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CONDITIONED CONTINUITY PROPERTIES OF THE N-PARAMETER WIENER PROCESS

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1. Introduction

We consider an N-parameter Wiener process $\{w^d(t): t\in R_+^N\}$ with values in R^d (see Definition 2.3); the parameter space R_+^N is the subset of points of R^N with all components nonnegative. We call a subset Δ of R_+^N an "*interval*", if Δ is the product of N one-dimensional intervals, and denote its volume by $|\Delta|$. Let l be a fixed integer. Our problem is the asymptotic behavior of increments $W^d(\Delta)$ over intervals Δ (see Definitions 2.2 and 2.3) as $|\Delta|\downarrow 0$ under the restriction that the ratios of length of the l-th shortest sides of Δ to the shortest sides are bounded. In case $l=1$, this restriction is trivial and Orey-Pruitt [3] already took up the problem and derived integral tests for the uniform continuity and the local one-sided growth. In connection with [5], however, we need integral tests for the growth rate of $w^d(\Delta)$ in case that $l=N$, and accordingly, when the ratios of length of the longest sides to the shortest sides are bounded. The growth rate in this case is different from the rate given in [3]. Motivated by this, we study the general case that $1\le l\le N$, and observe the relation between such restrictions and the asymptotic behaviors under restrictions.

Our results are stated in Section 2. Theorem 2.1 answers the question for the uniform continuity. In case $l=1$, this reduces to a result of [3]. Theorem 2.2 deals with the local two-sided growth problem in the sense of Jain-Taylor [2]. Section 3 contains proofs of the theorems. The outline of proof is the same as that of [3]; however, the manner of discretizing the unit interval $[0, 1]^N$ is simpler.

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2. Continuity properties

We begin with some definitions. Let $(\Omega, \mathcal{B}, P)$ be a complete probability space.

DEFINITION 2.1. *An N-parameter Wiener process* $\{w(t, \omega): t\in R_+^N\}$ is to be a separable real-valued Gaussian process with mean 0 and covariance

$$E[w(s)w(t)] = \prod_{\mu=1}^{N} s_\mu \wedge t_\mu ,$$

for $s=\langle s_1, \cdots, s_N\rangle$ and $t=\langle t_1, \cdots, t_N\rangle$ of R_+^N.

We sometimes write simply $t=\langle t_\mu\rangle$ for t of R_+^N, and denote the product of N one-dimensional intervals (s_μ, t_μ) by $\Delta(s, t)$ for $s=\langle s_\mu\rangle$ and $t=\langle t_\mu\rangle$ with $s_\mu \leq t_\mu$. Let $\alpha>1$ and be fixed. For $1\leq l\leq N$, a class $Q(l)$ of intervals is defined as follows: an interval $\Delta(s, t)$ belongs to $Q(l)$ if and only if $\Delta(s, t)$ is included in$[0,1]^N$ and

$$0< l\text{-}\min_\mu (t_\mu - s_\mu) \leq \alpha \min_\mu (t_\mu - s_\mu) ,$$

where l-min denotes the l-th smallest value. For example, if $a_1\leq a_2\leq \cdots \leq a_N$ then $l\text{-}\min_\mu a_\mu = a_l$.

Definition 2.2. The *increment* $w(\Delta(s, t))$ over an interval $\Delta(s, t)$ is defined by

$$\begin{aligned} w(\Delta(s, t)) = w(t) &- \sum_{\mu=1}^{N} w(\langle t_1, \cdots, s_\mu, \cdots, t_N\rangle) \\ &+ \sum_{1\leq \mu_1<\mu_2\leq N} w(\langle t_1, \cdots, s_{\mu_1}, \cdots, s_{\mu_2}, \cdots, t_N\rangle) \\ &- \cdots + (-1)^N w(s) . \end{aligned}$$

In this paper we consider an N-parameter Wiener process in R^d.

Definition 2.3. *An N-parameter Wiener process* $\{w^d(t, \omega): t\in R_+^N\}$ *in* R^d is defined by

$$w^d(t, \omega)=(w_1(t, \omega), \cdots, w_d(t, \omega)), \qquad t\in R_+^N ,$$

where $w_i(t)$, $i=1, \cdots, d$, are N-parameter Wiener processes, and they are independent. Furthermore its *increment* $w^d(\Delta)$ over an interval Δ is defined by

$$w^d(\Delta) = (w_1(\Delta), \cdots, w_d(\Delta)) .$$

Let $|\cdot|$ denote the N-dimensional Lebesgue measure and $||\cdot||$ denote the d-dimensional Euclidean norm. Our main results are the following two theorems.

