$-space $V(i)\otimes W$. Then $\mathrm{Ind}_{G(i)}^{G} V(i)\otimes W$ is irreducible, moreover each irreducible G-space is obtained in this way ([7; Proposition 25]).

We denote by $C_H(A)$ the centralizer of A in H, i.e.

$$C_H(A) = \{g\in H: g^{-1}ag = a \quad \text{for all } a\in A\} .$$

Proposition 4.1. *If the kernel of $\mathrm{Ind}_{G(i)}^{G} V(i)\otimes W$ is $\{1\}$, then the kernel of the A-space $V(i)$ is $\{1\}$, and $H(i)=C_H(A)$.*

Proof. This comes from the fact that $\ker V(i)\subset \ker \mathrm{Ind}_{G(i)}^{G} \{V(i)\otimes W\}$.

Since $C_H(A)$ is normal in H, H acts on $Irr(C_H(A))$ by

$$(\chi_{g*W})(h) = \chi_W(g^{-1}hg),$$

where $g \in H$, $h \in H$, and χ_W is the corresponding character to $W \in Irr(C_H(A))$.

Proposition 4.2. *Put $K = C_H(A)$, and let V be an irreducible A-space with the trivial kernel, W an irreducible K-space and h an element of H. Then we have*

$$\mathrm{Ind}_{AK}^G V \otimes (h_* W) = \mathrm{Ind}_{AK}^G (h^{-1}{}_* V) \otimes W$$

Proof. If we identify the representation spaces with the corresponding characters, by direct calculation we have

$$\{\mathrm{Ind}_{AK}^G V \otimes (h_* W)\}(g) = \mathrm{Ind}_{AK}^G \{(h^{-1}{}_* V) \otimes W\}(g) \qquad \text{for each } g \in G.$$

Proposition 4.3. *We have the following.*

(i) $\gamma\, \mathrm{Ind}_{G(i)}^G V(i) \otimes W = \mathrm{Ind}_{G(i)}^G (\gamma V(i)) \otimes (\gamma W)$ for $\gamma \in \Gamma$.

(ii) $\mathrm{Res}_H^G \mathrm{Ind}_{G(i)}^G V(i) \otimes W = \mathrm{Ind}_{H(i)}^H W$.

(iii) $\mathrm{Res}_A^G \mathrm{Ind}_{G(i)}^G V(i) \otimes W = \dim W \bigoplus_{[h] \in H/H(i)} h_* V(i)$

(iv) *If* $\ker \mathrm{Ind}_{G(i)}^G V(i) \otimes W = \ker \mathrm{Ind}_{G(j)}^G V(j) \otimes W'$, *then we have* $H(i) = H(j)$.

Proof. (i): This holds clearly.

(ii): Since $H \backslash G/G(i)$ consists of the only one coset, (ii) follows from the Mackey decomposition.

(iii): Since $A \backslash G/G(i)$ can be identified with $H/H(i)$, we have (iii) by the Mackey decomposition.

(iv): Put $U = \mathrm{Ind}_{G(i)}^G V(i) \otimes W$ and $U' = \mathrm{Ind}_{G(j)}^G V(j) \otimes W'$. From $\ker U = \ker U'$ we have $\ker \mathrm{Res}_A^G U = \ker \mathrm{Res}_A^G U'$. By (iii) we have $\ker V(i) = \ker V(j)$. This implies $H(i) = H(j)$.

Proposition 4.4. *Put $K = C_H(A)$, and let V be an irreducible A-space with the trivial kernel, U and W irreducible K-spaces. Set $M = \mathrm{Ind}_{AK}^G V \otimes U$ and $N = \mathrm{Ind}_{AK}^G V \otimes W$. Provided $\Gamma M \neq \Gamma N$ as subset of $Irr(G)$, then we have*

$$\langle \Gamma\, \mathrm{Res}_K^G M, \Gamma\, \mathrm{Res}_K^G N \rangle_K = \{0\}.$$

Proof. For $\gamma \in \Gamma$ we have

$$\gamma\, \mathrm{Res}_K^G M = \bigoplus_{[h] \in H/K} \gamma h_* U$$

by Proposition 4.3 (ii). Proposition 4.2 implies $\mathrm{Ind}_{AK}^G V \otimes (h_* U) \in \Gamma M$. Since $\Gamma M \neq \Gamma N$, we have

$$\langle \gamma h_* U, \gamma' h'_* W \rangle_K = 0$$

for each $\gamma \in \Gamma$, $\gamma' \in \Gamma$, $[h] \in H/K$ and $[h'] \in H/K$. This relation yields the consequence of Propoitions 4.4.

