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ΛG-STRUCTURE OF G-VECTOR BUNDLES AND GROUPS $KO_G(X)$, $KSp_G(X)$ and $J_G(X)$

Katsuo KAWAKUBO

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1. Introduction

Let G be a compact topological group. We say that X is a trivial G-space if X is a topological space with the G-action $gx=x$ for all $g\in G$ and all $x\in X$. Let V_i run over the inequivalent irreducible complex G-representations. For any complex G-representation V, there is a canonical isomorphism

$$(*) \qquad \bigoplus_i V_i \underset{G}{\otimes} \mathrm{Hom}_{CG}(V_i, V) \cong V .$$

Using this isomorphism, Atiyah and Segal had a decomposition of a complex G-vector bundle over a compact trivial G-space X [4]. As a consequence they showed that the equivariant complex K-group $K_G(X)$ is isomorphic to the tensor product $R(G)\otimes K(X)$ of the complex representation ring $R(G)$ and the complex K-group $K(X)$.

In the present paper, we first make real and symplectic versions of these for our later use, although they seem familiar to us all (Propositions 3.1 and 4.1).

Similar decompositions have been already obtained for some special cases; by Conner-Floyd [7] for G a cyclic group of odd prime order, by Atiyah-Singer [5] for G a monogenic group, and by Uchida [25] for semi-free S^1-and S^3-actions.

Moreover we show that the decompositions of real and symplectic G-vector bundles are unique up to isomorphism in respective category (Proposition 4.2).

As an application, we express the equivariant real K-group $KO_G(X)$ and the equivariant quaternionic K-group $KSp_G(X)$ in terms of irreducible G-representations and their types, the real K-group $KO(X)$, the complex K-group $K(X)$ and the quaternionic K-group $KSp(X)$ (Theorems 5.1 and 5.2) (Compare [21] for $KO_G(X)$). Consequently we know that the real version of the Atiyah-Segal theorem above does not hold in a similar form in general.

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Namely $KO_G(X)$ is not isomorphic to $RO(G)\otimes KO(X)$ in general (Remark 5.7). If all the irreducible representations are of the real type in the sense of Adams [2], then we have an isomorphism $KO_G(X)\cong RO(G)\otimes KO(X)$ of rings and an isomorphism $KSp_G(X)\cong RO(G)\otimes KSp(X)$ of additive groups (Corollary 5.3).

The normal bundle of the fixed point set of a smooth (symplectic) G-manifold is in the situation considered and the bordism group of real (or symplectic) G-vector bundles can be expressed uniquely in terms of the ordinary bordism groups of classifying spaces (Proposition 6.1 and Remark 6.2).

In order to clarify the substance of the discourse, we next deal with a special case, namely, semi-free G-actions. If G acts semi-freely, then G has to be a group which has a fixed point free representation except for two special cases, the trivial G-actions and free G-actions. These groups and their fixed point free representations are classified in [26].

Fortunately they have a desirable property for our purpose. Namely if the order of G is greater than two, then all the fixed point free representations of G come from complex or quaternionic representations and have the same degree (Proposition 6.5). Consequently, we have an exact sequence involving bordism groups of semi-free G-manifolds (Proposition 6.6).

The J_G-image of the normal bundle of the fixed point set of a smooth G-manifold M is an invariant of the G-homotopy type of M [13], [15] and we study $J_G(X)$ finally. This is in fact my motivation of the present paper.

Once we conjecture the present results and become aware of the formulations, the proofs are somewhat easy. So we only outline the proofs mostly.

In a forthcoming paper, we shall determine the centralizer of an arbitrary closed subgroup of the orthogonal group $O(n)$ along our line.

The real and symplectic versions of (*) were originally proven by case-by-case discussion. The unified proof given in this paper was shown to me by J.F. Adams.

I would like to thank Professor J.F. Adams for his kind advice and for permitting me to employ his argument.

2. Review of representation theory

In this paper, we make use of the book [2] of Adams freely.

First we recall some of it. Let G be a compact topological group and Λ be one of the classical fields R (the real numbers), C (the complex numbers) or Q (the quaternions). Then a *ΛG-space* is a finite-dimensional vector space V over Λ provided with a continuous homomorphism

$$\rho:\quad G\rightarrow \mathrm{Aut}\, V\,.$$

Such a V is also called a *representation* of G over Λ or a *G-space* over Λ.

Let V and W be ΛG-spaces. A *G-map* is a function $f: V \to W$ which commutes with the action of G, that is

$$f(gv) = gf(v) \quad \text{for} \quad g \in G, \ v \in V.$$

A *ΛG-map* is a G-map which is Λ-linear. The set of such ΛG-maps is written $\mathrm{Hom}_{\Lambda G}(V, W)$. It is a vector space over R if $\Lambda = R$ or Q, over C if $\Lambda = C$.

A *ΛG-isomorphism* is a ΛG-map which has an inverse. Two ΛG-spaces V and W are said to be *equivalent* (denoted by $V = W$) if they are isomorphic ($V \cong W$).

DEFINITION 2.1. (i) If V is a G-space over R, define $cV = C \underset{R}{\otimes} V$, regarded as a G-space over C.

(ii) Similarly, if V is a G-space over C, define $qV = Q \underset{C}{\otimes} V$, and regard it in the obvious way as a G-space and a left module over Q.

(iii) If V is a G-space over Q, let $c'V$ have the same underlying set as V and the same operations from G, but regard it as a vector space over C.

(iv) Similarly, if V is a G-space over C, let rV have the same underlying set as V and the same operations from G, but regard it as a vector space over R.

(v) Let V be a G-space over C. We define tV to have the same underlying set as V and the same operations from G, but we make C act in a new way: z acts on tV as $\bar{z}$ used to act on V.

DEFINITION 2.2. We say that a CG-space V is real (resp. symplectic or quaternionic) when there exists an RG-space V^r (resp. QG-space V^q) such that $V = cV^r$ (resp. $V = c'V^q$).

REMARK 2.3. V^r and V^q in Definition 2.2 are unique up to equivalence by the following lemma and we use these notations hereafter.

Lemma 2.4. $rc=2$, $cr=1+t$, $qc'=2$, $c'q=1+t$, $tc=c$, $rt=r$, $tc'=c'$, $qt=q$, $t^2=1$. *These equations are to be interpreted as saying that $rcV \cong V \oplus V$ for each RG-space V, $crV \cong V \oplus tV$ for each CG-space V, etc.*

DEFINITION 2.5. Given G-spaces V and W over the same field Λ, we can form $\mathrm{Hom}_{\Lambda}(V, W)$, the set of Λ-linear maps from V to W. It is a vector space over R if $\Lambda = R$ or Q, over C if $\Lambda = C$. We can make G act on it by

$$(gf)(v) = g(f(g^{-1}v)) \quad \text{for} \quad g \in G, \ f \in \mathrm{Hom}_{\Lambda G}(V, W).$$

The subspace of elements in $\mathrm{Hom}_{\Lambda}(V, W)$ which are invariant under G is precisely $\mathrm{Hom}_{\Lambda G}(V, W)$. We set

$$\mathrm{End}_{\Lambda G}(V) = \mathrm{Hom}_{\Lambda G}(V, V).$$

We now recall the following theorem of Adams [2].