Theorem 2.1. *Let ϕ be a nonnegative, non-increasing, continuous function defined for small pcsitive arguemenis. For almost all ω there exists $\delta(\omega)>0$ such that for all intervals Δ of $Q(l)$ with $|\Delta|\leq\delta(\omega)$,*

$$||w^d(\Delta, \omega)|| \leq |\Delta|^{1/2}\phi(|\Delta|) \tag{2.1}$$

if and only if the integral

$$\int_{+0} x^{-2}(\log 1/x)^{N-l}\phi^{4N+d-2}(x) \exp(-\phi^2(x)/2)dx \tag{2.2}$$

converges.

Theorem 2.2. *Let ϕ be a nonnegative, non-increasing, continuous function defined small positive arguements and let u be a fixed point of* $(0, 1)^N$. *For almost all ω there exists $\delta(\omega)>0$ such that for all intervals Δ of $Q(l)$ with $u\in\Delta$, $|\Delta|\leq\delta(\omega)$,*

$$\|w^d(\Delta, \omega)\| \leq |\Delta|^{1/2}\phi(|\Delta|) \tag{2.3}$$

if and only if the integral

$$\int_{+0} x^{-1}(\log 1/x)^{N-l}\phi^{4N+d-2}(x) \exp(-\phi^2(x)/2)dx \tag{2.4}$$

converges.

In addition to the above theorems, we mention a result about the local one-sided growth.

Theorem 2.3. *Let ϕ be a nonnegative, non-increasing, continuous function defined for small positive arguements and s be a fixed point in* $[0, 1)^N$. *For almost all ω there exists $\delta(\omega)>0$ such that for all intervals $\Delta(s, t)$ of $Q(l)$ with $|\Delta(s,t)|\leq \delta(\omega)$,*

$$\|w^d(\Delta(s, t), \omega)\| \leq |\Delta(s, t)|^{1/2}\phi(|\Delta(s, t)|)$$

if and only if the integral

$$\int_{+0} x^{-1}(\log 1/x)^{N-l}\phi^{2N+d-2}(x) \exp(-\phi^2(x)/2)\, dx$$

converges.

In case $l=1$, this theorem reduces to a result of Orey-Pruitt [3]. From this theorem we can derive information about the asymptotic behavior of $w^d(t)$ as $t\to\infty$, by using the following property: $\{w(t): t_\mu>0, \mu=1, \cdots, N\}$ and $\{|\Delta(t)|w(\langle t_\mu^{-1}\rangle): t_\mu>0, \mu=1, \cdots, N\}$ have the same distribution, where $\Delta(t)=\Delta(0, t)$, $0=\langle 0, \cdots, 0\rangle$. Let $Q'(l)$ be a class of intervals $\Delta(t)$ with $t_\mu\geq 1$, $\mu=1, \cdots, N$, and

$$\max_\mu t_\mu \leq \alpha\ l\text{-}\max_\mu t_\mu < \infty ,$$

where l-max denotes the l-th largest value.

Corollary. *Let ψ be a nonnegative, non-decreasing, continuous function defined for large arguements. For almost all ω there exists $M(\omega)>0$ such that for all t of R_+^N with $\Delta(t)\in Q'(l)$, $|\Delta(t)|\geq M(\omega)$,*

$$\|w^d(t, \omega)\| \leq |\Delta(t)|^{1/2}\psi(|\Delta(t)|)$$

if and only if the integral

$$\int^{+\infty} x^{-1}(\log x)^{N-l}\psi^{2N+d-2}(x)\exp(-\psi^2(x)/2)dx$$

converges.

3. Proofs

We deal with mainly the proof of Theorem 2.1. Our arguements follow closely Orey-Pruitt [3], and the main point of proof that requires some care is how to discretize the unit interval $[0, 1]^N$ for each problem. For Theorem 2.2 we only refer to a few relevant differences. As for Theorem 2.3 its proof is a mere variant of the proof of Theorem 2.2, so we omit it. In the following we use the convenient practice of letting c stand for unimportant positive constants which may even change from line to line.