Proposition 4.5. *Let L be a subgroup of H, then we have $N_G(L)=C_A(L)N_H(L)$.*

Proof. Let a and h are elements of A and H respectively. If $ah\in N_G(L)$, we have $(ah)^{-1}Lah=L$, consequently $a^{-1}La=hLh^{-1}$. For each $g\in L$, there exists $h'\in H$ such that $a^{-1}ga=h'$. Then we have $a^{-1}(gag^{-1})=h'g^{-1}\in A\cap H$. This means that $a^{-1}gag^{-1}=1$ and $h'g^{-1}=1$. Therefore we have $ga=ag$, that is, we have $a\in C_A(L)$. This yields $L=hLh^{-1}$. We obtain $h\in N_H(L)$. The above argument shows $N_G(L)\subset C_A(L)N_H(L)$. On the other hand $N_G(L)\supset C_A(L)N_H(L)$ holds obviously. Hence we have $N_G(L)=C_A(L)N_H(L)$.

Let h be an element of H, then h acts on the generators a of A by

$$h\cdot a = hah^{-1}\,.$$

Let L be the subset of H consisting of elements h such that

$$T(h) = \prod_{b\in\langle h\rangle\cdot a} b$$

is not equal to the unit element 1 of G, where a is a fixed generator of A, and $\langle h\rangle\cdot a$ is the orbit of a with respect to the above action of the group $\langle h\rangle$ generated by h. L is defined independently of the choice of a.

Proposition 4.6. *The above L is a subgroup of H.*

Proof. If $h\in K=C_H(A)$, we have $\langle h\rangle\cdot a=\{a\}$. This implies $T(h)\neq 1$. We get $L\supset K$, moreover we see that L is the union of several cosets of H/K. We remark that H/K is a cyclic p-group. If we can show that $h\in L$ implies $h^m\in L$ for $1\leqq m\leqq p$, we see that L is a subgroup of H.

Suppose $1\leqq m<p$. Since $\langle h\rangle\cdot a=\langle h^m\rangle\cdot a$, $h\in L$ implies $h^m\in L$.

Let h be an element of $H-K$, then we have the disjoint sum such that

$$\langle h\rangle\cdot a = \coprod_{j=0}^{p-1} h^j\langle h^p\rangle\cdot a\,.$$

If $T(h^p)=1$, we have

$$T(h) = \prod_{j=0}^{p-1} h^j T(h^p)h^{-j} = 1\,.$$

Therefore $h^p\notin L$ implies $h\notin L$; this means that $h\in L$ implies $h^p\in L$. This completes the proof of Proposition 4.6.

Proposition 4.7. *Put $K=C_H(A)$, and let V be an irreducible A space with the trivial kernel, W a K-space, a a generator of A and h an element of H. We have the following.*

(i) *Provided $h\in H-L$, the all eigenvalues of ah on $\mathrm{Ind}_{AK}^G V\otimes W$ are determined independently of the choice of the generator a of A.*

(ii) *Provided* $h \in L$, *ah does not have* 1 *as its eigenvalue on* $\mathrm{Ind}_{AK}^{G} V \otimes W$. *Here L is the group defined above.*

As we can prove this by direct calculation, we omit the proof.

5. On the case: *G* is generated by two elements

In this section $G=AH$ will be a hyperelementary group such that H is cyclic.

REMARK 5.1. Let K be a subgroup of H, then K is normal in H. If W is a K-space, then for any $h \in H$ we have $h_* W = W$.

Proposition 5.2. *We have the following.*

(i) *Let* $U=\mathrm{Ind}_{G(i)}^{G} V(i) \otimes W$ *be an irreducible G-space. Then* $\ker U = (\ker V(i))(\ker W)$ *holds, where* $\ker V(i) \subset A$ *and* $\ker W \subset H(i)$.

(ii) *If irreducible G-spaces U and U′ have the same kernel,* $\Gamma U = \Gamma U'$ *holds.*

(iii) *G has Property* 2.

Proof. (i): By the definition of the induced representation and Remark 5.1 we obtain $\ker U = (\ker V(i))(\ker W)$.

(ii): Suppose $U=\mathrm{Ind}_{G(i)}^{G} V(i) \otimes W$ and $U'=\mathrm{Ind}_{G(j)}^{G} V(j) \otimes W'$, then by (i) we have $\ker V(i) = \ker V(j)$ and $\ker W = \ker W'$ (see Proposition 4.3 (iv)). Since both A and $H(i)=H(j)$ are cyclic, we have $\Gamma V(i) = \Gamma V(j)$ and $\Gamma W = \Gamma W'$. From Proposition 4.3 (i) we obtain $\Gamma U = \Gamma U'$.

(iii): Lemma 2.1 and above (ii) imply (iii).

Proposition 5.3. *Let* $V(i)$ *be an irreducible A-space as before, W an* $H(i)$*-space and* γ *an element of* Γ. *Put* $x=\mathrm{Ind}_{G(i)}^{G}\{(\gamma V(i)) \otimes W - V(i) \otimes W\}$. *Then x belongs to* $R_h(G)$ *if and only if* $\mathrm{Res}_{G(i)}^{G} x$ *belongs to* $R_h(G(i))$.

Proof. The only if part is clear. We will prove the if part by induction on $|G|$. If $|A|=1$ or $|H|=1$ then Proposition 5.3 is trivial. Make the inductive hypothesis: for each hyperelementary group of the same type as G has and of smaller order than $|G|$ Proposition 5.3 is valid.