Theorem 2.6. *Suppose given a compact group G. Then it is possible to choose representations V_{Ri} over R, V_{Cj} over C and V_{Qk} over Q to satisfy the following conditions.*

(i) *The inequivalent irreducible representations over R are precisely the V_{Ri}, rV_{Cj} and $rc'V_{Qk}$.*

(ii) *The inequivalent irreducible representations over C are precisely the cV_{Ri}, V_{Cj}, tV_{Cj} and $c'V_{Qk}$.*

(iii) *The inequivalent irreducible representations over Q are precisely the qcV_{Ri}, qV_{Cj} and V_{Qk}.*

DEFINITION 2.7. When an irreducible RG-space V is equivalent to V_{Ri}, rV_{Cj} or $rc'V_{Qk}$, we call V an RG-space of *R-type, C-type* or *Q-type* respectively. When an irreducible QG-space V is equivalent to qcV_{Ri}, qV_{Cj} or V_{Qk}, we call V a QG-space of *R-type, C-type* or *Q-type* respectively.

DEFINITION 2.8. Let V be a CG-space. A *structure map* on V is a G-map $j\colon V\to V$ such that

(i) j is conjugate-linear, that is,

$$j(zv)=\bar{z}j(v)\ (z\in C)\,,\quad\text{and}$$

(ii) $j^2=\pm 1$.

3. ΛG-structure decompsition of RG- and QG-spaces

Let $\{V_i\}$ be a complete set of inequivalent irreducible CG-spaces. Then for a CG-space V, the evaluation map

$$\bigoplus_i V_i \underset{C}{\otimes} \mathrm{Hom}_{CG}(V_i, V)\to V$$

is a CG-isomorphism (e.g. Lemma 3.25 of [2]).

We wish to find the analogue of this result for real and symplectic representations, using structure map j. In the following, we use Lemma 2.4 freely.

For each i, let $\bar{i}$ be the index such that $V_{\bar{i}}$ is the complex conjugate of V_i. Choose a conjugate-linear isomorphism

$$j_i\colon V_i\to V_{\bar{i}}$$

such that

$$j_{\bar{i}}j_i=\varepsilon_i=\pm 1\colon V_i\to V_i\,.$$

If V_i is real, this is certainly possible with $\varepsilon_i=+1$; if V_i is symplectic, it is equally possible with $\varepsilon_i=-1$; and if V_i is not self-conjugate, we can choose j_i first and construct $j_{\bar{i}}$ from it, with either sign of ε_i. (Of course we get $\varepsilon_{\bar{i}}=\varepsilon_i$.)

Now suppose that V comes provided with a conjugate-linear structure map j_v such that $j_v^2=\varepsilon_v=\pm 1$. Then we define a kind of structure map

$$j_i': \operatorname{Hom}_{CG}(V_i, V) \to \operatorname{Hom}_{CG}(V_{\bar{i}}, V)$$

by sending $h_i \in \operatorname{Hom}_{CG}(V_i, V)$ to $j_v h_i j_{\bar{i}}^{-1}$. This map $j_v h_i j_{\bar{i}}^{-1}$ lies in $\operatorname{Hom}_{CG}(V_{\bar{i}}, V)$; the structure map j_i' is conjugate linear; and we have

$$j_{\bar{i}}' j_i' = \varepsilon_v \varepsilon_i \, .$$

For all this, compare [2], p. 31.

We can now define the structure map

$$j_i \otimes j_i': V_i \underset{C}{\otimes} \operatorname{Hom}_{CG}(V_i, V) \to V_{\bar{i}} \underset{C}{\otimes} \operatorname{Hom}_{CG}(V_{\bar{i}}, V) \, .$$

By construction, the evaluation map e commutes with the structure maps:

$$e(j_i \otimes j_i') = j_v e \, .$$

Therefore, we have an automatic answer to the question posed above: under the isomorphism e, the given structure map j_v on V corresponds to the structure map with components $j_i \otimes j_i'$. It remains only to make this description more explicit.

Consider first the case $i = \bar{i}$. In this case $\operatorname{Hom}_{CG}(V_i, V)$ gets a structure map whose square is $\varepsilon_v \varepsilon_i$, so that it is real or symplectic according as $\varepsilon_v \varepsilon_i$ is $+1$ or -1. That is,

(1) if V_i is real and V is real $\operatorname{Hom}_{CG}(V_i, V)$ is real,
(2) if V_i is real and V is symplectic $\operatorname{Hom}_{CG}(V_i, V)$ is symplectic,
(3) if V_i is symplectic and V is real $\operatorname{Hom}_{CG}(V_i, V)$ is symplectic,
(4) if V_i is symplectic and V is symplectic $\operatorname{Hom}_{CG}(V_i, V)$ is real.

(Compare [2] pp. 31–32.) The tensor product $V_i \underset{C}{\otimes} \operatorname{Hom}_{CG}(V_i, V)$ can then be interpreted as a tensor product over R in three cases and Q in one case (Compare [2] pp. 29–30). Explicitly $V_i \underset{C}{\otimes} \operatorname{Hom}_{CG}(V_i, V)$ is isomorphic to the following in respective case:

(1) $c(V_i^r \underset{R}{\otimes} \operatorname{Hom}_{RG}(V_i^r, V^r))$,

(2) $c'(V_i^r \underset{R}{\otimes} \operatorname{Hom}_{RG}(V_i^r, V^q))$,

(3) $c(V_i^q \underset{Q}{\otimes} \operatorname{Hom}_{RG}(V_i^q, V^r))$,

(4) $c'(V_i^q \underset{R}{\otimes} \operatorname{Hom}_{QG}(V_i^q, V^q))$.

Consider secondly the case $i \neq \bar{i}$. In this case we have put a structure map j on

$$[V_i \underset{C}{\otimes} \operatorname{Hom}_{CG}(V_i, V)] \oplus [V_{\bar{i}} \underset{C}{\otimes} \operatorname{Hom}_{CG}(V_{\bar{i}}, V)]$$

and its square is ε_v. If $\varepsilon_v = +1$, then the corresponding RG-module is the $+1$ eigenspace of j (compare [2] p. 25), and clearly this is isomorphic to the

RG-module underlying $V_i \underset{C}{\otimes} \mathrm{Hom}_{CG}(V_i, V)$ which is isomorphic to $V_i \underset{C}{\otimes} \mathrm{Hom}_{RG}(V_i, V^r)$. If $\varepsilon_v = -1$, then the corresponding QG-module is clearly

$$Q \underset{C}{\otimes} [V_i \underset{C}{\otimes} \mathrm{Hom}_{CG}(V_i, V)] .$$

Thus we have shown the following

Proposition 3.1. *For an RG-space (resp. QG-space) V, the evaluation map*

$$\mu : \left\{ \begin{array}{l} \bigoplus_i V_{Ri} \underset{R}{\otimes} \mathrm{Hom}_{RG}(V_{Ri}, V) \\ \oplus \bigoplus_j r[V_{Cj} \underset{C}{\otimes} \mathrm{Hom}_{RG}(V_{Cj}, V)] \\ \oplus \bigoplus_k V_{Qk} \underset{Q}{\otimes} \mathrm{Hom}_{RG}(V_{Qk}, V) \end{array} \right\} \to V$$

(resp.

$$\mu : \left\{ \begin{array}{l} \bigoplus_i V_{Ri} \underset{R}{\otimes} \mathrm{Hom}_{RG}(V_{Ri}, V) \\ \oplus \bigoplus_j q[V_{Cj} \underset{C}{\otimes} \mathrm{Hom}_{CG}(V_{Cj}, V)] \\ \oplus \bigoplus_k V_{Qk} \underset{R}{\otimes} \mathrm{Hom}_{QG}(V_{Qk}, V) \end{array} \right\} \to V)$$

is an RG-isomorphism (resp. QG-isomorphism).