Proof of Theorem 2.1. We assume that the integral (2.2) converges. Let $i=(i_1, \cdots, i_N)$, $m=(m_{l+1}, \cdots, m_N)$. Define the time sets

$$\begin{aligned} K(i, m, p) = \{(s, t)\in R_+^N\times R_+^N:\ & 2^{-p-1}\leq t_1-s_1\leq 2^{-p}\,, \\ & 2^{-p-1}\leq t_\mu-s_\mu\leq \alpha 2^{-p}, \qquad \mu=2, \cdots, l, \\ & 2^{-m_\mu-1}\leq t_\mu-s_\mu\leq 2^{-m_\mu}, \qquad \mu=l+1, \cdots, N\,, \\ & i_\mu 2^{-p-1}\leq t_\mu\leq (i_\mu+1)2^{-p-1}, \qquad \mu=1, \cdots, l\,, \\ & i_\mu 2^{-m_\mu-1}\leq t_\mu\leq (i_\mu+1)2^{-m_\mu-1}, \qquad \mu=l+1, \cdots, N\} \end{aligned}$$

and the events

$$E(i, m, p)=\{\omega: \sup_{(s,t)\in K(i,m,p)} \|w^d(\Delta(s, t), \omega)\|\,|\Delta(s, t)|^{-1/2}>\phi(\alpha^{l-1}2^{-r(m,p)})\}\,,$$

where $r(m, p)=lp+\sum_{\mu=l+1}^{N} m_\mu$. The parameters will be restricted to the following ranges:

$$\begin{aligned} & 0\leq i_\mu\leq 2^{p+1}-1, \qquad \mu=1, \cdots, l\,, \\ & 0\leq i_\mu\leq 2^{m_\mu+1}-1, \qquad \mu=l+1, \cdots, N\,, \\ & 0\leq m_\mu\leq p, \qquad \mu=l+1, \cdots, N\,,\ p\geq 3\,. \end{aligned}$$

Since $E[w(\Delta)w(\Delta')]=|\Delta\cap\Delta'|$ for any intervals Δ, Δ' by the definition of increment, it is easy to check that

$$\begin{aligned} & E[\{w(\Delta(s, t))\,|\Delta(s, t)|^{-1/2}-w(\Delta(s', t'))\,|\Delta(s', t')|^{-1/2}\}^2] \\ & \leq \alpha^{l-1}2^{N+1}\{\sum_{\mu=1}^{l}(|t_\mu-t'_\mu|+|s_\mu-s'_\mu|)2^p \\ & \qquad +\sum_{\mu=l+1}^{N}(|t_\mu-t'_\mu|+|s_\mu-s'_\mu|)2^{m_\mu}\} \end{aligned}$$

holds for all (s, t), (s', t') of $K(i, m, p)$. Define a metric λ in $K(i, m, p)$ by

$$\lambda((s,t),(s',t')) = \{\sum_{\mu=1}^{l}(|t_\mu-t'_\mu|^2+|s_\mu-s'_\mu|^2)2^{2p} + \sum_{\mu=l+1}^{N}(|t_\mu-t'_\mu|^2+|s_\mu-s'_\mu|^2)2^{2m_\mu}\}^{1/2}.$$

Using this metric λ, let $N(\varepsilon; B, \lambda)$ be the minimal number of sets of diameter at most 2ε which cover B. Then it holds that for $\eta^2=\alpha^{l-1}2^{N+1}\sqrt{2N}$, $N((2\eta^2a^2)^{-1}; K(i,m,p),\lambda)\leq c\,a^{4N}$ for $a>0$, where c does not depend on i, m and p. Applying Lemma 2.3 of [5] to $\{w^d(\Delta(s,t))/|\Delta(s,t)|^{1/2}; (s,t)\in K(i,m,p)\}$ and $(K(i,m,p),\lambda)$, we have

$$P(E(i,m,p))\leq c'\phi^{4N+d-2}(\alpha^{l-1}2^{-r(m,p)})\exp\{-\phi^2(\alpha^{l-1}2^{-r(m,p)})/2\},$$

where c' does not depend on i, m and p. Therefore

$$\sum_{i,m,p}P(E(i,m,p)) \leq c\sum_{m,p}2^{r(m,p)}\phi^{4N+d-2}(\alpha^{l-1}2^{-r(m,p)})\exp\{-\phi^2(\alpha^{l-1}2^{-r(m,p)})/2\}.$$

There are at most r^{N-l} ways of choosing m_μ, $\mu=l+1,\cdots,N$, and p to accomplish $r(m,p)=r$. Thus

$$\sum_{i,m,p}P(E(i,m,p)) \leq c\sum_{r=3}^{\infty}2^r r^{N-l}\phi^{4N+d-2}(\alpha^{l-1}2^{-r})\exp\{-\phi^2(\alpha^{l-1}2^{-r})/2\}.$$

This sum is seen to converge by comparison with the integral (2.2). We can easily verify (2.1) by using the Borel-Cantelli lemma and the same arguement as in [3].

