ON RIEMANNIAN MANIFOLDS ADMITTING CERTAIN
STRICTLY CONVEX FUNCTIONS

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1. Introduction. Let $M$ be an $m$-dimensional connected complete Riemannian manifold with metric $g$. For a smooth function $f$ on $M$, the Hessian $D^2f$ of $f$ is defined by $D^2f(X,Y) = X(Yf) - D_XYf(X,Y) \in TM$. By a theorem of H.W. Wissner ([5; Satz. II. 1.3]), if there is a smooth function $f$ on $M$ such that $D^2f = g$ on $M$, then $M$ is isometric to Euclidean space. In this note, we shall prove that if the Hessian of a smooth function $f$ on $M$ is close enough to $g$, then $M$ is quasi-isometric to Euclidean space in the following sense: There exists a diffeomorphism $F: M \to \mathbb{R}^m$ and some positive constant $\mu$ such that for each tangent vector $X$ on $M$, $\mu^{-1}||X||_M \leq ||F_*X||_{\mathbb{R}^m} \leq ||X||_M$. Our result contains the above theorem by Wissner as a special case ($\mu=1$), and generalises Yagi's theorem ([7]). Our theorem is stated as follows.

Theorem. Let $M$ be an $m$-dimensional connected complete Riemannian manifold with metric $g$. Suppose there exists a smooth function $f$ on $M$ which satisfies the following conditions:

(i) $(1 - H_i(f(x)))g(X,X) \leq \frac{1}{2} D^2f(X,Y) \leq (1 + H_i(f(x)))g(X,X)$, where $X \in T_xM(x \in M)$ and each $H_i \ (i=1, 2)$ is a nonnegative continuous function on $\mathbb{R}$,

(ii) $1 - H_i(t) > 0$ for $t \in \mathbb{R}$ and $\lim_{t \to \infty} H_i(t) = 0 (i = 1, 2)$,

(iii) $\left\{ \begin{array}{l}
\int_0^\infty H_i(s)/s \ ds < +\infty, \\
\int_{-\infty}^\infty \left( \int_0^s H_i(u)/u \ du \right) \ ds < +\infty \ (i = 1, 2).
\end{array} \right.$

Then $M$ is quasi-isometric to Euclidean space.

2. Proof of theorem and corollaries. Let $M$ be an $m$-dimensional connected complete Riemannian manifold with metric $g$.

Lemma 1. Let $M$ and $g$ be as above. Let $f$ be a smooth function on $M$ such
that the eigenvalues of $D^2 f$ are bounded from below by some positive constant $2\nu$ outside a compact subset $C$. Then $f$ is an exhaustion function, that is, \( \{ x \in M : f(x) \leq t \} \) is compact for each $t \in \mathbb{R}$. In particular, $f$ takes the minimum on $M$.

Proof. Suppose $f \equiv \lambda = \inf \{ f(x) : x \in M \} \ ( -\infty \leq \lambda < \infty )$. Then there is a divergent sequence $\{ p_n \}_{n \in \mathbb{N}}$ in $M$ with $\lim f(p_n) = \lambda$. Fix any point $o \in M$. Let $\gamma_n : [0, 1] \to M$ be a minimizing geodesic joining $o$ to $p_n$ for each $n \in \mathbb{N}$, where $1_n = \text{dis}(o, p_n)$. Then by the assumption of Lemma 1, we can choose sufficiently large $N$ and $T$ so that $\gamma_n(t) \in M - C$ and $f(\gamma_n)^\prime(t) = D^2 f(\gamma_n, \gamma_n)(t) \geq \nu$ for any $n \geq N$ and $t \in [T, 1]$. This implies $f(\gamma_n)(t) \geq f(\gamma_n)(T) + f(\gamma_n)'(T)(t - T) + \frac{\nu}{2}(t - T)^2$ for $t \in [T, 1]$. Taking $t = 1$, we have $f(p_n) \geq f(\gamma_n)(T) + f(\gamma_n)'(T)(1_n - T) + \frac{\nu}{2}(1_n - T)^2$. Since the distance between $o$ and $\gamma_n(T)$ equals $T$ for each $n \geq N$, $\{ f(\gamma_n)'(T) \}_{n \in \mathbb{N}}$ is bounded. In the preceding inequality, the left side tends to $\lambda$ and the right side goes to infinity as $n \to \infty$. This is a contradiction. Therefore $f$ takes the minimum at some points. By the same way, we see that $f$ is an exhaustion function on $M$. This completes the proof of Lemma 1.

Proof of Theorem. By (ii) in Theorem and Lemma 1, we can see that $f$ is a strictly convex exhaustion function on $M$. Let $\lambda$ be the minimum of $f$ on $M$ and $o \in M$ be the only one point such that $f(o) = \lambda$. Set $\tilde{f}(x) = f(x) - \lambda$, $k(x) = f(x)^{1/2}$, and $h_i(t) = H_i(t^2 + \lambda) \ (i = 1, 2)$. Then the conditions (i)~(iii) in Theorem can be rewritten as follows:

(i)' $$(1 - h_i(k(x)))g(X, X) \leq \frac{1}{2} D^2 \tilde{f}(X, X) \leq (1 + h_i(k(x)))g(X, X),$$

where $X \in T_\nu M \ (x \in M)$ and each $h_i(t) \ (i = 1, 2)$ is a nonnegative continuous function on $[0, \infty)$,

(ii)' $1 - h_i(t) > 0$ for $t \in [0, \infty)$ and $\lim_{t \to \infty} h_i(t) = 0 \ (i = 1, 2)$

(iii)' $$\int_0^\infty h_i(s) ds < +\infty \quad \text{and} \quad \int_0^\infty (u \cdot h_i(u) du/s^3) ds < +\infty \quad (i = 1, 2).$$

Since $o \in M$ is a nondegenerate critical point of $\tilde{f}$, there exists a coordinate system $x: U \to \mathbb{R}^n$, where $U$ is a neighborhood of $o$, with $x(o) = (0, \ldots, 0)$ and $\tilde{f}(p) = \sum_{i=1}^n x_i(p)^2$ for all $p \in U$ where $x(p) = (x_1(p), \ldots, x_n(p))$ (cf. [3; p. 6]). Let $\delta$ be a positive number such that $\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ \sum_{i=1}^n x_i^2 < \delta \} \subset x(U)$. We construct a metric $\bar{g}$ on $M$ with $\bar{g}\left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \delta_{ij}$ on $U_{\delta/2} = \{ p \in U \ : \ \sum_{i=1}^n x_i(p)^2 < \frac{\delta}{2} \}$ and
\( g = g \) on \( M - U \); such a metric can be constructed by the standard partition-of-
unity extension process. By the construction of \( g \), it suffices to prove that
\( M \) with the metric \( g \) is quasi-isometric to Euclidean space. Let \( a > 0 \) be
such that \( k^{-1}(a) = \{ x \in M : k(x) = a \} \subset U_b \). For each \( p \in k^{-1}(a) \), let \( \lambda_p(t) \) be the
maximal integral curve of \( \nabla k/||\nabla k||^2 \) with \( \lambda_p(a) = p \). Then we have
\[
\frac{d}{dt} k(\lambda_p(t)) = 1 \quad \text{and hence} \quad k(\lambda_p(t)) = t \quad (t > 0).
\]
Define \( F_1 : k^{-1}(a) \times (0, \infty) \to M - \{0\} \) by \( F_1(p, t) = \lambda_p(t) \), and \( F_2 : k^{-1}(a) \times (0, \infty) \to \mathbb{R}^m - \{(0, \ldots, 0)\} \) by \( F_2(p, t) = \frac{t}{a}(x_1(p), \ldots, x_m(p)) \). It follows that \( F_1 \) and \( F_2 \) are diffeomorphisms and \( F_2 \circ F_1^{-1} \) can be
extended to the diffeomorphism \( F : M \to \mathbb{R}^m \) (cf. [3; p. 221]). We shall now show
that \( F \) is a required quasi-isometry. Let \( \lambda : [0, \varepsilon] \to k^{-1}(a) \) be any smooth reg-
ular curve. Define a smooth map \( G : [0, \varepsilon] \times [0, \infty) \to M - \{0\} \) by \( G(t, s) = F_1 \)
(\( \lambda(t), s \)), and vector fields \( X \) and \( Y \) along \( G \) by \( X = F^*(\frac{\partial}{\partial t}) = \nabla k/||\nabla k||^2 \) and
\( Y = F^*(\frac{\partial}{\partial s}) \). Fix any \( b > 0 \) such that \( k^{-1}(b) \subset M - U_b \). Then we have
for \( s \geq b \),
\[
\begin{align*}
(1) \quad \frac{\partial}{\partial s} ||Y||(t, s) &= (D_X Y)/||Y||(t, s) \\
&= (D_Y X, Y)/||Y||(t, s) \\
&= (D_Y (\nabla k, Y)/||Y|| ||\nabla k||^2 (t, s) \\
&= D^2k(Y, Y)/||Y|| ||\nabla k||^2 (t, s)
\end{align*}
\]
By the assumption \((i)'\), we have on \( \{ x \in M : k(x) \geq b \} \)
\[
\{(1-h(t)) \cdot g - dk \cdot dk \}/k \leq D^2k \leq \{(1 + h(t)) \cdot g - dk \cdot dk \}/k.
\]
Therefore we get
\[
(2) \quad \{(1-h(t)) ||k||^2 \} (t, s) \leq D^2k(Y, Y)(t, s) \leq \{(1 + h(t)) ||k||^2 \}(t, s)
\]
for \( s \geq b \). Now we need the following

Lemma 2. On \( \{ x \in M : k(x) \geq b \} \), we have the following estimate:
\[
1 - 2 \int_s^b u \cdot h(u) \ du/||k^2 + (B - 4b^2)/4k^2 \leq
\]
\[
||\nabla k||^2 \leq 1 + 2 \int_b^t u \cdot h(u) \ du/||k^2 + (C - 4b^2)/4k^2 ,
\]
where \( B = \min \{ ||\nabla f||^2(x) : x \in k^{-1}(b) \} \) and \( C = \max \{ ||\nabla f||^2(x) : x \in k^{-1}(b) \} \).
We leave a proof of this lemma later. By (1), (2) and (3), we have
\[
(4) \quad (1 - X_1(s))/s ||Y|| \leq (t, s) \leq \frac{\partial}{\partial s} ||Y||(t, s) \leq (1 + X_2(s))/s ||Y|| \leq (t, s) ,
\]
where \( \chi_1(s) = \left( 8 \int_0^s u \cdot h_2(u) \, du + 4s^2 h_1(s) + C - 4b^2 \right) / \left( 8 \int_0^s u \cdot h_2(u) \, du + 4s^2 + C - 4b^2 \right) \)

and \( \chi_2(s) = \left( 8 \int_0^s u \cdot h_2(u) \, du + 4s^2 h_2(s) - B + 4b^2 \right) / \left( -8 \int_0^s u \cdot h_2(u) \, du + 4s^2 + B - 4b^2 \right) \).

It follows that

\[
\frac{s}{b} \exp \left( \int_b^s -\chi_1(u) \, du \right) \leq \|Y\|(t, s)/\|Y\|(t, b) \leq \frac{s}{b} \exp \left( \int_b^s \chi_2(u) \, du \right).
\]

By the assumption (iii)', there exists some positive constant \( \xi \) such that

\[
\exp \left( \int_b^s \chi_2(u) \, du \right) \leq \xi \quad \text{and} \quad \xi^{-1} \leq \exp \left( \int_b^s -\chi_1(u) \, du \right).
\]

By (5) and (6), we get

\[
\xi^{-1} \|Y\|(t, 0)/b \leq \|Y\|(t, s)/\xi \leq \xi \|Y\|(t, 0)/b.
\]

The assumption (iii)' implies that for some positive constant \( \zeta \)

\[
\xi^{-1} \leq \|\text{grad} \, k\| \leq \xi.
\]

Inequalities (7) and (8) show that \( F : M \rightarrow R^m \) is quasi-isometric. This completes the proof of Theorem.

Proof of Lemma 2. For each \( p \in k^{-1}(b) \), let \( \gamma_p(t) \) be the maximal integral curve of \( \text{grad} \, \bar{f} / \|\text{grad} \, \bar{f}\|^2 \) with \( \gamma_p(0) = p \). Then \( \frac{d}{dt} \bar{f}(\gamma_p(t)) = 1 \) and hence \( \bar{f}(\gamma_p(t)) = t + b^2 \) \( (t \geq 0) \). From the assumption (i)', we obtain the inequality:

\[
(1 - h_1(k)) \|\text{grad} \, \bar{f}\|^2 \leq \frac{1}{2} D^2 \bar{f}(\text{grad} \, \bar{f}, \text{grad} \, \bar{f}) \leq (1 + h_2(k)) \|\text{grad} \, \bar{f}\|^2.
\]

on \( \{x \in M : k(x) \geq b\} \). Noting \( D^2 \bar{f}(\text{grad} \, \bar{f}, \text{grad} \, \bar{f}) = \frac{1}{2} \text{grad} \, \bar{f}(\|\text{grad} \, \bar{f}\|^2) \) we see

\[
4(1 - h_1(\sqrt{t + b^2})) \leq \frac{d}{dt} \|\text{grad} \, \bar{f}\|^2(\gamma_p(t)) \leq 4(1 + h_2(\sqrt{t + b^2}))
\]

for \( t \geq 0 \). Therefore we get the inequalities:

\[
8 \int_b^{\sqrt{t + b^2}} u(1 - h_2(u)) \, du + \|\text{grad} \, \bar{f}\|^2(p) \leq \|\text{grad} \, \bar{f}\|^2(\gamma_p(t)) \leq 8 \int_b^{\sqrt{t + b^2}} u(1 + h_2(u)) \, du + \|\text{grad} \, \bar{f}\|^2(p)
\]

for \( t \geq 0 \). Since \( k(\gamma_p(t)) = \sqrt{t + b^2} \) and \( \text{grad} \, \bar{f} = 2 \, \text{grad} \, k \) we get the required
estimate. This completes the proof of Lemma 2.

By Moser's theorem ([4]), we have the following

**Corollary 1.** Let $M$ be as in Theorem. Then on $M$ there are no positive harmonic functions other than constants. If $M$ is in addition a Kaehler manifold, then it has no nonconstant bounded holomorphic functions.

We shall derive the theorem of Wissner ([5; Satz. II. 1.3]) as follows.

**Corollary 2.** If $M$ is a connected complete Riemannian manifold and $f$ is a smooth function on $M$ whose Hessian is equal to the metric tensor on $M$, then $M$ is isometric to Euclidean space.

Proof. By Lemma 1 and the strictly convexity of $f$, $f$ attains its minimum $\lambda$ at the one and only one point $o \in M$. Replacing $f$ for $\frac{1}{2} (f - \lambda)$, we may assume that $f$ is a smooth function such that $D^2 f = \frac{1}{2} g$ on $M$ and $f(x) \equiv f(o) = 0$ for any $x \in M - \{o\}$. Let $\gamma : [0, \infty) \to M$ be any arc-length parametrized geodesic issuing from $o$. Then $D^2 f(\dot{\gamma}, \dot{\gamma}) = f(\gamma(t))'' = \frac{1}{2}$ for $t \geq 0$ and hence $f(\gamma(t)) = t^2$.

That is, $f(x)$ equals $\text{dist}(x, o)^2$ near $o \in M$. Therefore the same arguments as in the proof of Theorem can be applied without any change of metric and we see that the exponential mapping at $o \in M$ is an isometry. This completes the proof of Corollary 2.

**Example.** Let $M$ be $C^m$ with the Kaehler metric $g$ defined by $g_{ij} = \frac{\partial^2}{\partial z_i \partial \overline{z}_j} (|z|^2 + \log(1 + |z|^2))$, where $(z_1, \ldots, z_m)$ is the canonical holomorphic coordinates on $C^m$ and $|z|^2 = |z_1|^2 + \cdots + |z_m|^2$. Then $M$ is a Kaehler manifold with a pole $o = (0, \ldots, 0)$, that is, the exponential mapping at $o$ induces a global diffeomorphism between $T_o M$ and $M$. Let $r(x)$ be the distance between $o$ and $x \in M$. By computing the (radial) curvatures, we can see $r^2$ satisfies all the conditions required in Theorem (cf. [1; Theorem C]).

**Corollary 3 ([7]).** Let $M$ be a Riemannian manifold with a pole $o \in M$ and $r(x)$ be the distance between $o$ and $x \in M$. Suppose there exists a continuous function $h : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(i) $(1 - h(r(x))) g(X, X) \leq \frac{1}{2} D^2 r(X, X) \leq (1 + h(r(x))) g(X, X)$

where $X \in T_x M (x \in M)$, and

(ii) $\int_1^\infty h(t) t \, dt < +\infty$. 
Then the exponential mapping at $o \in M$ is quasi-isometric.

Proof. Noting $||\text{grad} \, r|| = 1$ on $M - \{o\}$, we see the result easily from the proof of Theorem.

References


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