4. ΛG-structure decomposition of real and symplectic G-vector bundles

Once we have canonical isomorphisms μ for vector spaces, we get a corresponding result for vector bundles, by following the arguments of Atiyah and Bott [3], and Atiyah and Segal [4].

Proposition 4.1. *Let ξ be a real G-vector bundle over a trivial G-space X. Then* $\mathrm{Hom}_{RG}(\underline{\underline{V_{Ri}}}, \xi)$, $\mathrm{Hom}_{RG}(\underline{\underline{V_{Cj}}}, \xi)$, $\mathrm{Hom}_{RG}(\underline{\underline{V_{Qk}}}, \xi)$ *inherit canonically real, complex, symplectic vector bundle structures respectively and there is a canonical isomorphism of real G-vector bundles*:

$$\bar{\mu} : \left\{ \begin{array}{l} \bigoplus_i \underline{\underline{V_{Ri}}} \underset{R}{\otimes} \mathrm{Hom}_{RG}(\underline{\underline{V_{Ri}}}, \xi) \\ \oplus \bigoplus_j r[\underline{\underline{V_{Cj}}} \underset{C}{\otimes} \mathrm{Hom}_{RG}(\underline{\underline{V_{Cj}}}, \xi)] \\ \oplus \bigoplus_k \underline{\underline{V_{Qk}}} \underset{Q}{\otimes} \mathrm{Hom}_{RG}(\underline{\underline{V_{Qk}}}, \xi) \end{array} \right\} \to \xi .$$

Similarly for a symplectic G-vector bundle ξ, $\mathrm{Hom}_{RG}(\underline{\underline{V_{Ri}}}, \xi)$, $\mathrm{Hom}_{CG}(\underline{\underline{V_{Cj}}}, \xi)$, $\mathrm{Hom}_{QG}(\underline{\underline{V_{Qk}}}, \xi)$ *inherit canonically symplectic, complex, real vector bundle structures respectively and there is a canonical isomorphism of symplectic G-vector boundles*:

$$\bar{\mu}: \left\{ \begin{array}{l} \bigoplus_i \underline{\underline{V_{Ri}}} \otimes_R \operatorname{Hom}_{RG}(\underline{\underline{V_{Ri}}}, \xi) \\ \bigoplus \bigoplus_j q[\underline{\underline{V_{Cj}}} \otimes_C \operatorname{Hom}_{CG}(\underline{\underline{V_{Cj}}}, \xi) \\ \bigoplus \bigoplus_k \underline{\underline{V_{Qk}}} \otimes_R \operatorname{Hom}_{QG}(\underline{\underline{V_{Qk}}}, \xi) \end{array} \right\} \to \xi .$$

Here $\underline{\underline{V}}$ denotes the G-vector bundle $X \times V \to X$.

Moreover the decompositions in Proposition 4.1 are unique.

Proposition 4.2. *Let ξ_i, ξ'_i be real, ξ_j, ξ'_j complex and ξ_k, ξ'_k symplectic vector bundles over X with trivial G-action. Suppose that there is an isomorphism of real G-vector bundles:*

$$\left\{ \begin{array}{l} \bigoplus_i \underline{\underline{V_{Ri}}} \otimes_R \xi_i \\ \bigoplus \bigoplus_j r[\underline{\underline{V_{Cj}}} \otimes_C \xi_j] \\ \bigoplus \bigoplus_k \underline{\underline{V_{Qk}}} \otimes_Q \xi_k \end{array} \right\} \to \left\{ \begin{array}{l} \bigoplus_i \underline{\underline{V_{Ri}}} \otimes_R \xi'_i \\ \bigoplus \bigoplus_j r[\underline{\underline{V_{Cj}}} \otimes_C \xi'_j] \\ \bigoplus \bigoplus_k \underline{\underline{V_{Qk}}} \otimes_Q \xi'_k \end{array} \right\} .$$

Then we have

$$\xi_i \underset{R}{\cong} \xi'_i , \quad \xi_j \underset{C}{\cong} \xi'_j \quad \textit{and} \quad \xi_k \underset{Q}{\cong} \xi'_k .$$

Let ξ_i, ξ'_i be symplectic, ξ_j, ξ'_j complex and ξ_k, ξ'_k real vector bundles over X with trivial G-action. Suppose that there is an isomorphism of symplectic vector bundles:

$$\left\{ \begin{array}{l} \bigoplus_i \underline{\underline{V_{Ri}}} \otimes_R \xi_i \\ \bigoplus \bigoplus_j q[\underline{\underline{V_{Cj}}} \otimes_C \xi_j] \\ \bigoplus \bigoplus_k \underline{\underline{V_{Qk}}} \otimes_R \xi_k \end{array} \right\} \to \left\{ \begin{array}{l} \bigoplus_i \underline{\underline{V_{Ri}}} \otimes_R \xi'_i \\ \bigoplus \bigoplus_j q[\underline{\underline{V_{Cj}}} \otimes_C \xi'_j] \\ \bigoplus \bigoplus_k \underline{\underline{V_{Qk}}} \otimes_R \xi'_k \end{array} \right\}$$

Then we have

$$\xi_i \underset{Q}{\cong} \xi'_i , \quad \xi_j \underset{C}{\cong} \xi'_j \quad \textit{and} \quad \xi_k \underset{R}{\cong} \xi'_k .$$

In the real case, we can rewrite Propositions 4.1 and 4.2 in the following form. Let $\{V_i\}$ be a complete set of inequivalent irreducible RG-spaces. Then for a real G-vector bundle ξ over X with trivial G-action, we have a unique decomposition

$$\xi = \bigoplus_i \underline{\underline{V_i}} \otimes_{\Lambda_i} \xi_i$$

where $\Lambda_i = \operatorname{End}_{RG}(V_i)$ and ξ_i are Λ_i-vector bundles.