We now assume that the integral (2.2) diverges. It is sufficient to prove the theorem for ϕ satisfying

$$(\log 1/x)^{1/2}\leq\phi(x)\leq 2(\log 1/x)^{1/2}. \tag{3.1}$$

This is proved in the same way as in Sirao [4], so we do not repeat it. Let $i=(i_1,\cdots,i_N)$, $j=(j_1,\cdots,j_N)$, $k=(k_1,\cdots,k_N)$ and $m=(m_{l+1},\cdots,m_N)$. Define the events

$$F(i,j,k,m,p)=\{\omega: \|w^d(\Delta(s,t),\omega)\|>|\Delta(s,t)|^{1/2}\phi(|\Delta(s,t)|)\}$$

where

$$s_\mu=\begin{cases}(i_\mu+j_\mu/p)2^{-p}, & \mu=1,\cdots,l,\\ (i_\mu+j_\mu/p)2^{-m_\mu}, & \mu=l+1,\cdots,N,\end{cases}$$

$$t_\mu=\begin{cases}(i_\mu+k_\mu/p)2^{-p}, & \mu=1,\cdots,l,\\ (i_\mu+k_\mu/p)2^{-m_\mu}, & \mu=l+1,\cdots,N.\end{cases}$$

The parameters will be restricted as follows:

$$\begin{aligned}&0\le i_\mu\le 2^p-1\,, \qquad \mu=1,\cdots,l\,,\\&0\le i_\mu\le 2^{m_\mu}-1\,, \qquad \mu=l+1,\cdots,N\,,\\&0\le j_\mu\le p/10\,,\quad 9p/10\le k_\mu\le p\,, \qquad \mu=1,\cdots,N\,,\\&1\le m_\mu\le p\,, \qquad \mu=l+1,\cdots,N,\ \textit{and}\ \ p\ge 3\,.\end{aligned}$$

The above intervals $\Delta(s, t)$ belong to $Q(l)$ under the assumption that $\alpha\ge 2$. (To do without this assumption, only a slight modification of the ranges of j and k is needed.)

We apply the extension of the Borel-Cantelli lemma by Chung-Erdös to prove the latter part of the theorem. First, we verify the condition (i) of Theorem 1 of [1]. Since $|\Delta(s, t)|\sim c2^{-r(m,p)}$, a well known estimation ([3], Lemma 1.1, p. 141) leads to

$$P(F(i, j, k, m, p))\ge c\phi^{d-2}(2^{-r(m,p)})\exp\{-\phi^2(2^{-r(m,p)-1})/2\}\,,$$

where $r(m, p)=lp+\sum_{\mu=l+1}^{N} m_\mu$. On the other hand, the condition (3.1) ensures us the existence of positive constants a_1, a_2, independent of i, j, k, m and p, such that

$$a_1p^{1/2}\le\phi(|\Delta(s, t)|)\le a_2p^{1/2}\,, \tag{3.2}$$

where $\Delta(s, t)$ is the interval involved in $F(i, j,k, m, p)$. Thus

$$\begin{aligned}&\sum_{i,j,k,m,p} P(F(i, j, k, m, p))\\&\qquad\ge c\sum_{m,p} 2^{r(m,p)}\phi^{4N+d-2}(2^{-r(m,p)})\exp\{-\phi^2(2^{-r(m,p)-1})/2\}\,.\end{aligned}$$

There are at least $c'\, r^{N-l}$ ways of choosing m_μ, $l+1\le\mu\le N$, and p to accomplish $r(m, p)=r$. Therefore

$$\begin{aligned}&\sum_{i,j,k,m,p} P(F(i, j, k, m, p))\\&\qquad\ge c\sum_{r\ge 3N} 2^r r^{N-l}\phi^{4N+d-2}(2^{-r})\exp(-\phi^2(2^{-r})/2)\,.\end{aligned}$$