We assume that $\mathrm{Res}_{G(i)}^{G} x$ belongs to $R_h(G(i))$. By Lemma 2.1 and the inductive hypothesis it is sufficient to prove the proposition in the case: $V(i)(\{1\}) = V(i)$ and $W(\{1\}) = W$. In this case we have $x^L = 0$ in $R(N_G(L))$ for each non-trivial subgroup L of G. By Lemma 2.3 we complete the proof if we show $P=P(G;\, x) \equiv 0 \mod |G|$. Choose a positive integer s such that

$$\gamma(\exp(2\pi\sqrt{-1}/|A|)) = \exp(2\pi s\sqrt{-1}/|A|) \text{ and } s \equiv 1 \bmod |H|.$$

By (2.4) and (2.5) we have

$$P\equiv\sum_{g\in G}\{z(g)-Q(s;\ \mathrm{Ind}_{G(i)}^{G}\ V(i)\otimes W)(g)\}\ \mathrm{mod}\ |G|$$
$$=1-s^{n},$$

where $n=\dim\ \mathrm{Ind}_{G(i)}^{G}\ V(i)\otimes W$. Since $s\equiv 1\ \mathrm{mod}\ |H|$, we have $P\equiv 0\ \mathrm{mod}\ |H|$. On the other hand $\mathrm{Res}_{G(i)}^{G}\ x\in R_h(G(i))$ implies $\mathrm{Res}_{A}^{G}\ x\in R_h(A)$; we have $P(A;\ \mathrm{Res}_{A}^{G}\ x)\equiv\ 0\ \mathrm{mod}\ |A|$.From (2.5) we obtain

$$\sum_{g\in A}\{z(g)-Q(s;\ \mathrm{Ind}_{G(i)}^{G}\ V(i)\otimes W)(g)\}\equiv 0\ \mathrm{mod}\ |A|.$$

The left hand side of the above relation is equal to $1-s^{n}$. This means that $P\equiv 0\ \mathrm{mod}\ |A|$. Consequently we have $P\equiv 0\ \mathrm{mod}\ |G|$. This completes the proof.

Proposition 5.4. *Let $V(i)$ be an irreducible A-space as before, and U and W $H(i)$-spaces. Put $x=\mathrm{Ind}_{G(i)}^{G}(V(i)\otimes U-V(i)\otimes W)$. Then x belongs to $R_h(G)$ if and only if $\mathrm{Res}_{H}^{G}\ x$ belongs to $R_h(H)$.*

Proof. The only if part is clear. We will prove the if part by induction on $|G|$. If $|A|=1$ or $|H|=1$ then Proposition 5.4 is trivial. Make the inductive hypothesis: for each hyperelementary group of the same type as G and of smaller order than $|G|$ Proposition 5.4 is valid.

We assume that $\mathrm{Res}_{H}^{G}\ x$ belongs to $R_h(H)$. By Lemma 2.1 and the inductive hypothesis it is sufficient to prove the proposition in the case: $V(i)(\{1\})=V(i)$, $U(\{1\})=U$ and $W(\{1\})=W$. Since $K=C_H(A)$ is cyclic, those conditions imply

$$U-W\equiv\gamma W_0-W_0\ \mathrm{mod}\ R_1(K),$$

where W_0 is some irreducible K-space with the trivial kernel and γ is some element of Γ. Without loss of generality we may assume that $W=W_0$ and $U=\gamma W_0$. By this assumption we have $x^L=0$ for each non-trivial subgroup L of G. If we show that $P=P(G;\ x)\equiv 0\ \mathrm{mod}\,|G|$, by Lemma 2.4 we obtain Proposition 5.4. Choose a positive integer s such that

$$\gamma(\exp(2\pi\sqrt{-1}/|H|))=\exp(2\pi s\sqrt{-1}/|H|)\ \text{and}\ s\equiv 1\ \mathrm{mod}\ |A|.$$

By (2.4) and (2.5) we have

$$P\equiv\sum_{g\in G}\{z(g)-Q(s;\ \mathrm{Ind}_{G(i)}^{G}\ V(i)\otimes W)(g)\}\ \mathrm{mod}\ |G|$$
$$=1-s^{n},$$

where $n=\dim\ \mathrm{Ind}_{G(i)}^{G}\ V(i)\otimes W$. Since $s\equiv 1\ \mathrm{mod}\ |A|$, we have $P\equiv 0\ \mathrm{mod}\,|A|$. On the other hand, $\mathrm{Res}_{H}^{G}\ x\in R_h(H)$ implies $P(H;\ \mathrm{Res}_{H}^{G}\ x)\equiv 0\ \mathrm{mod}\,|H|$. From (2.5) we obtain

$$\sum_{g\in G}\{z(g)-Q(s;\ \mathit{Ind}_{G(i)}^{G}\ V(i)\otimes W)(g)\}\equiv 0\ \mathrm{mod}\ |H|.$$

The left hand side of the above relation is equal to $1-s^n$. This means that $P\equiv 0 \bmod |H|$. Consequently we have $P\equiv 0 \bmod |G|$.