5. $KO_G(X)$ and $KSp_G(X)$

Let X be a compact space with trivial G-action. Let V_{Ri}, V_{Cj} and V_{Qk} be as in Theorem 2.6. Denote by $KO(X)_i$, $K(X)_j$ and $KSp(X)_k$, the copies of $KO(X)$, $K(X)$ and $KSp(X)$ indexed by the set of irreducible RG-spaces $\{V_{Ri}\}$, the set of irreducible CG-spaces $\{V_{Cj}\}$ and the set of irreducible QG-spaces $\{V_{Qk}\}$ respectively. Then we have

Theorem 5.1. *We have an isomorphism of additive groups:*

$$\Phi\colon \bigoplus_i KO(X)_i \oplus \bigoplus_j K(X)_j \oplus \bigoplus_k KSp(X)_k \to KO_G(X).$$

Proof. Let ξ_i, η_i be real G-vector bundles and ξ_j, η_j be complex G-vector bundles and ξ_k, η_k be symplectic G-vector bundles. Denote by $[\xi]$ the equivalence class represented by ξ in respective category. Then we define Φ by

$$\begin{aligned} &\Phi(\bigoplus_i([\xi_i]-[\eta_i]) \oplus \bigoplus_j([\xi_j]-[\eta_j]) \oplus \bigoplus_k([\xi_k]-[\eta_k])) \\ &= [\bigoplus_i \underline{\underline{V_{Ri}}} \otimes_R \xi_i \oplus \bigoplus_j \underline{\underline{rV_{Cj}}} \otimes_C \xi_j \oplus \bigoplus_k \underline{\underline{rc'V_{Qk}}} \otimes_Q \xi_k] \\ &\quad -[\bigoplus_i \underline{\underline{V_{Ri}}} \otimes_R \eta_i \oplus \bigoplus_j \underline{\underline{rV_{Cj}}} \otimes_C \eta_j \oplus \bigoplus_k \underline{\underline{rc'V_{Qk}}} \otimes_Q \eta_k]. \end{aligned}$$

In the latter, $\oplus$ means the Whitney sum of G-vector bundles. It is easy to see that Φ is a well-defined homomorphism. It follows from Proposition 4.1 that Φ is surjective. The injectivity of Φ follows from Proposition 4.2.

This completes the proof of Theorem 5.1.

Denote by $KSp(X)_i$, $K(X)_j$ and $KO(X)_k$, the copies of $KSp(X)$, $K(X)$ and $KO(X)$ indexed by the set of irreducible RG-spaces $\{V_{Ri}\}$, the set of irreducible CG-spaces $\{V_{Cj}\}$ and the set of irreducible QG-spaces $\{V_{Qk}\}$ respectively. Then we have

Theorem 5.2. *We have an isomorphism of additive groups:*

$$\Phi\colon \bigoplus_i KSp(X)_i \oplus \bigoplus_j K(X)_j \oplus \bigoplus_k KO(X)_k \to KSp_G(X).$$

Proof. Note that the index set of $KSp(X)$ is $\{V_{Ri}\}$ and the index set of $KO(X)$ is $\{V_{Qk}\}$. Since the proof is quite similar to that of Theorem 5.1, we omit it.

Corollary 5.3. *Let G be a group, all of whose irreducible representations are of R-type. Then we have an isomorphism of rings:*

$$KO_G(X) \cong RO(G) \otimes KO(X)$$

and an isomorphism of additive groups:

$$KSp_G(X) \cong RO(G) \otimes KSp(X).$$

Proof. Isomorphisms of additive groups follow from Theorems 5.1 and 5.2. The ring isomorphism in the case of $KO_G(X)$ is verified in the manner of the proof of $K_G(X)$ [4].

As remarked in [17], every irreducible representation of the Weyl group of a compact connected Lie group is of R-type. Hence we have

EXAMPLE 5.4. If G is the Weyl group of a compact connected Lie group, then we have the isomorphisms in Corollary 5.3.

EXAMPLE 5.5. Let Z_{p^n} be the cyclic group of an odd prime power order p^n. Denote by $\rho: Z_{p^n} \to U(1)$ be the representation defined by $\rho(t) = \exp(2\pi t\sqrt{-1}/p^n)$. Then we have

$$\{V_{Ri}\} = R\text{: the trivial representation,}$$
$$\{V_{Cj}\} = \{\rho, \rho^2, \cdots, \rho^{(p^n-1)/2}\},$$
$$\{V_{Qk}\} = \phi\text{: empty.}$$

It follows from Theorems 5.1 and 5.2 that

$$KO_{Z_{p^n}}(X) \cong KO(X) \oplus \overbrace{K(X) \oplus \cdots \oplus K(X)}^{(p^n-1)/2},$$
$$KSp_{Z_{p^n}}(X) \cong KSp(X) \oplus \overbrace{K(X) \oplus \cdots \oplus K(X)}^{(p^n-1)/2}.$$

EXAMPLE 5.6. Let I_* be the binary icosahedral group [26]. As is well-known, I_* is isomorphic to $SL(2,5)$. In view of [2] and [11], one verifies that

$$\{V_{Ri}\} = \{\rho_1, \cdots, \rho_5\},$$
$$\{V_{Cj}\} = \phi,$$
$$\{V_{Qk}\} = \{\rho_6, \cdots, \rho_9\}.$$

Hence we have

$$KO_{I_*}(X) \cong \overbrace{KO(X) \oplus \cdots \oplus KO(X)}^{5} \oplus \overbrace{KSp(X) \oplus \cdots \oplus KSp(X)}^{4},$$
$$KSp_{I_*}(X) \cong \overbrace{KSp(X) \oplus \cdots \oplus KSp(X)}^{5} \oplus \overbrace{KO(X) \oplus \cdots \oplus KO(X)}^{4}.$$

REMARK 5.7. For p an odd prime integer, we have

$$KO_{Z_p}(S^6) \cong \overbrace{Z \oplus \cdots \oplus Z}^{p},$$
$$RO(Z_p) \otimes KO(S^6) \cong \overbrace{Z \oplus \cdots \oplus Z}^{(p+1)/2},$$

$$K_{Z_p}(S^6) \cong R(Z_p)\otimes K(S^6) \cong \overbrace{Z\oplus\cdots\oplus Z}^{2p},$$

$$KSp_{Z_p}(S^6) \cong Z_2\oplus\overbrace{Z\oplus\cdots\oplus Z}^{p}.$$

Hence they are quite different. In particular,

$$KO_{Z_p}(S^6) \not\cong RO(Z_p)\otimes KO(S^6).$$

REMARK 5.8. A formula similar to that of Theorem 5.1 holds in the case where G is a compact Lie group with an involution [27]. This was shown to the author by the referee.

6. Bordism groups of GS-bundles and semi-free G-manifolds

Let S be a family of irreducible real representations of G. Let $\eta\to X$ be a real G-vector bundle over a trivial G-space X. Each fiber η_x over $x\in X$ may be regarded as a representation space. Then η is called a GS-bundle when each irreducible representation which appears in η_x belongs to S for every $x\in X$.

Let M_i^m be closed oriented manifolds with trivial G-action and $\xi_i\to M_i^m$ be real GS-bundles over M_i^m of fiber dimension k ($i=1, 2$). The ξ_i are *bordant* if there is a real GS-bundle $E\to W^{m+1}$ over a compact oriented manifold with trivial G-action satisfying the following conditions:

(i) there is a diffeomorphism $\partial W^{m+1}\cong M_1\cup -M_2$ preserving the orientation,

(ii) there are G-vector bundle isomorphisms $E|M_i\cong\xi_i$ ($i=1, 2$).

We refer to this relation as the *bordism* relation. Then the bordism relation is an equivalence relation on the class of GS-bundles. The resulting set $B(\Omega_m, R^k)$ (G, S) of equivalence classes is an abelian group with addition induced by the disjoint union. We call $B(\Omega_m, R^k)$ (G,S) the oriented bordism group of real GS-bundles.

For a finite subset $\rho(i)$ ($i=1, 2, \cdots, s$) of S, let $n(\rho(i))$ be positive integers indexed by $\rho(i)$. For a Lie group H, we denote by BH the classifying space of H.