This sum is easily seen to diverge by comparison with the integral (2.2). Thus the condition (i) of Theorem 1 (extended Borel-Cantelli lemma) of [1] has been verified. Next, to verify the condition (iii) of the theorem by Chung-Erdös, we order the events so that the volume of the involved interval decreases. Fix an event $F=F(i, j, k, m, p)$ and let $F'=F(i', j', k', m', p')$ be an event following F in the order. We write $\Delta=\Delta(s, t)$ and $\Delta'=\Delta(s', t)$ for the intervals involved in F and F' respectively. Let $\rho=E[w_1(\Delta)w_1(\Delta')]$. Since by (3.2),

$$a_1^2(pp')^{1/2}\le\phi(|\Delta|)\phi(|\Delta'|)\le a_2^2(pp')^{1/2}\,,$$

Lemma 1.5 of [3] shows that $P(F\cap F')\leq cP(F)\,P(F')$, if $\rho^2<(a_2^4pp')^{-1}$. Thus as the events E_{ji} in (iii) of Theorem 1 of [1] (with taking F as E_j) we choose the events F' which give rise to values of ρ satisfying

$$(a_2^4pp')^{-1}\leq\rho^2\,. \tag{3.3}$$

We can easily see, using the same arguements as in [3], that there are at most $O(p^{5N-l}\log p)$ intervals which satisfy (3.3). In order to verify the condition (2) of (iii) of Theorem of [1], we divide the sum $\sum P(F\cap F')$ over F' satisfying (3.3) into $\sum_{(1)}P(F\cap F')$ and $\sum_{(2)}P(F\cap F')$, where $\sum_{(1)}$ means the summation over F' which satisfy (3.3) and

$$\rho^2\leq 1-p^{-1/2}\,, \tag{3.4}$$

and $\sum_{(2)}$ means the summation over F' which satisfy (3.3) and

$$\rho^2\geq 1-p^{-1/2}\,. \tag{3.5}$$

If F' satisfies (3.4), then by Lemma 1.6 of [3], $P(F\cap F')\leq c'\exp(-cp^{1/2})P(F)$. This shows that $\sum_{(1)}P(F\cap F')\leq c''P(F)$, since $p^{5N-l}\log p\exp(-cp^{1/2})=O(1)$. To estimate $\sum_{(2)}P(F\cap F')$, we subdivide the condition (3.5) as follows:

$$1-q/p\leq\rho^2\leq 1-(q-1)/p\,,\qquad q=1,\cdots,p^{1/2}\,. \tag{3.6}$$

Then, as before, we have by Lemma 1.6 of [3], $P(F\cap F')\leq c'\exp(-cq)P(F)$. We can also show that the number of events F' satisfying (3.6) is $O(q^{2N})$, by a variant of the arguement in [3] and the fact that

$$|k'_\mu-k_\mu|=O(q)\,,\quad |j'_\mu-j_\mu|=O(q)\,,\qquad \mu=1,\cdots,N\,. \tag{3.7}$$

Thus $\sum_{(2)}P(F\cap F')\leq c'\sum_{q=1}^{\infty}q^{2N}\exp(-cq)P(F)\leq c''P(F)$. This verifies the condition (2) of (iii). To check (3.7), it suffices to consider the next two cases:

(a) $$s_\mu\leq s'_\mu<t_\mu\leq t'_\mu\,,$$

(b) $$s_\mu\leq s'_\mu<t'_\mu\leq t_\mu\,.$$

Using the inequality

$$\rho^2=\frac{|\Delta\cap\Delta'|^2}{|\Delta|\,|\Delta'|}\leq\frac{(t_\mu\wedge t'_\mu-s_\mu\vee s'_\mu)^2}{(t_\mu-s_\mu)(t'_\mu-s'_\mu)} \tag{3.8}$$

we can easily obtain the estimates (3.7).

Finally, we verify the condition (ii) of Theorem 1 of [1], using the same arguements as in Orey-Pruitt [3]. The events $F(i,j,k,m,p)$ have been ordered so that $|\Delta(s,t)|$ decreases, and we use a single subscript. Then the events F_n are of the form $\{\omega\colon ||U_n||\geq c_n\}$. To compare $P(F_m\,|\,F_h^c\cdots F_n^c)$ with $P(F_m)$, since $F_h^c,\cdots,F_n^c$ are of the form $\{\omega\colon ||U_k||<c_k\}$, we replace F_m by

$$G_m = \{\omega : c_m < ||U_m|| \leq 2c_m\} ,$$

and use the well-known estimate

$$P(F_m) \leq 2P(G_m) .$$

To apply Lemma 4 of [4], it remains to check that

$$\rho_m = \max_{1 \leq i \leq n} E[U_i U_m] \to 0 , \qquad \text{as } m \to \infty ,$$