Proposition 5.5. *Let $V(i)$ be an irreducible A-space as before, W an irreducible $H(i)$-space, and γ and γ' elements of Γ. Put $x=\mathrm{Ind}_{G(i)}^G\{\gamma(V)\otimes(\gamma' W)-V\otimes W\}$. Then x belongs to $R_h(G)$ if and only if $\mathrm{Res}_{G(i)}^G x\in R_h(G(i))$ and $\mathrm{Res}_H^G x\in R_h(H)$.*

Proof. The only if part is clear. We prove the if part. Put

$$y = \mathrm{Ind}_{G(i)}^G\{(\gamma V(i))\otimes(\gamma' W)-(\gamma V(i))\otimes W\} \text{ and}$$
$$z = \mathrm{Ind}_{G(i)}^G\{(\gamma V(i))\otimes W-V(i)\otimes W\}.$$

We have $x=y+z$; we have $\mathrm{Res}_H^G x=\mathrm{Res}_H^G y$. $\mathrm{Res}_H^G x\in R_h(H)$ means that $\mathrm{Res}_H^G y\in R_h(H)$. By Proposition 5.4 we have $y\in R_h(G)$. This and $\mathrm{Res}_{G(i)}^G x\in R_h(G(i))$ imply $\mathrm{Res}_{G(i)}^G z\in R_h(G(i))$. By Proposition 5.3 we have $z\in R_h(G)$. Consequently we have $x=y+z\in R_h(G)$.

6. Hyperelementary groups and Property 2

In this section $G=AH$ will be a hyperelementary group such that all the elementary subgroups of the quotient groups of the subgroups of G have Property 2.

REMARK. If an elementary group $K=A\times H$ satisfies one of the conditions: (i) $(\phi(|A|),p)=1$, (ii) $|H|\leqq p^4$ and (iii) H is metacyclic, then K has Property 2.

Let $R(G,f)$ be the subgroup of $R(G)$ built from the irreducible G-spaces which yield faithful A-spaces when they are restricted to A. Put $R_h(G,f)=R(G,f)\cap R_h(G)$, and $R_0(G,f)=R(G,f)\cap R_0(G)$.

Proposition 6.1. *Let x be an element of $R_h(G,f)$, B a subgroup of A and K a subgroup of $C_H(B)$. Then for each $C\in X(G)=Irr(G)/\Gamma$ we have $\mathrm{Res}_{BK}^G f_C(x)\in R_h(BK)$.*

Proof. It is sufficient to prove the proposition in the case that $K=C_H(B)$. In this case we have $K\subset C_H(A)$. Put $L=C_H(A)$. Let V be an irreducible A-space with the trivial kernel, and U and W irreducible L-spaces. If $\Gamma\,\mathrm{Ind}_{AL}^G V\otimes U\neq\Gamma\,\mathrm{Ind}_{AL}^G V\otimes W$, we have

$$\langle\Gamma\,\mathrm{Res}_{BK}^G\mathrm{Ind}_{AL}^G V\otimes U,\ \Gamma\,\mathrm{Res}_{BK}^G\mathrm{Ind}_{AL}^G V\otimes W\rangle_{BK}=\{0\}$$

by Proposition 4.4. Since BK has Property 2 by the assumption, we have $\mathrm{Res}_{BK}^G f_C(x)\in R_h(BK)$ for each $C\in X(G)$.

Proposition 6.2. *Put $K=C_H(A)$, and let V be an irreducible A-space with* *the trivial kernel, W a K-space and γ an element of Γ. Put $x=\mathrm{Ind}_{AK}^{G}\{(\gamma V)\otimes W-V\otimes W\}$. Then x belongs to $R_h(G)$ if and only if for each subgroup B of A and $L=C_H(B)$ we have* $\mathrm{Res}_{BL}^{G}\, x\in R_h(BL)$.

Proof. The only if part is clear. We will prove the if part by induction on $|G|$. If $|A|=1$ or $|H|=1$, then Proposition 6.2 is trivial. Make the inductive hypothesis: for each hyperelementary group which satisfies the same condition as G satisfies and whose order is smaller than $|G|$ Proposition 6.2 is valid.

Assume that for each $B\subset A$ and $L=C_H(B)$ we have $\mathrm{Res}_{BL}^{G}\, x\in R_h(BL)$. Firstly we get $x\in R_0(G)$. By Propositions 3.1, 4.3, 4.4 and 6.1 it is sufficient to prove the proposition in the case that there exist a positive integer r, an irreducible K-space U and elements $h(m)$ of H, $1\leqq m\leqq r$, such that

$$W=\bigoplus_{m=1}^{r} h(m)_* U\,.$$

By Propositions 3.1 and 4.2 we have

$$\mathrm{Ind}_{AK}^{G}\{(\gamma V)\otimes h(m)_*U-V\otimes h(m)_*U\}\equiv \mathrm{Ind}_{AK}^{G}\{(\gamma V)\otimes U-V\otimes U\} \bmod R_1(G).$$

This enables us to assume that W itself is irreducible.

Assertion 6.3. *Let $M\neq\{1\}$ be a subgroup of G We have $x^M\in R_h(N_G(M))$.*

Proof. If $A\cap M\neq\{1\}$, then we have $x^M=0$ in $R(N_G(M))$. We assume $A\cap M=\{1\}$. In this case M is conjugate to a subgroup of H. By Proposition 2.2 we may assume $M\subset H$. By Proposition 4.5 we have $N_G(M)=C_A(M)N_H(M)$. The proof is divided into the following three cases.

Case 1. $C_A(M)\neq A$
Put $B=C_A(M)$, $L=C_H(B)$ and $y=\mathrm{Res}_{BH}^{G}\, x$. We have

$$y=\mathrm{Ind}_{BK}^{BH}\{(\gamma\ \mathrm{Res}_B^A\ V)\otimes W-(\mathrm{Res}_B^A\ V)\otimes W\}\,.$$

By Proposition 25 of [7; 8.2] we have y in another form as follows:

$$y=\mathrm{Ind}_{BL}^{BH}\{(\gamma\ \mathrm{Res}_B^A\ V)\otimes U-(\mathrm{Res}_B^A\ V)\otimes U\}\,,$$

where U is an L-space. For a subgroup C of B, we put $N=C_H(C)$; we have $\mathrm{Res}_{CN}^{BH}\, y=\mathrm{Res}_{CN}^{G}\, x\in R_h(CN)$ by the assumption. By the inductive hypothesis y belongs to $R_h(BH)$. This implies $x^M=y^M\in R_h(N_G(M))$.

Case 2. $C_A(M)=A$ and $N_H(M)\neq H$
Put $N=N_G(M)$, $D=H\cap N$, $E=K\cap N$ and $y=\mathrm{Res}_N^G\, x$, then we have

$$y=\sum_{[h]\in H/DK}\mathrm{Ind}_{AE}^{N}\{(\gamma h_*V)\otimes(\mathrm{Res}_E^K\, h_*W)-(h_*V)\otimes(\mathrm{Res}_E^K\, h_*W)\}\,.$$

By Proposition 3.1 we have

$$y \equiv \sum_{[h]\in H/DK} \mathrm{Ind}_{AE}^{N} \{(\gamma V)\otimes(\mathrm{Res}_{E}^{K} h_* W) - V\otimes(\mathrm{Res}_{E}^{K} h_* W) \bmod R_1(N)$$
$$= \mathrm{Ind}_{AE}^{N} \{(\gamma V)\otimes U - V\otimes U\} ,$$

where

$$U = \bigoplus_{[h]\in H/DK} \mathrm{Res}_{E}^{K} h_* W .$$

For a subgroup B of A and $L=C_D(B)$ we have $\mathrm{Res}_{BL}^{N}\, y=\mathrm{Res}_{BL}^{G}\, x\in R_h(BL)$. We have $y\in R_h(N_G(M))$ by the inductive hypothesis. This implies $x^M=y^M\in R_h(N_G(M))$.

Case 3. $N_G(M)=G$

We have reduced the problem to the case that W is irreducible. In this case $\mathrm{Ind}_{AX}^{G}(\gamma V)\otimes W$ and $\mathrm{Ind}_{AK}^{G}\, V\otimes W$ are irreducible. If $(\mathrm{Ind}_{AK}^{G}\, V\otimes W)^M \neq \{0\}$, then we have $(\mathrm{Ind}_{AK}^{G}\, V\otimes W)^M=\mathrm{Ind}_{AK}^{G}\, V\otimes W$. We get $\ker \mathrm{Ind}_{AX}^{G}\, V\otimes W \supset M$. By the inductive hypothesis we have $x\in R_h(G)$. This completes the proof of Assertion 6.3.

If we show $P=P(G;\, x)\equiv 0 \bmod |G|$, we complete the proof of Proposition 6.2. Choose a positive integer s such that

$$\gamma(\exp(2\pi\sqrt{-1}/|A|)) = \exp(2\pi s\sqrt{-1}/|A|) \text{ and } s\equiv 1 \bmod |H| .$$

By (2.4) and (2.5) we have

$$P\equiv \sum_{g\in G} \{z(g)-Q(s;\, \mathrm{Ind}_{AK}^{G}\, V\otimes W)(g)\} \bmod |G| .$$

Since $s\equiv 1 \bmod |H|$, we have $P\equiv 0 \bmod |H|$. On the other hand there exist integers n_C for the cyclic subgroups C of H such that

$$P = \sum_{C<H\,:\,\mathrm{cyclic}} n_C P(G;\, \mathrm{Res}_{AC}^{G}\, x) .$$

If we can show $P(G;\, \mathrm{Res}_{AC}^{G}\, x)\equiv 0 \bmod |A|$, we see that $P\equiv 0 \bmod |A|$; consequently we obtain $P\equiv 0 \bmod |G|$. $P(G;\, \mathrm{Res}_{AC}^{G}\, x)\equiv 0 \bmod |A|$, follows from the following assertion.