Put

$$\Lambda(\rho(i)) = R,\ C,\ Q$$

and

$$B\Lambda(\rho(i))(n(\rho(i))) = BO(n(\rho(i))),\ BU(n(\rho(i))),\ BSp(n(\rho(i)))$$

according as $\rho(i)$ is of R-type, C-type, Q-type respectively. Denote by $\Omega_m(X)$ the oriented bordism group of X (see [7]). Then we have

Proposition 6.1. *There is an isomorphism*:

$$\Phi\colon \oplus\Omega_m(B\Lambda(\rho(1))(n(\rho(1)))\times\cdots\times B\Lambda(\rho(s))(n(\rho(s))))\to B(\Omega_m, R^k)(G, S)$$

where the direct sum is taken over all s, $\rho(i)\in S$, $n(\rho(i))$ *with*

$$\sum_{i=1}^{s}(\dim_R \rho(i))n(\rho(i)) = k\,.$$

Proof. An element of

$$\oplus\Omega_m(B\Lambda(\rho(1))(n(\rho(1)))\times\cdots\times B\Lambda(\rho(s))(n(\rho(s))))$$

is represented by

$$\oplus(\xi_{\rho(1)}^{n(\rho(1))}, \cdots, \xi_{\rho(s)}^{n(\rho(s))})$$

where $\xi_{\rho(i)}^{n(\rho(i))}$ are $\Lambda(\rho(i))$-vector bundles of fiber dimension $n(\rho(i))$ over a closed oriented manifold M_ρ^m. Then we set

$$\Phi(\oplus(\xi_{\rho(1)}^{n(\rho(1))}, \cdots, \xi_{\rho(s)}^{n(\rho(s))})) = \sum_i \oplus \underline{\rho(i)} \underset{\Lambda(\rho(i))}{\otimes} \xi_{\rho(i)}^{n(\rho(i))}\,.$$

The inverse map Φ^{-1} is given by the unique decomposition of G-vector bundles of Propositions 4.1 and 4.2.

Once we have correspondences Φ, Φ^{-1}, Proposition 6.1 is easily proven.

REMARK 6.2. For $A=\mathfrak{N}$, Ω, Ω^U, Ω^{Sp} and for $\Lambda=R$, C, Q, the bordism groups $B(A_m, \Lambda^k)(G, S)$ are defined similarly and those versions of Proposition 6.1 hold.

As the set S, we take for examples: the set of all irreducible representations, the set of non trivial irreducible representations, the set of fixed point free irreducible representations (see below).

DEFINITION 6.3. If ρ is a ΛG-representation and if $e\neq g\in G$ implies that $\rho(g)$ does not have $+1$ for an eigenvalue, then ρ is *fixed point free*. Let $F_\Lambda(G)$ denote the set of all equivalence classes of irreducible fixed point free ΛG-representations.

DEFINITION 6.4. A fixed point free group is a finite group which has a fixed point free Λ-representation.

It is easy to see that the definition does not depend on the choice of Λ. Fixed point free groups G and the set $F_C(G)$ are studied in [26] and the following theorem is deduced easily from [26].

Proposition 6.5. *The elements of* $F_\Lambda(G)$ *all have the same* $\Lambda(G)$*-type and the same degree* $d_\Lambda(G)$. *Moreover if* G *is not isomorphic to* Z_2, *then* $\Lambda(G)\neq R$.

As a typical case, we now consider bordism groups of weakly symplectic semi-free G-actions.

A *weakly symplectic structure* for a vector bundle ξ is a symplectic vector bundle structure on the stable bundle of ξ. A *weakly symplectic manifold* is a pair consisting of a differentiable manifold M and a weakly symplectic structure on the tangent bundle TM of M [19], [20], [24]. Then the weakly symplectic bordism group $\Omega_n^{Sp}(X)$ is defined as usual (Compare [7], [23]). A *weakly symplectic G-action* on a weakly symplectic manifold M is a G-action such that the differential $dg\colon TM \to TM$ is stably symplectic linear for all $g \in G$. Then the fixed point set F becomes canonically a weakly symplectic manifold and the normal bundle ν to F in M is given canonically a QG-bundle structure (Compare [8]). The isotropy group G_x of $x \in M$ is the subgroup $\{g \in G \mid gx = x\}$ of G. If $G_x = \{e\}$ (resp. $\{e\}$ or G) for all $x \in M$, the action is called free (resp. semi-free). The symplectic bordism group

$$\Omega_n^{Sp}(G, F) \quad (\text{resp. } \Omega_n^{Sp}(G, SF))$$

of free (resp. semi-free) symplectic G-manifolds is defined as the reader understands without ambiguity.

Recall the symplectic bordism group

$$B(\Omega_m^{Sp}, Q^k)\,(G, S)$$

of symplectic GS-bundles in Remark 6.2.

We now consider the case where S is the set $F_Q(G)$ of irreducible fixed point free QG-representations.

Put

$$BA(G)\,(n(\rho(i))) = BU(n(\rho(i))), \quad BO(n(\rho(i)))\,,$$

according as $A(G) = C, Q$ respectively.

Obviously for a finite group G, there are an isomorphism:

$$\Omega_n^{Sp}(G, F) \cong \Omega_n^{Sp}(BG)$$

and an exact sequence:

$$\cdots \to \Omega_n^{Sp}(G, F) \to \Omega_n^{Sp}(G, SF) \to \bigoplus_{m+4k=n} B(\Omega_m^{Sp}, Q^k)\,(G, F_Q(G))) \to \Omega_{n-1}^{Sp}(G, F) \to \cdots\,.$$

In view of Remark 6.2, we have

Proposition 6.6. *Let G be a fixed point free group which is not isomorphic to Z_2. Then we have the following exact sequence:*

$$\cdots \to \Omega_n^{Sp}(BG) \to \Omega_n^{Sp}(G, SF) \to \oplus\Omega_m^{Sp}(BA(G)\,(n(\rho(1))) \times \cdots \times BA(G)\,(n(\rho(s)))) \to \Omega_{n-1}^{Sp}(BG) \to \cdots\,,$$

where the summation is taken over all s, $\rho(i)\in F_Q(G)$, $n(\rho(i))$ *with*

$$m+4d_Q(G)\sum_{i=1}^{s} n(\rho(i)) = n .$$

REMARK 6.7. Although we dealt only with weakly symplectic case in Proposition 6.6, unoriented, oriented, weakly complex versions hold similarly.

REMARK 6.8. When a finite group G is not a fixed point free group, we have an isomorphism:

$$\Omega_n^{Sp}(G, SF) \cong \Omega_n^{Sp}(BG)\oplus\Omega_n^{Sp} .$$

REMARK 6.9. In case $G=S^1$, the exact sequence splits [25]. However the exact sequence in our case does not split in general.

EXAMPLE 6.10. Let I_* be the binary icosahedral group. According to [11], [26], I_* has two fixed point free representations of Q-type whose degree is 1. It follows from Proposition 6.6 that we have the following exact sequence:

$$\begin{aligned}&\to \Omega_n^{Sp}(BI_*) \to \Omega_n^{Sp}(I_*, SF) \to \\ &\bigoplus_{n_1,n_2} \Omega_{n-4(n_1+n_2)}^{Sp}(BO(n_1)\times BO(n_2)) \to \Omega_{n-1}^{Sp}(BI_*) \to \cdots\end{aligned}$$

7. Equivariant J-group $J_G(X)$

First we recall the definition of $J_G(X)$ [13], [15].