and $2c_m \leq \rho_m^{-\gamma}$, for some $\gamma < 1$. Let p' be the p parameter corresponding to the event F_m and p'' be larger than the p parameters corresponding to $F_h, \cdots, F_n$. Then we have

$$\rho_m \leq 2^{(p''-p'+1)/2} ,$$

by (3.7). This implies that $\rho_m \to 0$ as $m \to \infty$, and that

$$2c_m = 2\phi(|\Delta(s, t)|) \leq cp'^{1/2} \leq \rho_m^{-1/2} ,$$

for sufficiently large p', where $\Delta(s, t)$ denotes the interval involved in F_m. Thus the proof of Theorem 2.1 has been completed.

Next we give a brief proof for Theorem 2.2.

Proof of Theorem 2.2. We assume that the integral (2.4) converges. Let $m = (m_{l+1}, \cdots, m_N)$ and define the time sets

$$\begin{aligned} K(m, p) = \{(s, t) \in R_+^N \times R_+^N :& \\ & 2^{-p-1} \leq t_1 - u_1 \leq 2^{-p+1} ,\ 0 \leq u_1 - s_1 \leq 2^{-p} , \\ & 2^{-p-1} \leq t_\mu - u_\mu \leq \alpha 2^{-p+1} ,\ 0 \leq u_\mu - s_\mu \leq \alpha 2^{-p} , \qquad \mu = 2, \cdots, l , \\ & 2^{-m_\mu - 1} \leq t_\mu - u_\mu \leq 2^{-m_\mu + 1} ,\ 0 \leq u_\mu - s_\mu \leq 2^{-m_\mu} , \qquad \mu = l+1, \cdots, N\} \end{aligned}$$

and the events

$$E(m, p) = \{\omega : \sup_{(s,t) \in K(m,p)} ||w^d(\Delta(s, t), \omega)|| \, |\Delta(s, t)|^{-1/2} > \phi(3^N \alpha^{l-1} 2^{-lp \Sigma m_\mu})\} .$$

The parameters will be restricted as follows:

$$1 \leq m_\mu \leq p , \qquad \mu = l+1, \cdots, N , \ p \geq 3 .$$

Define a metric λ in $K(m, p)$ by

$$\lambda((s, t), (s', t')) = \{\sum_{\mu=1}^{l} (|t_\mu - t'_\mu|^2 + |s_\mu - s'_\mu|^2) 2^{2p} + \sum_{\mu=l+1}^{N} (|t_\mu - t'_\mu|^2 + |s_\mu - s'_\mu|^2) 2^{2m_\mu}\}^{1/2} .$$

Then the proof goes as before.

We now assume that the integral (2.4) diverges. It suffices to prove the theorem for ϕ satisfying

$$(\log\log 1/x)^{1/2}\leq\phi(x)\leq 2N(\log\log 1/x)^{1/2}.$$

Let $j=(j_1,\cdots,j_N)$, $k=(k_l,\cdots,k_N)$ and $m=(m_{l+1},\cdots,m_N)$. Define the events

$$F(j,k,m,p)=\{\omega:\|w^d(\Delta(s,t),\omega)\|>|\Delta(s,t)|^{1/2}\phi(|\Delta(s,t)|)\}$$

where

$$s_\mu=\begin{cases}(u_\mu-j_\mu/\log p)2^{-p-1}, & \mu=1,\cdots,l,\\ (u_\mu-j_\mu/\log p)2^{-m_\mu-1}, & \mu=l+1,\cdots,N,\end{cases}$$

$$t_\mu=\begin{cases}(u_\mu+k_\mu/\log p)2^{-p-1}, & \mu=1,\cdots,l,\\ (u_\mu+k_\mu/\log p)2^{-m_\mu-1}, & \mu=l+1,\cdots,N.\end{cases}$$

The parameters will be restricted as follows:

$$\begin{aligned}&9\log p\ 10\leq j_\mu,\ k_\mu\leq\log p, \qquad \mu=1,\cdots,N,\\ &1\leq m_\mu\leq p,\ \mu=l+1,\cdots,N, \qquad p\geq 3.\end{aligned}$$

We have, here, assumed that $1/4<u_\mu<3/4$, $\mu=1,\cdots,N$. This is, however, not essential and it is clear how to do without this assumption. Then the proof goes as before except that p is replaced by $\log p$ in many estimates; this is done, for example, in (3.2)~(3.6).

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