Assertion 6.4. *For each cyclic subgroup C of H, we have* $\mathrm{Res}_{AC}^{G}\, x\in R_h(AC)$.

Proof. Put $y=\mathrm{Res}_{AC}^{G}\, x$ and $M=C\cap K$. We have

$$y = \sum_{[h]\in H/CK} \mathrm{Ind}_{AM}^{AC} \{(\gamma h_* V)\otimes(\mathrm{Res}_{M}^{K} h_* W) - (h_* V)\otimes(\mathrm{Res}_{M}^{K} h_* W)\}$$
$$\equiv \sum_{[h]\in H/CK} \mathrm{Ind}_{AM}^{AC} \{(\gamma V)\otimes(\mathrm{Res}_{M}^{K} h_* W) - V\otimes(\mathrm{Res}_{M}^{K} h_* W)\} \bmod R_1(AC)$$
$$= \mathrm{Ind}_{AM}^{AC} \{(\gamma V)\otimes U - V\otimes U\} ,$$

where

$$U = \bigoplus_{[h]\in H/CK} \mathrm{Res}_{M}^{K} h_* W .$$

Since we have $\mathrm{Res}^{AC}_{AM}\, y = \mathrm{Res}^{G}_{AM}\, x \in R_h(AM)$ by the assumption, we have $\mathrm{Res}^{G}_{AC}\, x = y \in R_h(AC)$ by Proposition 5.3. This completes the proof of Assertion 6.4.

Proposition 6.5. *Put $K = C_H(A)$, and let V be an irreducible A-space with the trivial kernel, W a K-space and γ an element of Γ. Put $x = \mathrm{Ind}^{G}_{AX}\{V \otimes (\gamma W) - V \otimes W\}$. Then x belongs to $R_h(G)$ if and only if $\mathrm{Res}^{G}_{H}\, x$ belongs to $R_h(H)$.*

Proof. The only if part is clear. We will prove the if part by induction on $|G|$. If $|A|=1$ or $|H|=1$, then the proposition is trivial. Make the inductive hypothesis: for each hyperelementary group which satisfies the condition stated at the beginning of this section and whose order is smaller than $|G|$ Proposition 6.5 is valid.

We assume $\mathrm{Res}^{G}_{H}\, x \in R_h(H)$ and $|A| \neq 1$. Firstly we have $x \in R_0(G)$. By Propositions 3.1, 4.3, 4.4 and 6.1 it is sufficient to prove the proposition in the case that there exist a positive integer r, an irreducible K-space U and elements $h(m)$ of H, $1 \leqq m \leqq r$, such that

$$W = \bigoplus_{m=1}^{r} h(m)_* U .$$

By Propositions 3.1 and 4.2 we have

$$\mathrm{Ind}^{G}_{AK}\{V \otimes (\gamma h(m)_* U) - V \otimes (h(m)_* U)\} \equiv \mathrm{Ind}^{G}_{AK}\{V \otimes (\gamma U) - V \otimes U\} \bmod R_1(G).$$

This enables us to assume that W itself is irreducible.

Assertion 6.6. *Let L be a non-trivial subgroup of G. We have $x^L \in R_h(N_G(L))$.*

Proof. Since A acts freely on $\mathrm{Ind}^{G}_{AK}\, V \otimes \gamma W$ and on $\mathrm{Ind}^{G}_{AK}\, V \otimes W$ except the origins, it is sufficient to prove the assertion in the case that $L \cap A = \{1\}$. In this case L is conjugate to a subgroup of H. By Proposition 2.2 we may assume $L \subset H$. Then we have $N_G(L) = C_A(L) N_H(L)$ by Proposition 4.5. We divide the proof into the following three cases.

Case 1. $C_A(L) \neq A$

We put $B = C_A(L)$ and $y = \mathrm{Res}^{G}_{BH}\, x$. We have

$$y = \mathrm{Ind}^{BH}_{BK}\{(\mathrm{Res}^{A}_{B}\, V) \otimes (\gamma W) - (\mathrm{Res}^{A}_{B}\, V) \otimes W\} .$$

Put $M = C_H(B)$, then we have

$$y = \mathrm{Ind}^{BH}_{BM}\{(\mathrm{Res}^{A}_{B}\, V) \otimes (\gamma\, \mathrm{Ind}^{M}_{K}\, W) - (\mathrm{Res}^{A}_{B}\, V) \otimes (\mathrm{Ind}^{M}_{K}\, W)\} .$$

On the other hand we have $\mathrm{Res}^{BH}_{B}\, y = \mathrm{Res}^{G}_{H}\, x \in R_h(H)$. By the inductive hypothesis we have $y \in R_h(BH)$. This implies $x^L = y^L \in R_h(N_G(L))$.

Case 2. $C_A(L)=A$ and $N_H(L)\neq H$

Put $M=N_H(L)$, $N=N_G(L)$, $D=K\cap M$ and $y=\mathrm{Res}^G_N\, x$. We have $N=AM$ and

$$\begin{aligned} y &= \sum_{[h]\in H/KM} \mathrm{Ind}^N_{AD} \{(h_* V)\otimes(\gamma\, \mathrm{Res}^K_D\, h_* W)-(h_* V)\otimes(\mathrm{Res}^K_D\, h_* W)\} \\ &\equiv \sum_{[h]\in H/KM} \mathrm{Ind}^N_{AD} \{V\otimes(\gamma\, \mathrm{Res}^K_D\, h_* W)-V\otimes(\mathrm{Res}^K_D\, h_* W)\} \bmod R_1(N) \\ &= \mathrm{Ind}^N_{AD} \{V\otimes(\gamma U)-V\otimes U\}\,, \end{aligned}$$

where

$$U=\bigoplus_{[h]\in H/KM} \mathrm{Res}^K_D\, h_* W\,.$$

Since we have $\mathrm{Res}^N_M\, y=\mathrm{Res}^G_M\, x\in R_h(M)$, by the inductive hypothesis we get $y\in R_h(N)$. This implies $x^L=y^L\in R_h(N_G(L))$.

Case 3. $N_G(L)=G$

When W is irreducible, $\mathrm{Ind}^G_{AK}\, V\otimes W$ and $\mathrm{Ind}^G_{AR}\, V\otimes\gamma W$ are irreducible. This implies that $x^L=x$ or 0 in $R(G)$. If $x^L=0$, Assertion 6.6 is clearly valid. If $x^L=x$, then L is included in the kernel of x. By the inductive hypothesis we obtain $x\in R_h(G)$. This completes the proof of Assertion 6.6.

If we show $P=P(G;\, x)\equiv 0 \bmod |G|$, we complete the Proof of Proposition 6.5. As usual choose a positive integer s such that

$$\gamma(\exp(2\pi\sqrt{-1}/|H|))=\exp(2\pi s\sqrt{-1}/|H|) \text{ and } s\equiv 1 \bmod |A|\,.$$

By (2.5) we have

$$P\equiv\sum_{g\in G}\{z(g)-Q(s;\, \mathrm{Ind}^G_{AK}\, V\otimes W)(g)\} \bmod |G|\,.$$

By the inductive hypothesis, for each proper subgroup B of A we have $\mathrm{Res}^G_{BH}\, x\in R_h(BH)$. This implies $P(BH;\, \mathrm{Res}^G_{BH}\, x)\equiv 0 \bmod |BH|$. Therefore we have

$$P\equiv\sum_{ah\in AH\,:\,\langle a\rangle=A}\{z(ah)-Q(s;\, \mathrm{Ind}^G_{AK}\, V\otimes W)(ah)\} \bmod |H|\,.$$

By Propositions 4.6 and 4.7 we have

$$\begin{aligned} P &\equiv \sum_{\substack{a\in A\,:\,\langle a\rangle=A \\ h\in H-L}}\{z(ah)-Q(s;\, \mathrm{Ind}^G_{AK}\, V\otimes W)(ah)\} \bmod |H| \\ &\equiv \phi(|A|)\sum_{h\in H-L}\{z(h)-Q(s;\, \mathrm{Ind}^G_{AK}\, V\otimes W)(h)\} \bmod |H|, \end{aligned}$$

where L is the group given in Proposition 4.6, ϕ is the Euler function. $\mathrm{Res}^G_L\, x\in R_h(L)$ and (2.5) imply

$$\sum_{h\in L}\{z(h)-Q(s;\, \mathrm{Ind}^G_{AK}\, V\otimes W)(h)\}\equiv 0 \bmod |L|\,.$$

Since $\phi(|A|)$ is a multiple of $|H/K|$ and $|L|$ a multiple of $|K|$, we have

$$P \equiv \phi(|A|) \sum_{h \in H} \{z(h) - Q(s; \operatorname{Ind}_{AK}^{G} V \otimes W)(h)\} \bmod |H| .$$

From $\operatorname{Res}_H^G x \in R_h(H)$, we have $P \equiv 0 \bmod |H|$. On the other hand for the cyclic subgroups C of H there exist integers n_C such that

$$P = \sum_{C < H\,:\,\text{cyclic}} n_C \sum_{g \in AC} z(g) .$$

We obtain $P \equiv 0 \bmod |A|$ from the following assertion; consequently we get $P \equiv 0 \bmod |G|$.

Assertion 6.7. *For each cyclic subgroup C of H, we have* $\operatorname{Res}_{AC}^G x \in R_h(AC)$.

Proof. Put $y = \operatorname{Res}_{AC}^G x$ and $D = C \cap K$, then we have

$$\begin{aligned} y &= \sum_{[h] \in H/CK} \operatorname{Ind}_{AD}^{AC} \{(h_* V) \otimes (\gamma \operatorname{Res}_D^K h_* W) - (h_* V) \otimes (\operatorname{Res}_D^K h_* W)\} \\ &\equiv \sum_{[h] \in H/CK} \operatorname{Ind}_{AD}^{AC} \{V \otimes (\gamma \operatorname{Res}_D^K h_* W) - V \otimes (\operatorname{Res}_D^K h_* W)\} \bmod R_1(AC) \\ &= \operatorname{Ind}_{AD}^{AC} (V \otimes \gamma U - V \otimes U) , \end{aligned}$$

where

$$U = \bigoplus_{[h] \in H/CK} \operatorname{Res}_D^K h_* W .$$

Moreover we have $\operatorname{Res}_C^{AC} y = \operatorname{Res}_C^G x \in R_h(C)$. By Proposition 5.4 we have $y \in R_h(AC)$. This completes the proof of Assertion 6.9 consequently completes the proof of Proposition 6.5.

Proposition 6.10. *Put $K = C_H(A)$, and let V be an irreducible A-space with the trivial kernel, W a K-space and γ an element of Γ. Put $x = \operatorname{Ind}_{AK}^G \{\gamma(V \otimes W) - V \otimes W\}$. Then x belongs to $R_h(G)$ if and only if for each subgroup B of A and $L = C_H(B)$ we have* $\operatorname{Res}_{BL}^G x \in R_h(BL)$.

Proof. The only if part is clear. We prove the if part. Put $y = \operatorname{Ind}_{AK}^G \{\gamma(V \otimes W) - (\gamma V) \otimes W\}$ and $z = \operatorname{Ind}_{AK}^G \{(\gamma V) \otimes W - V \otimes W\}$, then we have $x = y + z$. Since $\operatorname{Res}_H^G z = 0$, we have $\operatorname{Res}_H^G y \in R_h(H)$ by the assumption. From Proposition 6.5 we obtain $y \in R_h(G)$. This yields that

$$\operatorname{Res}_{BL}^G z = \operatorname{Res}_{BL}^G x - \operatorname{Res}_{BL}^G y \in R_h(BL) .$$

Proposition 6.2 implies $z \in R_h(G)$. Hence we conclude that $x \in R_h(G)$.

Theorem 6.11. *Let G be a hyperelementary group such that all the elementary subgroups of the quotient groups of the subgroups of G have Property 2. Then G has Property 2.*

Proof. We prove it by induction on $|G|$. If $|A|=1$ or $|H|\leqq p$, we are aware that G has Property 2. Make the inductive hypothesis: each hyper-elementary group which satisfies the same condition as G satisfies and whose order is smaller than $|G|$ has Property 2.

Let x be an elemant of $R_h(G)$. By Lemma 2.1 and the inductive hypothesis we may assume $x(\{1\})=x$. This implies $x\in R_h(G, f)$. Put $K=C_H(A)$. For a fixed element C of $X(G)$, there exist $\gamma\in\Gamma$, an irreducible A-space V and an irreducible K-space W such that

$$f_C(x)\equiv \mathrm{Ind}_{AK}^G \{\gamma(V\otimes W)-V\otimes W\} \bmod R_1(G)\,.$$

By Propositions 6.1 and 6.10 we get $f_C(x)\in R_h(G)$.

For a subgroup B of A, we get an elementary subgroup $BC_H(B)$ of G. Varying B, we obtain several elementary groups. Let $E(G)$ be the set of all those elementary groups. Lemma 2.1 and Propositions 6.1 and 6.10 yield the following theorem.

Theorem 6.12. *In the same situation as in Theorem* 6.11

$$\mathrm{Res}\colon R_0(G, f)/R_h(G, f) \to \bigoplus_{K\in E(G)} j(K)$$

is injective. Therefore we obtain a naturally defined injection

$$j(G) \to \bigoplus_{B} \bigoplus_{K\in E(G/B)} j(K)$$

where B runs over the subgroups of A.

7. A closing example

Let A (resp. H) be the cyclic group of order 7 (resp. 5) which consists of the 7-th (resp. 5-th) roots of unity, and G the direct product of A and H. For each integer i (resp. j) with $0\leqq i\leqq 6$ (resp. $0\leqq j\leqq 4$) define the A-(resp. H-) representation v_i (resp. w_j) by

$$v_i(z) = z^i \text{ for } z\in A$$
$$(\text{resp. } w_j(z) = z^j \text{ for } z\in H)\,.$$

We denote by V_i (resp. W_j) the corresponding representation space to v_i (resp. w_j). Define an element x of $R(G)$ by

$$x = V_2\otimes W_1+V_2\otimes W_0+V_2\otimes W_0-V_1\otimes W_1-V_1\otimes W_0-V_1\otimes W_0\,.$$

Then we have $x\in R_0(G)\cap R(G, f)$; moreover we have $\mathrm{Res}_A^G\, x\in R_h(A)$ and $\mathrm{Res}_H^G\, x\in R_h(H)$. The x does not, however, belong to $R_h(G)$. This is a counter example to [1; Proposition 5.2].

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