Let G be a compact topological group and X be a compact G-space. Let ξ and η be G-vector bundles over X. Denote by $S(\xi)$ (resp. $S(\eta)$) the sphere bundle associated with ξ (resp. η).

DEFINITION 7.1. $S(\xi)$ and $S(\eta)$ are said to be of the same G-fiber homotopy type if there exist fiber-preserving G-maps:

$$f\colon S(\xi)\to S(\eta)\,,\quad f'\colon S(\eta)\to S(\xi)$$

and fiber-preserving G-homotopies:

$$h\colon S(\xi)\times I\to S(\xi)\,,\quad h'\colon S(\eta)\times I\to S(\eta)$$

with

$$\begin{aligned}h\,|\,S(\xi)\times 0 &= f'\cdot f\,, & h\,|\,S(\xi)\times 1 &= \text{identity}\\ h'\,|\,S(\eta)\times 0 &= f\cdot f'\,, & h'\,|\,S(\eta)\times 1 &= \text{identity}\,.\end{aligned}$$

Let $KO_G(X)$ be the Grothendieck-Atiyah-Segal group [4] defined in terms of real G-vector bundles over X. Let $T_G(X)$ be the additive subgroup of $KO_G(X)$ generated by elements of the form $[\xi]-[\eta]$, where ξ and η are G-vector bundles whose associated sphere bundles are G-fiber homotopy equivalent.

DEFINITION 7.2. We define our equivariant J-group $J_G(X)$ by

$$J_G(X) = KO_G(X)/T_G(X)$$

and define our equivariant J-homomorphism J_G by the natural epimorphism

$$J_G\colon KO_G(X) \to J_G(X)\,.$$

When X is a point $*$, $J_G(*)$ is studied in [9], [10], [12], [14], [16] and [17]. Similar groups $JO(G)$ and $jO(G)$ are studied in [6], [18] and [22].

We now recall [13], [15].

Theorem 7.3. *Let M_1, M_2 be closed smooth G-manifolds. If there is a G-homotopy equivalence $f\colon M_1 \to M_2$, then*

$$J_G([TM_1]) = J_G([f^*TM_2])$$

where TM_i denote the tangent G-vector bundles of M_i $(i=1, 2)$.

Let $f\colon M_1 \to M_2$ be a G-homotopy equivalence. Denote by F_1^μ each component of the fixed point set of M_1. Set $F_2^\mu = f(F_1^\mu)$. Then the union $\bigcup_\mu F_2^\mu$ is the fixed point set of M_2 and each F_2^μ is a component of $\bigcup_\mu F_2^\mu$. Denote by N_i^μ the normal bundles of F_i^μ in M_i $(i=1, 2)$.

As a corollary to Theorem 7.3, we have

Corollary 7.4.

$$J_G([N_1^\mu]) = J_G([(f\,|\,F_1^\mu)^* N_2^\mu])\,.$$

Namely each $J_G([N_1^\mu])$ is a G-homotopy type invariance. The normal bundle N_1^μ is a G-vector bundle over a trivial G-space F_1^μ and from now on we will deal with $J_G\colon KO_G(X) \to J_G(X)$ in the case where X is a trivial G-space.

Denote by $\underline{\underline{\{0\}}}$ the zero dimensional vector bundle over X. Let ξ be a G-vector bundle over X and $\{\rho_i\}$ be the set of irreducible G-representations which appear in ξ. Denote by V_i the representation space of ρ_i. By Propositions 4.1 and 4.2, we have a unique decomposition

$$\xi = \bigoplus_i \underline{\underline{V_i}} \otimes_{\Lambda_i} \xi_i$$

where $\Lambda_i = \mathrm{End}_{RG}(V_i)$ and ξ_i are Λ_i-vector bundles. Let H be a normal subgroup of G. Then, ξ^H is a vector sub-bundle of ξ. Since H is a normal subgroup, ξ^H is even a G-vector bundle. Then we have

Lemma 7.5. *If H is a normal subgroup of G, then we have*

$$\xi^H = \bigoplus_{\mathrm{Ker}\rho_i \supset H} \underline{\underline{V_i}} \otimes_{\Lambda_i} \xi_i\,.$$

Proof. It is easy to see that

$$(\bigoplus_i \underline{\underline{V_i}} \underset{\Lambda_i}{\otimes} \xi_i)^H = \bigoplus_i \underline{\underline{V_i^H}} \underset{\Lambda_i}{\otimes} \xi_i$$

in general. Obviously we have

$$\xi^H \supset \bigotimes_{\mathrm{Ker}\rho_i \supset H} \underline{\underline{V_i}} \underset{\Lambda_i}{\otimes} \xi_i .$$

Suppose that they are different. Denote by ξ' the complementary G-vector sub-bundle of $\bigoplus_{\mathrm{Ker}\rho_i \supset H} \underline{\underline{V_i}} \underset{\Lambda_i}{\otimes} \xi_i$ in ξ^H, that is

$$\xi^H = (\bigoplus_{\mathrm{Ker}\rho_i \supset H} \underline{\underline{V_i}} \underset{\Lambda_i}{\otimes} \xi_i) \oplus \xi' .$$

Decompose ξ' as before:

$$\xi' = \bigoplus_i \underline{\underline{V_i}} \underset{\Lambda^i}{\otimes} \xi'_i .$$

It follows from the uniqueness of the decomposition that Ker ρ_i does not include H for i with $\xi'_i \neq \underline{\underline{\{0\}}}$. Hence H acts non-trivially on such V_i. It follows that

$$\xi'^H = \bigoplus_i \underline{\underline{V_i^H}} \underset{\Lambda_i}{\otimes} \xi'_i \neq \bigoplus_i \underline{\underline{V_i}} \underset{\Lambda_i}{\otimes} \xi'_i = \xi' .$$

This is a contradiction. Namely $\xi' = \underline{\underline{\{0\}}}$.

This completes the proof of Lemma 7.5.

Let ξ and η be real G-vector bundles over X. Denote by L the set of irreducible RG-representations which appear in ξ and η. Then we define a set $\{H_i\}$ of subgroups of G by

$$\{H_i \mid i = 1, \cdots, k\} = \{\mathrm{Ker}\ \rho \mid \rho \in L\} .$$

It is possible to arrange $\{H_i\}$ in such order that $H_i \supseteqq H_j$ implies $i \leqq j$. We classify the set L by kernels such that Ker $\rho_{it} = H_i$. Denote by V_{it} the representation space of ρ_{it}. Set

$$\Lambda_{it} = \mathrm{End}_{RG}(V_{it}) .$$

By Propositions 4.1 and 4.2, we have unique decompositions:

$$\xi = \bigoplus_i \bigoplus_t \underline{\underline{V_{it}}} \underset{\Lambda_{it}}{\otimes} \xi_{it} \quad \text{and} \quad \eta = \bigoplus_i \bigoplus_t \underline{\underline{V_{it}}} \underset{\Lambda_{it}}{\otimes} \eta_{it} .$$

Then we set

$$\xi^i = \xi \oplus \bigoplus_{s=1}^{i} \bigoplus_t \underline{\underline{V_{st}}} \underset{\Lambda_{st}}{\otimes} \eta_{st}$$

and

$$\eta^i = \eta \oplus \bigoplus_{s=1}^{i} \bigoplus_t \underline{\underline{V_{st}}} \underset{\Lambda_{st}}{\otimes} \xi_{st}\,.$$

Lemma 7.6. *If there exists a G-fiber homotopy equivalence $f\colon S(\xi)\to S(\eta)$, then there exist G/H_i-vector bundles α_i and G-vector bundles β_i over X for $i=1, \cdots, k$ such that*

(a)$_i$ $S(\bigoplus_t \underline{\underline{V_{it}}} \underset{\Lambda_{it}}{\otimes} \xi_{it} \oplus \alpha_i)$ and $S(\bigoplus_t \underline{\underline{V_{it}}} \underset{\Lambda_{it}}{\otimes} \eta_{it} \oplus \alpha_i)$ *are G/H_i-fiber homotopy equivalent,*

(b)$_i$ $S(\xi^i \oplus \beta_i)$ *and* $S(\eta^i \oplus \beta_i)$ *are G-fiber homotopy equivalent.*

Proof. We prove Lemma 7.6 by induction on i. Note that H_i is a normal subgroup of G and is maximal in $\{H_i, H_{i+1}, \cdots, H_k\}$. It follows from Lemma 7.5 that

$$\xi^{H_1} = \bigoplus_t \underline{\underline{V_{1t}}} \underset{\Lambda_{1t}}{\otimes} \xi_{1t}$$

and

$$\eta^{H_1} = \bigoplus_t \underline{\underline{V_{1t}}} \underset{\Lambda_{1t}}{\otimes} \eta_{1t}\,.$$

Note that the restriction

$$f^{H_1}\colon\ S(\xi)^{H_1} \to S(\eta)^{H_1}$$

is a G-fiber homotopy equivalence. We can also regard f^{H_1} as a G/H_1-fiber homotopy equivalence. Hence we get a G/H_1-fiber homotopy equivalence

$$f^{H_1}\colon\ S(\bigoplus_t \underline{\underline{V_{1t}}} \underset{\Lambda_{1t}}{\otimes} \xi_{1t}) \to S(\bigoplus_t \underline{\underline{V_{1t}}} \underset{\Lambda_{1t}}{\otimes} \eta_{1t})\,.$$

Thus we have $(a)_1$ by taking $\alpha_1 = \underline{\underline{\{0\}}}$. Let f_1' be a G-fiber homotopy inverse of f^{H_1}. Then the map

$$\begin{aligned} f*f_1'\colon\ S(\xi^1) &= S(\xi)*S\,(\bigoplus_t \underline{\underline{V_{1t}}} \underset{\Lambda_{1t}}{\otimes} \eta_{1t}) \\ \to S(\eta^1) &= S(\eta)*S\,(\bigoplus_t \underline{\underline{V_{1t}}} \underset{\Lambda_{1t}}{\otimes} \xi_{1t}) \end{aligned}$$

gives a G-fiber homotopy equivalence where $*$ denotes the join. Thus we have $(b)_1$ taking $\beta_1 = \{0\}$.

Suppose that Lemma 7.6 is true for all $j \leqq i$. By the induction hypothesis $(b)_i$, there is a G-fiber homotopy equivalence

$$f_i\colon\ S(\xi^i \oplus \beta_i) \to S(\eta^i \oplus \beta_i)\,.$$

Then the restriction

$$f_i^{H_{i+1}}\colon\ S(\xi^i \oplus \beta_i)^{H_{i+1}} \to S(\eta^i \oplus \beta_i)^{H_{i+1}}$$

is a G/H_{i+1}-fiber homotopy equivalence. In virtue of Lemma 7.5, we have

$$(\xi^i)^{H_{i+1}} = \bigoplus_t \underline{\underline{V_{i+1t}}} \otimes_{\Lambda_{i+1t}} \xi_{i+1t} \oplus \bigoplus_{H_s \supsetneqq H_{i+1}} \{\bigoplus_t \underline{\underline{V_{st}}} \otimes_{\Lambda_{st}} (\xi_{st} \oplus \eta_{st})\}$$

and

$$(\eta^i)^{H_{i+1}} = \bigoplus_t \underline{\underline{V_{i+1t}}} \otimes_{\Lambda_{i+1t}} \eta_{i+1t} \oplus \bigoplus_{H_s \supsetneqq H_{i+1}} \{\bigoplus_t \underline{\underline{V_{st}}} \otimes_{\Lambda_{st}} (\eta_{st} \oplus \xi_{st})\} .$$

We now set

$$\alpha_{i+1} = \bigoplus_{H_s \supsetneqq H_{i+1}} \{\bigoplus_t \underline{\underline{V_{st}}} \otimes_{\Lambda_{st}} (\xi_{st} \oplus \eta_{st})\} \oplus \beta_i^{H_{i+1}} .$$

Then we obtain a G/H_{i+1}-fiber homotopy equivalence

$$f_i^{H_{i+1}}\colon (S \bigoplus_t \underline{\underline{V_{i+1t}}} \otimes_{\Lambda_{i+1t}} \xi_{i+1t} \oplus \alpha_{i+1}) \to S(\bigoplus_t \underline{\underline{V_{i+1t}}} \otimes_{\Lambda_{i+1t}} \eta_{i+1t} \oplus \alpha_{i+1}) .$$

Thus we have $(a)_{i+1}$. Let f_i' be a G-fiber homotopy inverse of $f_i^{H_{i+1}}$. We now set

$$\beta_{i+1} = \beta_i \oplus \alpha_{i+1} .$$

Then we obtain a G-fiber homotopy equivalence

$$\begin{aligned} f_i * f_i'\colon S(\xi^{i+1} \oplus \beta_{i+1}) &= S(\xi^i \oplus \beta_i) * S(\bigoplus_t \underline{\underline{V_{i+1t}}} \otimes_{\Lambda_{i+1t}} \eta_{i+1t} \oplus \alpha_{i+1}) \\ \to S(\eta^{i+1} \oplus \beta_{i+1}) &= S(\eta^i \oplus \beta_i) * S(\bigoplus_t \underline{\underline{V_{i+1t}}} \otimes_{\Lambda_{i+1t}} \xi_{i+1t} \oplus \alpha_{i+1}) . \end{aligned}$$

Thus we have $(b)_{i+1}$.

This makes the proof of Lemma 7.6 complete.

REMARK 7.7. $(a)_i$ of Lemma 7.6 is what we need and $(b)_i$ is what we used in order to put forward the inductive step of $(a)_i$.

Denote by $\{H_\lambda\}$ the set of all Ker ρ where ρ are irreducible RG-representations. We classify the set of all irreducible RG-representations by the kernels such that Ker $\rho_{\lambda\mu} = H_\lambda$. It follows from Theorem 5.1 that we have a decomposition

$$KO_G(X) = \bigoplus_\lambda A_\lambda$$

corresponding to the set $\{H_\lambda\}$ where X is a trivial G-space. Moreover we can regard A_λ as a subgroup of $KO_{G/H_\lambda}(X)$. Then we have

Proposition 7.8.

$$J_G(X) \cong \bigoplus_\lambda J_G(A_\lambda) \cong \bigoplus_\lambda J_{G/H_\lambda}(A_\lambda) .$$

Proof. For the first isomorphism, it suffices to prove that

$$\operatorname{Ker} J_G = \bigoplus_\lambda \operatorname{Ker} (J_G | A_\lambda) .$$

Obviously $\operatorname{Ker} J_G \supset \bigoplus_\lambda \operatorname{Ker}(J_G|A_\lambda)$. On the other hand, Lemma 7.6 $(a)_i$ means nothing but

$$\operatorname{Ker} J_G \subset \bigoplus_\lambda \operatorname{Ker}(J_G|A_\lambda) .$$

Furthermore Lemma 7.6 $(a)_i$ means that

$$\operatorname{Ker}(J_G|A_\lambda) = \operatorname{Ker}(J_{G/H_\lambda}|A_\lambda) .$$

This completes the proof of Proposition 7.8.

For a compact topological group G, we define a subset $\lambda(G)$ (resp. $\tilde{\lambda}(G)$) of $\{R, C, Q\}$ by

$$\{\operatorname{End}_{RG}(V) \mid V\text{: irreducible } RG\text{-space}\}$$
$$(\text{resp. } \{\operatorname{End}_{RG}(V) \mid V\text{: non-trivial irreducible } RG\text{-space}\}) .$$

We define $KA(X)$ for $A=R$, C and Q by $KR(X)=KO(X)$, $KC(X)=K(X)$ and $KQ(X)=KSp(X)$. Denote by $\widetilde{RO}(G)$ the subgroup of $RO(G)$ generated by non-trivial irreducible RG-representations. Then we set

$$\tilde{J}_G(*) = J_G(\widetilde{RO}(G)) .$$

By making use of Propositions 4.1 and 4.2, we deduce easily the following

Proposition 7.9. (i) *If $KA(X)\cong Z$ for all $A\in\lambda(G)$, then $J_G(X)\cong J_G(*)$.* (ii) *If $KA(X)\cong Z$ for all $A\in\tilde{\lambda}(G)$, then $J_G(X)\cong J(X)+\tilde{J}_G(*)$.*

Denote by Z_n the cyclic group Z/nZ of order n where n is an integer greater than one. Let $n=2^k\cdot p_1^{r(1)}\cdots p_t^{r(t)}$ be the prime decomposition of n. Then we define a group $J'_{Z_n}(*)$ as follows.

Case 1. $k\geqq 2$. We set

$$J'_{Z_n}(*) = Z\oplus Z_{2^{k-2}}\oplus \bigoplus_{i=1}^t Z_{(p_i^{r(i)}-p_i^{r(i)-1})}$$

Case 2. $k=0$ or 1. We set

$$J'_{Z_n}(*) = Z\oplus \{\bigoplus_{i=1}^t Z_{(p_i^{r(i)}-p_i^{r(i)-1})}\}/Z_2$$

where the inclusion of Z_2 into $\bigoplus_{i=1}^t Z_{(p_i^{r(i)}-p_i^{r(i)-1})}$ is given by $1\mapsto \bigoplus_{i=1}^t (p_i^{r(i)}-p_i^{r(i)-1})/2$.

Let G be a compact abelian topological group and F_0, F_1 and F_2 be the family of all closed subgroups H of G such that G/H is isomorphic to the circle S^1, Z_n for some $n>2$ and Z_2 respectively. For a set F and for an abelian group H, we denote by $H(F)$ the direct sum of copies of H indexed by F. Let S^n be the n-dimensional sphere with trivial G-action.

Then we have

Corollary 7.10. *We have the following isomorphisms:*

$$J_G(S^{2n+1}) \cong \begin{cases} Z\oplus Z(F_0)\oplus \bigoplus_{H\in F_1} J'_{G/H}(*)\oplus Z(F_2) & \text{for } n\not\equiv 0, \bmod 4, \\ Z\oplus Z_2\oplus Z(F_0)\oplus \bigoplus_{H\in F_1} J'_{G/H}(*)\oplus (Z\oplus Z_2)(F_2) & \text{for } n\equiv 0, \bmod 4. \end{cases}$$

Proof. It is easy to see that any irreducible representation of a compact abelian topological group is either of R-type or of C-type. Moreover R-type occurs only in the form: $\rho\colon G\to O(1)$. If $n\not\equiv 0 \bmod 4$, then $KO(S^{2n+1})\cong Z$ and $K(S^{2n+1})\cong Z$. Hence the isomorphism in this case follows from Proposition 7.9 (i) and the result of [14]. If $n\equiv 0 \bmod 4$, then $KO(S^{2n+1})\cong Z\oplus Z_2$ and $K(S^{2n+1})\cong Z$. Hereafter we assume that $n\equiv 0 \bmod 4$. Let A (resp. B) be the subgroup of $KO_G(S^{2n+1})$ generated by the elements of the form $\rho\underset{=R}{\otimes}\xi$ (resp. $\rho\underset{=C}{\otimes}\xi$) where ρ is of R-type (resp. of C-type) and ξ is a real (resp. complex) vector bundle. Since the kernel of ρ which appears in A and that of ρ which appears in B are different, we have a direct sum decomposition by Proposition 7.8,

$$J_G(S^{2n+1}) \cong J_G(A)\oplus J_G(B).$$

Concerning B, an argument similar to the proof of Proposition 7.9 is valid, since $K(S^{2n+1})\cong Z$. Concerning A, we have only to prove that

$$J_{Z_2}\colon KO_{Z_2}(S^{2n+1}) \to J_{Z_2}(S^{2n+1})$$

is an isomorphism. As is well-known, $J\colon KO(S^{2n+1})\to J(S^{2n+1})$ is an isomorphism [1]. It follows that $J(S^{2n+1})\cong Z\oplus Z_2$. Denote by $\alpha\colon Z_2\to O(1)$ the non-trivial irreducible representation. Suppose that $S(\xi_1\oplus\alpha\underset{=R}{\otimes}\xi_2)$ and $S(\eta_1\oplus\alpha\underset{=R}{\otimes}\eta_2)$ are Z_2-fiber homotopy equivalent where ξ_i and η_i are real vector bundles. Restricting to the fixed point set and forgetting the Z_2-action, we have that

$$J(\xi_1) = J(\eta_1) \quad\text{and}\quad J(\xi_1\oplus\xi_2) = J(\eta_1\oplus\eta_2).$$

Hence $J(\xi_2)=J(\eta_2)$. Since J is an isomorphism in this case,

$$J_{Z_2}\colon KO_{Z_2}(S^{2n+1}) \to J_{Z_2}(S^{2n+1})$$

is also an isomorphism.

This completes the proof of Corollary 7.10.

EXAMPLE 7.11. Z_{p^n} be the cyclic group of an odd prime power order p^n.

Recall Example 5.5. If $K(X)\cong Z$, then we have by Proposition 7.9 (ii) and [14] that

$$J_{Z_{p^n}}(X) \cong I(X) \oplus \bigoplus_{i=1}^{n} (Z \oplus Z_{(p^i-p^{i-1})/2}) .$$

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Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan