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SPECTRA OF LAPLACE-BELTRAMI OPERATORS ON $SO(n+2)/SO(2)\times SO(n)$ AND $Sp(n+1)/Sp(1)\times Sp(n)$

CHIAKI TSUKAMOTO

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Introduction. Let $M=G/K$ be a compact symmetric space with G compact and semisimple. We assume that the Riemannian metric on M is the metric induced from the Killing form sign-changed. We consider the Laplace-Beltrami operator Δ^p acting on p-forms and its spectrum $\mathrm{Spec}^p(M)$.

Ikeda and Taniguchi [3] computed $\mathrm{Spec}^p(M)$ for $M=S^n$ and $P^n(C)$, studying representations of G and K. They showed that $\Delta^p=-$Casimir operator when we consider the space of p-forms $C^\infty(\Lambda^p M)$ as a G-module. Each irreducible G-submodule of $C^\infty(\Lambda^p M)$ is included in some eigenspace of Δ^p and the sum of irreducible G-submodules of $C^\infty(\Lambda^p M)$ equals to the sum of eigenspaces of Δ^p. We can compute eigenvalues from Freudenthal's formula and multiplicities from Weyl's dimension formula. Thus to compute $\mathrm{Spec}^p(M)$, we have only to decompose $C^\infty(\Lambda^p M)$ into irreducible G-submodules and count out them.

But generally it is not easy. Though Beers and Millman [1] determined $\mathrm{Spec}^p(M)$ when M is a Lie group of a low rank such as $SU(3)$ or $SO(5)$ by the similar method, these seem to be all we know.

Frobenius' reciprocity law enables us to reduce the problem into the following two: How does an irreducible G-module decompose into irreducible K-modules? How does the p-th exterior product of (complexified) cotangent space decompose into irreducible K-modules? The former is usually called a branching law.

In this paper, we give a branching law for $G=SO(n+2)$ and $K=SO(2)\times SO(n)$, which enables us to compute $\mathrm{Spec}^p(M)$. As a matter of fact, we should distinguish between the case $n=$odd and the case $n=$even. Almost in parallel, we get a branching law for $G=Sp(n+1)$ and $K=Sp(1)\times Sp(n)$, which reproduces the result of Lepowsky [4] obtained in a different way.

The latter problem, i.e., the decomposition of an exterior power of an isotropy representation is a rather technical (but indispensable) part in computing $\mathrm{Spec}^p(M)$. We give a complete list of members in the decomposition for $G=SO(n+2)$ and $K=SO(2)\times SO(n)$. For $G=Sp(n+1)$ and $K=Sp(1)\times Sp(n)$, we confine ourselves to indicating a procedure to determine the decomposition and giving lists for some n and p.

Throughout this paper, modules are assumed to be over the complex number field C.

1. Branching laws

We state branching laws in terms of highest weights.

We denote by $M(n,C)$ the set of all $n\times n$-matrices of complex coefficients.

Let $G=SO(n+2)$ and $K=SO(2)\times SO(n)$. We adopt the following conventions:

$$\begin{aligned}
\mathfrak{g} &= \mathfrak{o}(n+2,C) = \{X\in M(n+2,C);\, {}^tX+X=0\}\,,\\
\mathfrak{k} &= \mathfrak{o}(2,C)\times\mathfrak{o}(n,C)\\
&= \left\{\begin{pmatrix} X & 0\\ 0 & Y\end{pmatrix};\ \begin{matrix} X\in M(2,C),\ {}^tX+X=0\\ Y\in M(n,C),\ {}^tY+Y=0\end{matrix}\right\},\\
\mathfrak{t} &= \left\{\begin{pmatrix} R(\lambda_0) & & & & \\ & R(\lambda_1) & & & \\ & & \cdots\cdots & & \\ & & & R(\lambda_m) & \\ & & & & (0)\end{pmatrix};\ \begin{matrix} R(\lambda)=\begin{pmatrix} 0 & -\sqrt{-1}\lambda\\ \sqrt{-1}\lambda & 0\end{pmatrix}\\ \lambda_i\in C\end{matrix}\right\},
\end{aligned}$$

where $n=2m$ or $n=2m+1$. Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ and also one of $\mathfrak{k}$. We regard λ_i as a form on $\mathfrak{t}$ giving the value of λ_i. We take a Weyl chamber for $(\mathfrak{g}, \mathfrak{t})$ so that the simple roots of $\mathfrak{g}$ are $\alpha_0=\lambda_0-\lambda_1$, $\alpha_1=\lambda_1-\lambda_2$, $\cdots$, $\alpha_{m-1}=\lambda_{m-1}-\lambda_m$, and $\alpha_m=\lambda_{m-1}+\lambda_m$ when $n=2m$, $\alpha_m=\lambda_m$ when $n=2m+1$. We take a Weyl chamber for $(\mathfrak{k}, \mathfrak{t})$ so that the simple roots of $\mathfrak{k}$ are those of $\mathfrak{g}$ excluding α_0.

We first treat the case $n=2m$.

Any dominant integral form for $(\mathfrak{g}, \mathfrak{t})$ which corresponds to an irreducible representation of $G=SO(2m+2)$ is uniquely expressed as

$$\Lambda = h_0\lambda_0+h_1\lambda_1+\cdots+h_{m-1}\lambda_{m-1}+\varepsilon h_m\lambda_m\,, \tag{1.1}$$

where $\varepsilon=1$ or -1 and $h_0, h_1, \cdots, h_m$ are integers satisfying

$$h_0\geqslant h_1\geqslant\cdots\geqslant h_{m-1}\geqslant h_m\geqslant 0\,. \tag{1.2}$$

Any dominant integral form for $(\mathfrak{k}, \mathfrak{t})$ which corresponds to an irreducible representation of $K=SO(2)\times SO(2m)$ is uniquely expressed as

$$\Lambda' = k_0\lambda_0+k_1\lambda_1+\cdots+k_{m-1}\lambda_{m-1}+\varepsilon' k_m\lambda_m\,, \tag{1.3}$$

where $\varepsilon'=1$ or -1 and $k_0, k_1, \cdots, k_m$ are integers satisfying

$$k_1\geqslant\cdots\geqslant k_{m-1}\geqslant k_m\geqslant 0\,. \tag{1.4}$$

For integers $h_0, h_1, \cdots, h_m$ and $k_1, k_2, \cdots, k_m$, we define integers $l_0, l_1, \cdots, l_m$ by

$$l_0 = h_0 - \max(h_1, k_1)\,, \tag{1.5}$$
$$l_i = \min(h_i, k_i) - \max(h_{i+1}, k_{i+1}) \quad \text{for } 1 \leqslant i \leqslant m-1\,,$$
$$l_m = \min(h_m, k_m)\,.$$

Theorem 1.1. *Let $G=SO(2m+2)$ and $K=SO(2)\times SO(2m)$. Let Λ be the highest weight of an irreducible G-module V. Then the irreducible decomposition of V as a K-module contains an irreducible K-module V' with the highest weight Λ' if and only if;*

a)
$$h_{i-1} \geqslant k_i \geqslant h_{i+1} \quad \textit{for } 1 \leqslant i \leqslant m-1\,,$$
$$h_{m-1} \geqslant k_m \ (\geqslant 0)\,,$$

expressing Λ and Λ' as (1.1) *and* (1.3), *and*

b) the coefficient of X^{k_0} in the (finite) power series expansion in X of

$$X^{\varepsilon\varepsilon' l_m}\left(\prod_{i=0}^{m-1}((X^{l_i+1} - X^{-l_i-1})/(X - X^{-1}))\right)$$

does not vanish.

Moreover, the number of the times V' appearing in the decomposition is equal to the coefficient of X^{k_0} in the expansion.

REMARK. Suppose a) is satisfied. Then all the integers $l_0, l_1, \cdots, l_m$ are non-negative and all the coefficients in the power series are also non-negative.

The proof is given in the next section.

Next we treat the case $n=2m+1$.

Any dominant integral form for $(\mathfrak{g}, \mathfrak{t})$ which corresponds to an irreducible representation of $G=SO(2m+3)$ is uniquely expressed as

$$\Lambda = h_0\lambda_0 + h_1\lambda_1 + \cdots + h_m\lambda_m\,, \tag{1.6}$$

where $h_0, h_1, \cdots, h_m$ are integers satisfying (1.2). Any dominant integral form for $(\mathfrak{k}, \mathfrak{t})$ which corresponds to an irreducible representation of $K=SO(2)\times SO(2m+1)$ is uniquely expressed as

$$\Lambda' = k_0\lambda_0 + k_1\lambda_1 + \cdots + k_m\lambda_m\,, \tag{1.7}$$

where $k_0, k_1, \cdots, k_m$ are integers satisfying (1.4).

In this case we also define integers $l_0, l_1, \cdots, l_m$ by (1.5).

Theorem 1.2. *Let $G=SO(2m+3)$ and $K=SO(2)\times SO(2m+1)$. Let Λ be the highest weight of an irreducible G-module V. Then the irreducible decomposition of V as a K-module contains an irreducible K-module V' with the highest weight Λ' if and only if;*

a) $$h_{i-1}\geqslant k_i\geqslant h_{i+1} \text{ for } 1\leqslant i\leqslant m-1\,,$$
$$h_{m-1}\geqslant k_m\ (\geqslant 0)\,,$$

expressing Λ *and* Λ' *as* (1.6) *and* (1.7), *and*

b) the coefficient of X^{k_0} *in the (finite) power series expansion in* X *of*

$$(X-X^{-1})^{-m}(\textstyle\prod_{i=0}^{m-1}(X^{l_i+1}-X^{-l_i-1}))(X^{1/2}-X^{-1/2})^{-1}(X^{l_m+1/2}-X^{-l_m-1/2})$$

does not vanish.

Moreover, the number of the times V' *appearing in the decomposition is equal to the coefficient of* X^{k_0} *in the expansion.*

REMARK. Suppose a) is satisfied. Then all the integers $l_0, l_1, \cdots, l_m$ are non-negative and all the coefficients in the power series are also non-negative.

For the sake of completeness we state the branching law for $G=Sp(m+1)$ and $K=Sp(1)\times Sp(m)$. We adopt the following convensions:

$$\mathfrak{g} = \mathfrak{sp}(m+1, C)$$
$$= \left\{\begin{pmatrix} X & Z \\ Y & -{}^tX\end{pmatrix};\ \begin{matrix} X, Y, Z\in M(m+1, C) \\ {}^tY = Y,\ {}^tZ = Z\end{matrix}\right\},$$
$$\mathfrak{k} = \mathfrak{sp}(1, C)\times\mathfrak{sp}(m, C)$$
$$= \left\{\begin{pmatrix} x & 0 & z & 0 \\ 0 & X & 0 & Z \\ y & 0 & -x & 0 \\ 0 & Y & 0 & -{}^tX\end{pmatrix};\ \begin{matrix} x, y, z\in C \\ X, Y, Z\in M(m, C) \\ {}^tY = Y,\ {}^tZ = Z\end{matrix}\right\},$$
$$\mathfrak{t} = \{\mathrm{diag}(\lambda_0, \lambda_1, \cdots, \lambda_m, -\lambda_0, -\lambda_1, \cdots, -\lambda_m);\ \lambda_i\in C\}\,.$$

Then $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ and also one of $\mathfrak{k}$. We regard λ_i as a form on $\mathfrak{t}$. We take a Weyl chamber for $(\mathfrak{g}, \mathfrak{t})$ so that the simple roots of $\mathfrak{g}$ are $\alpha_0=\lambda_0-\lambda_1$, $\alpha_1=\lambda_1-\lambda_2, \cdots, \alpha_{m-1}=\lambda_{m-1}-\lambda_m$, $\alpha_m=2\lambda_m$. We take a Weyl chamber for $(\mathfrak{k}, \mathfrak{t})$ so that the simple roots of $\mathfrak{k}$ are $\alpha_0'=2\lambda_0$ and α_i $(1\leqslant i\leqslant m)$.

Since G and K are simply connected, each representation of their Lie algebras can be lifted to a group representation. Hence each dominant integral form corresponds to an irreducible representation and vice versa.

Any dominant integral form for $(\mathfrak{g}, \mathfrak{t})$ is uniquely expressed as (1.6), where $h_0, h_1, \cdots, h_m$ are integers satisfying (1.2). Any dominant integral form for $(\mathfrak{k}, \mathfrak{t})$ is uniquely expressed as (1.7), where $k_0, k_1, \cdots, k_m$ are integers satisfying (1.4) and $k_0\geqslant 0$. We again define integers $l_0, l_1, \cdots, l_m$ by (1.5).

Theorem 1.3 (Lepowsky). *Let* $G=Sp(m+1)$ *and* $K=Sp(1)\times Sp(m)$. *Let* Λ *be the highest weight of an irreducible* G*-module* V. *Then the irreducible decomposition of* V *as a* K*-module contains an irreducible* K*-module* V' *with the* *highest weight* Λ' *if and only if*;

a) $$h_{i-1} \geqslant k_i \geqslant h_{i+1} \text{ for } 1 \leqslant i \leqslant m-1\,,$$
$$h_{m-1} \geqslant k_m \ (\geqslant 0)\,,$$

expressing Λ *and* Λ' *as* (1.6) *and* (1.7), *and*
b) *the coefficient of* X^{k_0+1} *in the* (*finite*) *power series expansion in* X *of*

$$(X - X^{-1})^{-m} (\textstyle\prod_{i=0}^{m} (X^{l_i+1} - X^{-l_i-1}))$$

does not vanish.

Moreover, the number of the times V' *appearing in the decomposition is equal to the coefficient of* X^{k_0+1} *in the expansion.*

REMARK. Suppose a) is satisfied. Then all the integers $l_0, l_1, \cdots, l_m$ are non-negative and all the coefficients of X^k $(k>0)$ are also non-negative. The coefficient of X^{-k} is equal to the negation of the coefficient of X^k.

2. Proof of branching laws

Let G be a compact connected semisimple Lie group, K a closed subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the complexified Lie algebras of G and K. We assume that $\mathfrak{g}$ contains a Cartan subalgebra $\mathfrak{t}$ which is also a Cartan subalgebra of $\mathfrak{k}$.

We consider a group algebra over Z generated by an additive group of integral forms for $(\mathfrak{g}, \mathfrak{t})$ and one for $(\mathfrak{k}, \mathfrak{t})$. Since an integral form for $(\mathfrak{g}, \mathfrak{t})$ is also integral for $(\mathfrak{k}, \mathfrak{t})$, the group algebra for $(\mathfrak{g}, \mathfrak{t})$ is included in the group algebra for $(\mathfrak{k}, \mathfrak{t})$.

A formal character of a G-module V is an element of the group algebra for $(\mathfrak{g}, \mathfrak{t})$ defined by the formal sum of all the weights of V. (See, for example, Humphreys [2].) For an irreducible G-module V with the highest weight Λ, we denote its formal character by $\chi_G(\Lambda)$. We do the same for a K-module.

In terms of formal characters, a branching law for G and K means to determine the set S (which counts multiplicities) in the following formula:

(2.1) $$\chi_G(\Lambda) = \Sigma \chi_K(\Lambda') \quad (\Lambda' \in S)\,,$$

where Λ is a dominant integral form for $(\mathfrak{g}, \mathfrak{t})$ and Λ' is one for $(\mathfrak{k}, \mathfrak{t})$.

We will rewrite (2.1). Let W_G be the Weyl group of $(\mathfrak{g}, \mathfrak{t})$ acting on integral forms. We denote by $e(\Lambda)$ a generator of the group algebra corresponding to an integral form Λ. We define $\xi_G(\Lambda)$ by

$$\xi_G(\Lambda) = \Sigma(-1)^{\sigma} e(\sigma\Lambda) \quad (\sigma \in W_G)\,.$$

We set $\delta_G = (\Sigma\alpha)/2$ $(\alpha \in \Delta_G^+)$, where Δ_G^+ denotes the set of positive roots of $\mathfrak{g}$.

Then, by Weyl's character formula, we have

$$\xi_G(\Lambda+\delta_G) = \xi_G(\delta_G)\cdot\chi_G(\Lambda)\,.$$

We get in parallel

$$\xi_K(\Lambda'+\delta_K) = \xi_K(\delta_K)\cdot\chi_K(\Lambda')\,.$$

Now (2.1) is reduced to

$$\xi_G(\Lambda+\delta_G)\cdot\xi_K(\delta_K) = \xi_G(\delta_G)\cdot\Sigma\xi_K(\Lambda'+\delta_K) \quad (\Lambda'\in S)\,. \tag{2.2}$$

Our task is to devide $\xi_G(\Lambda+\delta_G)$ by $\xi_G(\delta_G)/\xi_K(\xi_K)$ and set it in the form $\Sigma\xi_K(\Lambda'+\delta_K)$. Since $\xi_K(\Lambda')$ for dominant integral forms Λ' are linearly independent, the set S is uniquely determined.

We may calculate in a larger group algebra generated by an additive group of forms. We can write

$$\xi_G(\delta_G) = \Pi(e(\alpha/2)-e(-\alpha/2)) \quad (\alpha\in\Delta_G^+)\,,$$
$$\xi_K(\delta_K) = \Pi(e(\alpha/2)-e(-\alpha/2)) \quad (\alpha\in\Delta_K^+)\,,$$

and so

$$\xi_G(\delta_G)/\xi_K(\delta_K) = \Pi(e(\alpha/2)-e(-\alpha/2)) \quad (\alpha\in\Delta_G^+\backslash\Delta_K^+)\,.$$

We will exhibit ξ_G and ξ_K in terms of λ_i in the cases of our branching laws. We set $s(\Lambda)=e(\Lambda)-e(-\Lambda)$, $c(\Lambda)=e(\Lambda)+e(-\Lambda)$. We denote by $[a_{ij}]_{p:q}$ a square matrix whose suffixes i, j range from p to q.

a) $G=SO(2m+2)$, $K=SO(2)\times SO(2m)$.

When we express $\Lambda+\delta_G$ as in (1.1), $\varepsilon=1$ or -1 and $h_0, h_1, \cdots, h_m$ are integers satisfying

$$h_0>h_1>\cdots>h_{m-1}>h_m\geqslant 0\,.$$

When we express $\Lambda'+\delta_K$ as in (1.3), $\varepsilon'=1$ or -1 and $k_0, k_1, \cdots, k_m$ are integers satisfying

$$k_1>\cdots>k_{m-1}>k_m\geqslant 0\,.$$

We get

$$\xi_G(\Lambda+\delta_G) = (1/2)(\det[c(h_i\lambda_j)]_{0:m}+\varepsilon\det[s(h_i\lambda_j)]_{0:m})\,,$$
$$\xi_K(\Lambda'+\delta_K) = e(k_0\lambda_0)\cdot(1/2)(\det[c(k_i\lambda_j)]_{1:m}+\varepsilon'\det[s(k_i\lambda_j)]_{1:m})\,,$$
$$\xi_G(\delta_G)/\xi_K(\delta_K) = \Pi_{i=1}^m(s((\lambda_0+\lambda_i)/2)s((\lambda_0-\lambda_i)/2))\,.$$

b) $G=SO(2m+3)$, $K=SO(2)\times SO(2m+1)$.

When we express $\Lambda+\delta_G$ as in (1.6), $h_0, h_1, \cdots, h_m$ are integers $+1/2$ satisfying

$$h_0>h_1>\cdots>h_m>0\,.$$

When we express $\Lambda'+\delta_K$ as in (1.7), k_0 is an integer and $k_1, \cdots, k_m$ are integers $+1/2$ satisfying

$$k_1>k_2>\cdots>k_m>0 .$$

We get

$$\xi_G(\Lambda+\delta_G) = \det[s(h_i\lambda_j)]_{0:m} ,$$
$$\xi_K(\Lambda'+\delta_K) = e(k_0\lambda_0)\cdot\det[s(k_i\lambda_j)]_{1:m} ,$$
$$\xi_G(\delta_G)/\xi_K(\delta_K) = s(\lambda_0/2)\cdot\Pi_{i=1}^{m}(s((\lambda_0+\lambda_i)/2)s((\lambda_0-\lambda_i)/2)) .$$

c) $G=Sp(m+1)$, $K=Sp(1)\times Sp(m)$.

When we express $\Lambda+\delta_G$ as in (1.6), $h_0, h_1, \cdots, h_m$ are integers satisfying

$$h_0>h_1>\cdots>h_{m-1}>h_m>0 .$$

When we express $\Lambda'+\delta_K$ as in (1.7), $k_0, k_1, \cdots, k_m$ are integers satisfying

$$k_0>0,\ k_1>\cdots>k_{m-1}>k_m>0 .$$

We get

$$\xi_G(\Lambda+\delta_G) = \det[s(h_i\lambda_j)]_{0:m} ,$$
$$\xi_K(\Lambda'+\delta_K) = s(k_0\lambda_0)\cdot\det[s(k_i\lambda_j)]_{1:m} ,$$
$$\xi_G(\delta_G)/\xi_K(\delta_K) = \Pi_{i=1}^{m}(s((\lambda_0+\lambda_i)/2)s(\lambda_0-\lambda_i)/2)) .$$

The crucial point in the proofs of our branching laws is that the quotient of $\det[s(h_i\lambda_j)]_{0:m}$ or $\det[c(h_i\lambda_j)]_{0:m}$ devided by $\Pi_{i=1}^{m}(s((\lambda_0+\lambda_i)/2)s((\lambda_0-\lambda_i)/2))$ is a sum of (a finite power series in $e(\lambda_0)$) $\times$ ($\det[s(k_i\lambda_j)]_{1:m}$ or $\det[c(k_i\lambda_j)]_{1:m}$). The next lemma enables us to excute the division. The substitution of the obtained result in (2.2), using the above expressions, completes the proofs of the branching laws.

Lemma 2.1. *Let $(h_0, h_1, \cdots, h_m)$ be a set of integers satisfying $h_0>h_1>\cdots>h_m\geqslant 0$. Then*

$$\det[s(h_i\lambda_j)]_{0:m}/\Pi_{i=1}^{m}(s((\lambda_0+\lambda_i)/2)s((\lambda_0-\lambda_i)/2)) = (s(\lambda_0))^{-m}\Sigma(\Pi_{i=0}^{m}s(l_i\lambda_0))\cdot\det[s(k_i\lambda_j)]_{1:m} , \tag{2.3}$$

$$\det[c(h_i\lambda_j)]_{0:m}/\Pi_{i=1}^{m}(s((\lambda_0+\lambda_i)/2)s((\lambda_0-\lambda_i)/2)) = (s(\lambda_0))^{-m}\Sigma(\Pi_{i=0}^{m-1}(s(l_i\lambda_0))\cdot H\cdot c(l_m\lambda_0)\cdot\det[c(k_i\lambda_j)]_{1:m} , \tag{2.4}$$

where the summation is taken over all the sets of integers $(k_1, k_2, \cdots, k_m)$ satisfying $k_1>k_2>\cdots>k_m\geqslant 0$ and

$$h_{i-1}>k_i>h_{i+1} \text{ for } 1\leqslant i\leqslant m-1 , \quad h_{m-1}>k_m\ (\geqslant 0) , \tag{2.5}$$

and integers $l_0, l_1, \cdots, l_m$ *are defined by* (1.5) *from* $h_0, h_1, \cdots, h_m$ *and* $k_1, k_2, \cdots, k_m$, *and further*

$$H = \begin{cases} 1 & \text{for } k_m > 0\,, \\ 1/2 & \text{for } k_m = 0\,. \end{cases}$$

The equalities (2.3) *and* (2.4) *are also valid when* $(h_0, h_1, \cdots, h_m)$ *is a set of integers*$+1/2$ *satisfying* $h_0 > h_1 > \cdots > h_m > 0$. *Then the summations should be taken over all the sets of integers*$+1/2$ $(k_1, k_2, \cdots, k_m)$ *satisfying* $k_1 > k_2 > \cdots > k_m > 0$ *and* (2.5).

REMARK. The assumption on $h_0, h_1, \cdots, h_m$ and $k_1, k_2, \cdots, k_m$ ensures us that $l_0, l_1, \cdots, l_{m-1}$ are positive integers.

Proof. We prove the case (2.4) where $(h_0, h_1, \cdots, h_m)$ is a set of integers. By sight changes, we can prove the other cases.

We transform $[c(h_i\lambda_j)]_{0:m}$ by subtracting "the $(i-1)$-th row $\times\, c(h_i\lambda_0)/c(h_{i-1}\lambda_0)$" from the i-th row in turn.

$$\begin{aligned} &\det[c(h_i\lambda_j)]_{0:m} \\ &= (\textstyle\prod_{i=1}^{m-1} c(h_i\lambda_0))^{-1} \det[c(h_{i-1}\lambda_0)c(h_i\lambda_j) - c(h_i\lambda_0)c(h_{i-1}\lambda_j)]_{1:m} \\ &= (\textstyle\prod_{i=1}^{m-1} c(h_i\lambda_0))^{-1} \\ &\quad \times \det[s((h_{i-1}+h_i)(\lambda_0+\lambda_j)/2)s((h_{i-1}-h_i)(\lambda_0-\lambda_j)/2) \\ &\qquad + s((h_{i-1}-h_i)(\lambda_0+\lambda_j)/2)s((h_{i-1}+h_i)(\lambda_0-\lambda_j)/2)]_{1:m}\,. \end{aligned}$$

We devide the (i, j)-element of the last matrix by $s((\lambda_0+\lambda_j)/2)\, s((\lambda_0-\lambda_j)/2)$. The result is

$$s(\lambda_0)^{-1} \sum P_i(k_i) c(k_i\lambda_j) \qquad (k_i \in Z)\,,$$

where $P_i(k)$ is given by

$$P_i(k) = \begin{cases} c(h_i\lambda_0)s((h_{i-1}-k)\lambda_0) & \text{if } h_{i-1} > k > h_i\,, \\ c(k\lambda_0)s((h_{i-1}-h_i)\lambda_0) & \text{if } h_i \geqslant k > 0\,, \\ s((h_{i-1}-h_i)\lambda_0) & \text{if } k = 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we get

$$\begin{aligned} &\det[c(h_i\lambda_j)]_{0:m} / \textstyle\prod_{i=1}^{m} (s((\lambda_0+\lambda_i)/2)s((\lambda_0-\lambda_i)/2)) \\ &= s(\lambda_0)^{-m} (\textstyle\prod_{i=1}^{m-1} c(h_i\lambda_0))^{-1} \sum (\prod_{i=1}^{m} P_i(k_i)) \det[c(k_i\lambda_j)]_{1:m} \\ &\qquad\qquad ((k_1, k_2, \cdots, k_m) \in Z^m) \\ &= s(\lambda_0)^{-m} (\textstyle\prod_{i=1}^{m-1} c(h_i\lambda_0))^{-1} \sum \det[P_i(k_j)]_{1:m} \det[c(k_i\lambda_j)]_{1:m} \\ &\qquad\qquad (k_1 > k_2 > \cdots > k_m \geqslant 0)\,. \end{aligned}$$

Note that if $k_1, k_2, \cdots, k_m$ do not satisfy (2.5), then $\det[P_i(k_j)]_{1:m}$ vanishes. Indeed, if $k_i \geqslant h_{i-1}$ $(1 \leqslant i \leqslant m)$, then the first i columns are linearly dependent. If $h_{i+1} \geqslant k_i$ $(1 \leqslant i \leqslant m-1)$, then the first $i+1$ rows are linearly dependent.

Assuming that $k_1, k_2, \cdots, k_m$ satisfy (2.5), we transform $[P_i(k_j)]_{1:m}$ by subtracting "the $(j-1)$-th column $\times c(k_j\lambda_0)/c(k_{j-1}\lambda_0)$" (or its half when $k_m=0$ and $j=m$) from the j-th column for $j=m, m-1, \cdots, 2$ in this order. The resulting matrix $[P_{ij}]_{1:m}$ is a tridiagonal matrix such that $P_{i,i+1}P_{i+1,i}=0$ for $1 \leqslant i \leqslant m-1$. This means that its determinant is equal to the product of the diagonal elements.

$$P_{ii} = c(h_{i-1}\lambda_0)s(l_{i-1}\lambda_0)c(p_i\lambda_0)/c(p_{i-1}\lambda_0)\,.$$

We defined $p_0, p_1, \cdots, p_m$ by $p_0=h_0$, $p_i=\min(h_i, k_i)$ for $1 \leqslant i \leqslant m$ $(p_m=l_m)$. Therefore

$$\begin{aligned}&\det[P_i(k_j)]_{1:m}\\ &= \left(\textstyle\prod_{i=0}^{m-1} c(h_i\lambda_0)\right)\left(\textstyle\prod_{i=0}^{m-1} s(l_i\lambda_0)\right)\cdot H\cdot c(p_m\lambda_0)/c(p_0\lambda_0)\\ &= \left(\textstyle\prod_{i=1}^{m-1} c(h_i\lambda_0)\right)\left(\textstyle\prod_{i=0}^{m-1} s(l_i\lambda_0)\right)\cdot H\cdot c(l_m\lambda_0)\,,\end{aligned}$$

which proves (2.4).

3. Decomposition of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$

We identify a complexified contangent space of $M=G/K$ at $o=[K]$ with $(\mathfrak{g}/\mathfrak{k})^*$, the dual space of $\mathfrak{g}/\mathfrak{k}$.

First we treat the case $G=SO(n+2)$ and $K=SO(2)\times SO(n)$.

The space $(\mathfrak{g}/\mathfrak{k})^*$ decomposes into two irreducible K-modules, V_+ and V_-, with the highest weights $\lambda_0+\lambda_1$ and $-\lambda_0+\lambda_1$. This decomposition of $(\mathfrak{g}/\mathfrak{k})^*$ gives a rough decomposition of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$:

$$\Lambda^p(\mathfrak{g}/\mathfrak{k})^* \cong \Sigma\Lambda^{r,s} \qquad (r+s=p)\,, \tag{3.1}$$

where $\Lambda^{r,s}=(\Lambda^r V_+)\otimes(\Lambda^s V_-)$. Then the $SO(2)$-parts of weights in $\Lambda^{r,s}$ are $(r-s)\lambda_0$. In order to decompose $\Lambda^{r,s}$ as a K-module, we should decompose it as an $SO(n)$-module.

Let $\Lambda_1, \Lambda_2, \cdots, \Lambda_m$ be the fundamental weights of $SO(n)$ dual to the simple roots $\alpha_1, \alpha_2, \cdots, \alpha_m$. We set $\Lambda_0=0$. We denote by $V(\Lambda)$ an irreducible $SO(n)$-module with the highest weight Λ.

The space $\Lambda^{r,s}$ is isomorphic to $(\Lambda^r V(\Lambda_1))\otimes(\Lambda^s V(\Lambda_1))$ as an $SO(n)$-module. Since $\Lambda^{r,s}\cong\Lambda^{s,r}$ and $\Lambda^{r,s}\cong\Lambda^{n-r,s}$ as $SO(n)$-modules, we may restrict our attention to the case $0 \leqslant r \leqslant s \leqslant m$.

When $n=2m$, we define $V_{i,j}$ by

$$\begin{aligned}V_{i,j} &= V(\Lambda_i+\Lambda_j) \qquad \text{for } 0 \leqslant i \leqslant j \leqslant m-2\,,\\ V_{i,m-1} &= V(\Lambda_i+\Lambda_{m-1}+\Lambda_m) \qquad \text{for } 0 \leqslant i \leqslant m-2\,,\\ V_{m-1,m-1} &= V(2\Lambda_{m-1}+2\Lambda_m)\,,\end{aligned}$$

$$\begin{aligned} V_{i,m} &= V(\Lambda_i+2\Lambda_{m-1})\oplus V(\Lambda_i+2\Lambda_m) \quad \text{for } 0\leqslant i\leqslant m-2\,, \\ V_{m-1,m} &= V(3\Lambda_{m-1}+\Lambda_m)\oplus V(\Lambda_{m-1}+3\Lambda_m)\,, \\ V_{m,m} &= V(4\Lambda_{m-1})\oplus V(4\Lambda_m)\,, \\ V_{i,j} &= V_{i,n-j} \quad \text{for } m+1\leqslant j\leqslant n-i\,. \end{aligned}$$

When $n=2m+1$, we define $V_{i,j}$ by

$$\begin{aligned} V_{i,j} &= V(\Lambda_i+\Lambda_j) \quad \text{for } 0\leqslant i\leqslant j\leqslant m-1\,, \\ V_{i,m} &= V(\Lambda_i+2\Lambda_m) \quad \text{for } 0\leqslant i\leqslant m-1\,, \\ V_{m,m} &= V(4\Lambda_m)\,, \\ V_{i,j} &= V_{i,n-j} \quad \text{for } m+1\leqslant j\leqslant n-i\,. \end{aligned}$$

Proposition 3.1. *An $SO(n)$-module $\Lambda^{r,s}(0\leqslant r\leqslant s\leqslant m)$ decomposes into irreducible modules as follows:*

$$\Lambda^{r,s}\cong\Sigma V_{i,j} \quad ((i,j)\in S)\,,$$

where the set S consists of pairs of non-negative integers (i,j) satisfying $s-r\leqslant j-i$, $i+j\leqslant r+s$ and $i+j\equiv r+s$ (mod 2).

This proposition and (3.1) give an $SO(n)$-irreducible decomposition of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$, which is also the K-irreducible decomposition.

The proof of Proposition 3.1 resembles that of the primitive decomposition of $\Lambda^p(C^n+\bar{C}^n)$ via $U(n)$ and uses it.

The $SO(n)$-module $V(\Lambda_1)$ is isomorphic to C^n, the complexification of R^n with a canonical $SO(n)$-action, and posesses a natural $SO(n)$-invariant symmetric inner product. We take an orthonormal basis $\{x_i\}$ $(1\leqslant i\leqslant n)$ in R^n. Then $\Omega=\Sigma_{i=1}^n x_i\otimes x_i$ is the unique $SO(n)$-invariant element in $V(\Lambda_1)\otimes V(\Lambda_1)$ up to a constant factor. We set $e_i=(x_{2i-1}-\sqrt{-1}x_{2i})/\sqrt{2}$, $e_{n-i+1}=(x_{2i-1}+\sqrt{-1}x_{2i})/\sqrt{2}$ for $1\leqslant i\leqslant m$ and $e_{m+1}=x_n$ when $n=2m+1$. Then we have for $H\in\mathfrak{t}\cap\mathfrak{o}(n,C)$

$$\begin{aligned} \rho(H)(e_i) &= \lambda_i(H)e_i \quad \text{for } 1\leqslant i\leqslant m\,, \\ \rho(H)(e_{n-i+1}) &= -\lambda_i(H) \quad \text{for } 1\leqslant i\leqslant m\,, \\ \rho(H)(e_{m+1}) &= 0 \quad \text{when } n=2m+1\,, \end{aligned}$$

where ρ denotes the action of $\mathfrak{o}(n,C)$. We can rewrite Ω as $\Sigma_{i=1}^n e_i\otimes e_{n-i+1}$. We define an $SO(n)$-homomorphism

$$L:\ \Lambda^{r,s}\to\Lambda^{r+1,s+1}$$

by $L\omega=\Omega\wedge\omega$ $(\omega\in\Lambda^{r,s})$.

Lemma 3.2. *For $r+s<n$ $(0\leqslant r,\ s\leqslant n)$, L: $\Lambda^{r,s}\to\Lambda^{r+1,s+1}$ is injective.*

In fact, L^{n-r-s}: $\Lambda^{r,s} \to \Lambda^{n-s,n-r}$ is an $SO(n)$-isomorphism. For the proof, see Weil [5]. Notice that the $SO(n)$-action on C^n can be extended to $U(n)$-actions in two manners; a canonical one and a complex conjugate one. When we take a canonical action for a $U(n)$-action on V_+ and a complex conjugate action for a $U(n)$-action on V_-, L is the same $U(n)$-homomorphism used in [5].

There is an $SO(n)$-isomorphism

$$*\colon\ \Lambda^p V(\Lambda_1) \to \Lambda^{n-p} V(\Lambda_1)$$

given by

$$(*\alpha, \beta) e_1 \wedge e_2 \wedge \cdots \wedge e_n = \alpha \wedge \beta,\ \alpha \in \Lambda^p V(\Lambda_1),\ \beta \in \Lambda^{n-p} V(\Lambda_1)\,,$$

where (,) denotes the symmetric inner product. If $(i_1, i_2, \cdots, i_n)$ is a permutation of $(1, 2, \cdots, n)$,

$$*(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \operatorname{sgn}(i_1, i_2, \cdots, i_n) e_{n-i_{r+1}+1} \wedge \cdots \wedge e_{n-i_n+1}\,.$$

We define an $SO(n)$-homomorphism

$$T\colon\ \Lambda^{r,s} \to \Lambda^{r+1,s-1}$$

by the composition of the following three $SO(n)$-homomorphisms:

$$\begin{aligned} (-1)^{s-1} Id \otimes *\colon&\ \Lambda^{r,s} \to \Lambda^{r,n-s}\,, \\ L\colon&\ \Lambda^{r,n-s} \to \Lambda^{r+1,n-s+1}\,, \\ Id \otimes *^{-1}\colon&\ \Lambda^{r+1,n-s+1} \to \Lambda^{r+1,s-1}\,. \end{aligned}$$

An explicit formula for T is given by

$$\begin{aligned} &T(e_{i_1} \wedge \cdots \wedge e_{i_r} \otimes e_{j_1} \wedge \cdots \wedge e_{j_s}) \\ &= \textstyle\sum_{t=1}^{s} (-1)^{t-1} e_{j_t} \wedge e_{i_1} \wedge \cdots \wedge e_{i_r} \otimes e_{j_1} \wedge \cdots \wedge \widehat{e_{j_t}} \wedge \cdots \wedge e_{j_s}\,. \end{aligned}$$

The following lemmas are easily verified.

Lemma 3.3. *For* $0 \leqslant r < s \leqslant n$, T: $\Lambda^{r,s} \to \Lambda^{r+1,s-1}$ *is an injective* $SO(n)$-*homomorphism.*

Lemma 3.4. *For* $2 \leqslant r \leqslant s \leqslant n-r$, *the following diagram commutes*:

$$\begin{array}{ccc} \Lambda^{r-2,s} & \xrightarrow{\ L\ } & \Lambda^{r-1,s+1} \\ T\downarrow & & T\downarrow \\ \Lambda^{r-1,s-1} & \xrightarrow{\ L\ } & \Lambda^{r,s}\,. \end{array}$$

Lemma 3.5. *Let* T^* *be the adjoint of* T *with respect to the invariant symmetric inner product.*

a) *For* $2 \leqslant r \leqslant s \leqslant n-r$, *the following diagram commutes*:

$$\begin{array}{ccc} \Lambda^{r-2,s} & \xrightarrow{L} & \Lambda^{r-1,s+1} \\ T^*\uparrow & & T^*\uparrow \\ \Lambda^{r-1,s-1} & \xrightarrow{L} & \Lambda^{r,s}\,. \end{array}$$

b) For $1 \leqslant s \leqslant r-1$, *the following diagram commutes*:

$$\begin{array}{ccc} & & \Lambda^{0,s+1} \\ & \overset{0}{\nearrow} & T^*\uparrow \\ \Lambda^{0,s} & \xrightarrow{L} & \Lambda^{1,s} \end{array} \qquad (0 \textit{ denotes } 0\textit{-map}).$$

Notice that an explicit formula for T^* is given by

$$T^*(e_{i_1}\wedge\cdots\wedge e_{i_r}\otimes e_{j_1}\wedge\cdots\wedge e_{j_s})$$
$$=\sum_{t=1}^{r}(-1)^{t-1}e_{i_1}\wedge\cdots\wedge \widehat{e_{i_t}}\wedge\cdots\wedge e_{i_r}\otimes e_{i_t}\wedge e_{j_1}\wedge\cdots\wedge e_{j_s}\,.$$

From these lemmas we can deduce that $\Lambda^{r,s}$ contains submodules isomorphic to $\Lambda^{r-1,s-1}$ and $\Lambda^{r-1,s+1}$ with the intersection isomorphic to $\Lambda^{r-2,s}$ (or $\{0\}$ if $r=1$). The space $\Lambda^{r,s}$ must also contain $V_{r,s}$ which corresponds to the highest weight of $\Lambda^{r,s}$. It is obvious that $V_{r,s}$ can intersect with the sum of $\Lambda^{r-1,s-1}$ and $\Lambda^{r-1,s+1}$ only by $\{0\}$. Computing the dimension of the above modules, we can obtain

Proposition 3.6. *We have the following $SO(n)$-isomorphisms*:

$$\Lambda^{1,s}\cong V_{1,s}\oplus\Lambda^{0,s-1}\oplus\Lambda^{0,s+1} \quad (1\leqslant s\leqslant m)\,,$$
$$\Lambda^{r,s}\oplus\Lambda^{r-2,s}\cong V_{r,s}\oplus\Lambda^{r-1,s-1}\oplus\Lambda^{r-1,s+1} \quad (2\leqslant r\leqslant s\leqslant m)\,.$$

It is easy to see that this proposition is equivalent to Proposition 3.1.

Remark. We may call V_- the holomorphic part and V_+ the anti-holomorphic part by the following reason. Let H_0 be an element of $\mathfrak{t}$ satisfying $\lambda_0(H_0)=\sqrt{-1}$, $\lambda_i(H_0)=0$ for $1\leqslant i\leqslant m$. Then $ad\ H_0$ defines a complex structure on $\mathfrak{g}/\mathfrak{k}$. The space V_- is an eigenspace of $ad\ H_0$ in $(\mathfrak{g}/\mathfrak{k})^*$ with an eigenvalue $-\sqrt{-1}$ and the space V_+ is one with an eigenvalue $\sqrt{-1}$. Because $ad\ H_0$ commutes with the action of K, it defines on $M=G/K$ a G-invariant almost complex structure, with which the metric we assumed defines a Kaehler structure.

Note that Frobenius' reciprosity law gives an explicit correspondence between a K-submodule of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$ and a G-submodule of $C^\infty(\Lambda^p M)$. In our case, the holomorphic [anti-holomorphic] part $V_-[V_+]$ corresponds to holomorphic [anti-holomorphic] forms and $V^{r,s}$ to forms of type (s,r).

We proceed to the case $G=Sp(n+1)$ and $K=Sp(1)\times Sp(n)$. The K-module $(\mathfrak{g}/\mathfrak{k})^*$ is an irreducible module with the highest weight $\lambda_0+\lambda_1$. We take a maximal torus T in $Sp(1)$ whose complexified Lie algebra is contained in $\mathfrak{t}$.

We set $K'=T\times Sp(n)$. If we consider $(\mathfrak{g}/\mathfrak{k})^*$ as a K'-module, it decomposes into two irreducible K'-modules V_+ and V_- with the highest weights $\lambda_0+\lambda_1$ and $-\lambda_0+\lambda_1$. We first study a K'-irreducible decomposition of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$ and next reconstruct a K-irreducible decomposition.

First we have the following rough decomposition as K'-modules:

$$\Lambda^p(\mathfrak{g}/\mathfrak{k})^* \cong \Sigma\Lambda^{r,s} \quad (r+s=p),$$

where $\Lambda^{r,s}=(\Lambda^r V_+)\otimes(\Lambda^s V_-)$. The T-parts of weights in $\Lambda^{r,s}$ are $(r-s)\lambda_0$. We should decompose $\Lambda^{r,s}$ as an $Sp(n)$-module. Let $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ be the fundamental weights of $Sp(n)$ dual to the simple roots $\alpha_1, \alpha_2, \cdots, \alpha_n$. We set $\Lambda_0=0$. Both V_+ and V_- are irreducible $Sp(n)$-modules with the same highest weight $\Lambda_1=\lambda_1$. We denote by $V(\Lambda)$ an irreducible $Sp(n)$-module with the highest weight Λ.

Proposition 3.7. *For* $0\leqslant r\leqslant n$, *we have*

$$\begin{aligned} \Lambda^r V(\Lambda_1) &\cong V(\Lambda_r)\oplus V(\Lambda_{r-2})\oplus\cdots\oplus V(\Lambda_1) \quad \textit{when } r=odd,\\ &\cong V(\Lambda_r)\oplus V(\Lambda_{r-2})\oplus\cdots\oplus V(\Lambda_0) \quad \textit{when } r=even;\\ \Lambda^r V(\Lambda_1) &\cong \Lambda^{2n-r}V(\Lambda_1)\,. \end{aligned}$$

Proof. The $Sp(n)$-module $V(\Lambda_1)$ is isomorphic to C^{2n}, the complexification of R^{2n} with a canonical $Sp(n)$-action, and posesses natural $Sp(n)$-invariant inner product and symplectic form ω. We take an orthonormal basis $\{x_i\}$ $(1\leqslant i\leqslant 2n)$ in R^{2n}, which satisfies $\omega(x_i, x_j)=0$, $\omega(x_{n+i}, x_{n+j})=0$, $\omega(x_i, x_{n+j})=\delta_{ij}$ for $1\leqslant i, j\leqslant n$. We set $\Omega=\Sigma_{i=1}^n x_i\wedge x_{n+i}$, which is the unique $Sp(n)$-invariant element in $\Lambda^2 V(\Lambda_1)$ up to a constant factor. We define an $Sp(n)$-homomorphism $L\colon \Lambda^p V(\Lambda_1)\to\Lambda^{p+2}V(\Lambda_1)$ by $L\alpha=\Omega\wedge\alpha$ $(\alpha\in\Lambda^p V(\Lambda_1))$. Then L is injective for $0\leqslant p<n$, as is seen in the proof of the primitive decomposition in [5]. The space $\Lambda^p V(\Lambda_1)$ includes a submodule isomorphic to $\Lambda^{p-2}V(\Lambda_1)$ and one isomorphic to $V(\Lambda_p)$, and they can intersect only by $\{0\}$. Computing the dimensions of these modules, we can prove

$$\Lambda^p V(\Lambda_1)\cong V(\Lambda_p)\oplus\Lambda^{p-2}V(\Lambda_1) \quad (2\leqslant p\leqslant n)\,,$$

which is equivalent to the top half of the proposition.

The remainder is obvious.

Thus to decompose $\Lambda^{r,s}$ as $Sp(n)$-modules, we have only to decompose $V(\Lambda_r)\otimes V(\Lambda_s)$ for $0\leqslant r\leqslant s\leqslant n$.

Proposition 3.8. *An* $Sp(n)$*-module* $V(\Lambda_r)\otimes V(\Lambda_s)$ $(0\leqslant r\leqslant s\leqslant n)$ *decomposes into irreducible modules as follows:*

$$V(\Lambda_r)\otimes V(\Lambda_s)\cong\Sigma V(\Lambda_i+\Lambda_j) \quad ((i,j)\in S)\,,$$

where the set S *consists of pairs of non-negative integers* (i,j) *satisfying* $s-r\leqslant j-i\leqslant 2n-s-r$, $i+j\leqslant r+s$ *and* $i+j\equiv r+s$ (mod 2).

Proof. As in the $SO(n)$ case, it is enough to prove

$$(V(\Lambda_p)\otimes V(\Lambda_q))\oplus(V(\Lambda_{p-2})\otimes V(\Lambda_q))$$
$$\cong V(\Lambda_p+\Lambda_q)\oplus(V(\Lambda_{p-1})\otimes V(\Lambda_{q-1}))\oplus(V(\Lambda_{p-1})\otimes V(\Lambda_{q+1}))$$
$$(0\leqslant p\leqslant q\leqslant n),$$

where the terms including $V(\Lambda_r)$ with $r<0$ or $r>n$ should be omitted. It is equivalent to the following relation among the formal characters: $(\chi=\chi_{Sp(n)})$

$$\chi(\Lambda_p)\chi(\Lambda_q)+\chi(\Lambda_{p-2})\chi(\Lambda_q)$$
$$=\chi(\Lambda_p+\Lambda_q)+\chi(\Lambda_{p-1})\chi(\Lambda_{q-1})+\chi(\Lambda_{p-1})\chi(\Lambda_{q+1}).$$

We can rewrite the above, using Weyl's character formula. We set $\xi=\xi_{Sp(n)}$ and $\delta=\delta_{Sp(n)}$ and factor out $(\xi(\delta))^2$. Then we have

$$(3.2)\qquad \xi(\Lambda_p+\delta)\xi(\Lambda_q+\delta)+\xi(\Lambda_{p-2}+\delta)\xi(\Lambda_q+\delta)$$
$$=\xi(\Lambda_p+\Lambda_q+\delta)\xi(\delta)+\xi(\Lambda_{p-1}+\delta)\xi(\Lambda_{q-1}+\delta)+\xi(\Lambda_{p-1}+\delta)\xi(\Lambda_{q+1}+\delta).$$

Let $\Lambda_1', \Lambda_2', \cdots, \Lambda_n'$ be the fundamental weights for $SO(2n)$, $\xi'=\xi_{SO(2n)}$ and $\delta'=\delta_{SO(2n)}$. We consider ξ and ξ' as finite power sereis in $e(\lambda_1), e(\lambda_2), \cdots, e(\lambda_n)$. We can represent ξ as $(\prod_{i=1}^n s(\lambda_i))\times$(a linear combination of ξ'). For example, $(D=\prod_{i=1}^n s(\lambda_i))$

$$(3.3)\qquad \begin{aligned}\xi(\delta)&=D\cdot\xi'(\delta'),\\ \xi(\Lambda_1+\delta)&=D\cdot\xi'(\Lambda_1'+\delta'),\\ \xi(\Lambda_p+\delta)&=D\cdot(\xi'(\Lambda_p'+\delta')-\xi'(\Lambda_{p-2}'+\delta'))\quad(2\leqslant p\leqslant n-2),\\ \xi(\Lambda_{n-1}+\delta)&=D\cdot(\xi'(\Lambda_{n-1}'+\Lambda_n'+\delta')-\xi'(\Lambda_{n-3}'+\delta')),\\ \xi(\Lambda_n+\delta)&=D\cdot(\xi'(2\Lambda_{n-1}'+\delta')+\xi'(2\Lambda_n+\delta')-\xi'(\Lambda_{n-2}'+\delta')),\\ \xi(\Lambda_p+\Lambda_q+\delta)&=D\cdot(\xi'(\Lambda_p'+\Lambda_q'+\delta')-\xi'(\Lambda_{p-2}'+\Lambda_q'+\delta')\\ &\qquad-\xi'(\Lambda_p'+\Lambda_{q-2}'+\delta')+\xi'(\Lambda_{p-2}'+\Lambda_{q-2}'+\delta))\\ &\qquad(4\leqslant p+2\leqslant q\leqslant n-2).\end{aligned}$$

On the other hand, Proposition 3.6 provides us relations among ξ'. For example,

$$(3.4)\qquad \begin{aligned}&\xi'(\Lambda_p'+\delta')\xi'(\Lambda_q'+\delta')+\xi'(\Lambda_{p-2}'+\delta')\xi'(\Lambda_q'+\delta')\\ &=\xi'(\Lambda_p'+\Lambda_q'+\delta')\xi'(\delta')+\xi'(\Lambda_{p-1}'+\delta')\xi'(\Lambda_{q-1}'+\delta')\\ &\qquad+\xi'(\Lambda_{p-1}'+\delta')\xi'(\Lambda_{q+1}'+\delta')\\ &\qquad(2\leqslant p\leqslant q\leqslant n-2).\end{aligned}$$

By combining four equations of the (3.4) type, we can get an equation of the (3.2) type substituted the expressions (3.3).

Proposition 3.7 and 3.8 enables us to decompose $\Lambda^{r,s}$ as an $Sp(n)$-module, which completes the K'-irreducible decomposition of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$.

Let us return to the K-irreducible decomposition. We note that an irreducible K-module with the highest weight $k\lambda_0+\Lambda$ decomposes into irreducible K'-modules with the highest weights $(k-2i)\lambda_0+\Lambda$ ($i=0, 1, \cdots, k$). Conversely we gather the highest weights in the K'-irreducible decomposition in bunches of the above form: Take the highest weight $k\lambda_0+\Lambda$ of the biggest T-part for a fixed $Sp(n)$-part. Then make up the highest weights of the form $(k-2i)\lambda_0+\Lambda$ ($i=0, 1, \cdots, k$) into a bunch. Next do the same in the remaining highest weights, and so on. This procedure exhausts the highest weights without fail and the member of the biggest T-part in each bunch gives the highest weight in the K-irreducible decomposition of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$.

We present here a table of the highest weights of the irreducible K-submodules of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$ when $G=Sp(n+1)$ and $K=Sp(1)\times Sp(n)$ for some p.

$p=0$		0.
$p=1$		$\lambda_0+\Lambda_1$.
$p=2$	$n\geqslant 2$	$2\lambda_0+\Lambda_2$, $2\lambda_0$, $2\Lambda_1$.
	$n=1$	$2\lambda_0$, $2\Lambda_1$.
$p=3$	$n\geqslant 3$	$3\lambda_0+\Lambda_3$, $3\lambda_0+\Lambda_1$, $\lambda_0+\Lambda_1+\Lambda_2$, $\lambda_0+\Lambda_1$.
	$n=2$	$3\lambda_0+\Lambda_1$, $\lambda_0+\Lambda_1+\Lambda_2$, $\lambda_0+\Lambda_1$.
$p=4$	$n\geqslant 4$	$4\lambda_0+\Lambda_4$, $4\lambda_0+\Lambda_2$, $4\lambda_0$, $2\lambda_0+\Lambda_1+\Lambda_3$, $2\lambda_0+2\Lambda_1$, $2\lambda_0+\Lambda_2$, $2\Lambda_2$, Λ_2, 0.
	$n=3$	$4\lambda_0+\Lambda_2$, $4\lambda_0$, $2\lambda_0+\Lambda_1+\Lambda_3$, $2\lambda_0+2\Lambda_1$, $2\lambda_0+\Lambda_2$, $2\Lambda_2$, Λ_2, 0.
	$n=2$	$4\lambda_0$, $2\lambda_0+2\Lambda_1$, $2\lambda_0+\Lambda_2$, $2\Lambda_2$, Λ_2, 0.

REMARK. For a compact symmetric space $M=G/K$ with G compact and semisimple, Δ^p preserves a decomposition of $C^\infty(\Lambda^p M)$ corresponding to a decomposition of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$ under Frobenius' reciprocity law.

4. Examples

In the cases $G=SO(n+2)$ and $K=SO(2)\times SO(n)$, the cases $n=1$ and 2 are exceptional. When $n=1$, all our computation becomes trivial, and when $n=2$, we need some modification, for K is abelian. Anyway, since $M=G/M$ is homothetic to the standard sphere S^2 when $n=1$, and to $S^2\times S^2$ when $n=2$, the spectra are well-known.

Our first example is the case $G=SO(5)$ and $K=SO(2)\times SO(3)$ ($n=3$, $m=1$). We set $\Lambda_1=\lambda_1/2$, $\bar{\Lambda}_0=\lambda_0$ and $\bar{\Lambda}_1=(\lambda_0+\lambda_1)/2$. We denote by $I(k, s)$ for non-negative integers k and s the irreducible G-module with the highest weight $k\bar{\Lambda}_0+2s\bar{\Lambda}_1$. The Casimir operator acts on $I(k, s)$ by the multiplication of $-\{(k+s)(k+2s+3)+s(s+1)\}/6$. The dimension of $I(k, s)$ is $(2k+2s+3)(k+2s+2)(k+1)(2s+1)/6$.

We give in Table A the highest weights that irreducible K-submodules of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$ have and the G-modules which includes an irreducible K-submodule of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$ at least once. We denote by μ the multiplicity, the number of the times a K-module appearing in the K-irreducible decomposition of a G-module. Integers r and s may take any non-negative value and $\mu=1$ unless otherwise denoted.

Table A.

p	H.W.	G-module	
0	0	$I(2r, s)$	
1	$\lambda_0+2\Lambda_1$	$I(2r, s)$ $r\geqq 1$ or $s\geqq 1$	$\mu=2$ if $r\geqq 1$, $s\geqq 1$
	$-\lambda_0+2\Lambda_1$	$I(2r+1, s)$ $s\geqq 1$	
2	$2\lambda_0+2\Lambda_1$	$I(2r, s)$ $r\geqq 1$, $s\geqq 1$	
	$-2\lambda_0+2\Lambda_1$	$I(2r+1, s)$ $r\geqq 1$ or $s\geqq 1$	$\mu=2$ if $r\geqq 1$, $s\geqq 1$
	$4\Lambda_1$	$I(2r, s)$ $r\geqq 1$ or $s\geqq 2$	$\mu=2$ if $s=1$
			$\mu=3$ if $r\geqq 1$, $s\geqq 2$
		$I(2r+1, s)$ $s\geqq 1$	$\mu=2$ if $s\geqq 2$
	$2\Lambda_1$	$I(2r, s)$ $s\geqq 1$	
		$I(2r+1, s)$	$\mu=2$ if $s\geqq 1$
	0	see above	
3	$3\lambda_0$ $-3\lambda_0$	$I(2r+1, s)$ $r\geqq 1$	
	$\lambda_0+4\Lambda_1$	$I(2r, s)$ $r\geqq 1$ or $s\geqq 2$	$\mu=2$ if $r\geqq 1$, $s\geqq 2$
	$-\Lambda_0+4\Lambda_1$	$I(2r+1, s)$ $r\geqq 1$ or $s\geqq 1$	$\mu=2$ if $r=0$, $s\geqq 2$ or $r\geqq 1$, $s=1$
			$\mu=3$ if $r\geqq 1$, $s\geqq 2$
	$\lambda_0+2\Lambda_1$ $-\lambda_0+2\Lambda_1$	see above	
	λ_0 $-\lambda_0$	$I(2r+1, s)$	

Next we give the information on $\mathrm{Spec}^p(M)$ for the case $G=SO(6)$ and $K=SO(2)\times SO(4)$ ($n=4$, $m=2$) in Table B. We set $\Lambda_1=(\lambda_1-\lambda_2)/2$, $\Lambda_2=(\lambda_1+\lambda_2)/2$, $\bar{\Lambda}_0=\lambda_0$, $\bar{\Lambda}_1=(\lambda_0+\lambda_1-\lambda_2)/2$ and $\bar{\Lambda}_2=(\lambda_0+\lambda_1+\lambda_2)/2$. We denote by $I_i(r,s)$ for non-negative integers r and s the irreducible G-module with the highest weight given in Table B-1. There we have also given the eigenvalue of the $(-8)\times$Casimir operator and the dimension of the module. We list the highest weight of irreducible submodules of $\Lambda^p(\mathfrak{g}/\mathfrak{k})^*$ in Table B-2. Table B-3 indicates G-modules which contain a K-module in their K-irreducible decomposition at least once and the number of times they contain the K-module. In Table B-3, integers r and s may take any non-negative value and multiplicity $\mu=1$ unless otherwise denoted.

Table B-1.

Module	Highest Weight	
$I_0(r, s)$	$2r\overline{\Lambda_0}+s(\overline{\Lambda_1}+\overline{\Lambda_2})$	e.v. $=(2r+s)(2r+s+4)+s(s+2)$ dim. $=(2r+s+2)^2(2r+2s+3)(2r+1)(s+1)^2/12$
$I_1(r, s)$	$(2r+1)\overline{\Lambda_0}+s(\overline{\Lambda_1}+\overline{\Lambda_2})+2\overline{\Lambda_1}$	e.v. $=(2r+s+2)(2r+s+6)+(s+2)^2$
$I_2(r, s)$	$(2r+1)\overline{\Lambda_0}+s(\overline{\Lambda_1}+\overline{\Lambda_2})+2\overline{\Lambda_2}$	dim. $=(2r+2s+6)(2r+s+5)(2r+s+3)(2r+2)$ $(s+3)(s+1)/12$
$I_3(r, s)$	$2r\overline{\Lambda_0}+s(\overline{\Lambda_1}+\overline{\Lambda_2})+4\overline{\Lambda_1}$	e.v. $=(2r+s+3)(2r+s+5)+(s+3)^2$
$I_4(r, s)$	$2r\overline{\Lambda_0}+s(\overline{\Lambda_1}+\overline{\Lambda_2})+4\overline{\Lambda_2}$	dim. $=(2r+2s+7)(2r+s+6)(2r+s+2)(2r+1)$ $(s+5)(s+1)/12$

Table B-2.

p	Highest Weight
0	0
1	$\lambda_0+\Lambda_1+\Lambda_2$, $-\lambda_0+\Lambda_1+\Lambda_2$.
2	$2\lambda_0+2\Lambda_1$, $2\lambda_0+2\Lambda_2$, $2\Lambda_1+2\Lambda_2$, $2\Lambda_1$, $2\Lambda_2$, 0, $-2\lambda_0+2\Lambda_1$, $-2\lambda_0+2\Lambda_2$.
3	$3\lambda_0+\Lambda_1+\Lambda_2$, $\lambda_0+3\Lambda_1+\Lambda_2$, $\lambda_0+\Lambda_1+3\Lambda_2$, $\lambda_0+\Lambda_1+\Lambda_2$ (twice), $-\lambda_0+\Lambda_1+\Lambda_2$ (twice), $-\lambda_0+\Lambda_1+3\Lambda_2$, $-\lambda_0+3\Lambda_1+\Lambda_2$, $-3\lambda_0+\Lambda_1+\Lambda_2$.
4	$4\lambda_0$, $2\lambda_0+2\Lambda_1+2\Lambda_2$, $2\lambda_0+2\Lambda_1$, $2\lambda_0+2\Lambda_2$, $2\lambda_0$, $4\Lambda_1$, $4\Lambda_2$, $2\Lambda_1+2\Lambda_2$ (twice), $2\Lambda_1$, $2\Lambda_2$, 0 (twice), $-2\lambda_0+2\Lambda_1+2\Lambda_2$, $-2\lambda_0+2\Lambda_1$, $-2\lambda_0+2\Lambda_2$, $-2\lambda_0$, $-4\lambda_0$.

Table B-3.

H.W.	G-module	
0	$I_0(r, s)$	
$\lambda_0+\Lambda_1+\Lambda_2$	$I_0(r, s)$ $r\geqslant 1$ or $s\geqslant 1$	$\mu=2$ if $r\geqslant 2$, $s=1$
$-\lambda_0+\Lambda_1+\Lambda_2$	$I_1(r, s)$	
	$I_2(r, s)$	
$2\lambda_0+2\Lambda_1$	$I_0(r, s)$ $r\geqslant 1$, $s\geqslant 1$	
$-2\lambda_0+2\Lambda_2$	$I_1(r, s)$	
	$I_2(r, s)$ $r\geqslant 1$	

Table B-3 (continued).

H.W.	G-module	
$2\lambda_0+2\Lambda_2$ $-2\lambda_0+2\Lambda_1$	$I_0(r, s)$ $r\geqslant 1$, $s\geqslant 1$	
	$I_1(r, s)$ $r\geqslant 1$	
	$I_2(r, s)$	
$2\Lambda_1+2\Lambda_2$	$I_0(r, s)$ $r\geqslant 1$ or $s\geqslant 2$	$\mu=2$ if $r\geqslant 2$, $s=1$
		$\mu=3$ if $r\geqslant 1$, $s\geqslant 2$
$2\Lambda_1$ $2\Lambda_2$	$I_0(r, s)$ $s\geqslant 1$	
	$I_1(r, s)$	
	$I_2(r, s)$	
$3\lambda_0+\Lambda_1+\Lambda_2$ $-3\lambda_0+\Lambda_1+\Lambda_2$	$I_0(r, s)$ $r=1$, $s\geqslant 1$ or $r\geqslant 2$	$\mu=2$ if $r\geqslant 2$, $s\geqslant 1$
	$I_1(r, s)$ $r\geqslant 1$	
	$I_2(r, s)$ $r\geqslant 1$	
$\lambda_0+3\Lambda_1+\Lambda_2$ $-\lambda_0+\Lambda_1+3\Lambda_2$	$I_0(r, s)$ $r\geqslant 1$ or $s\geqslant 2$	$\mu=2$ if $r\geqslant 1$, $s\geqslant 2$
	$I_1(r, s)$	$\mu=2$ if $s\geqslant 1$
	$I_2(r, s)$ $r\geqslant 1$ or $s\geqslant 1$	$\mu=2$ if $r\geqslant 1$, $s\geqslant 1$
$\lambda_0+\Lambda_1+3\Lambda_2$ $-\lambda_0+3\Lambda_1+\Lambda_2$	$I_0(r, s)$ $r\geqslant 1$ or $s\geqslant 2$	$\mu=2$ if $r\geqslant 1$, $s\geqslant 2$
	$I_1(r, s)$ $r\geqslant 1$ or $s\geqslant 1$	$\mu=2$ if $r\geqslant 1$, $s\geqslant 1$
	$I_2(r, s)$	$\mu=2$ if $s\geqslant 1$
$4\lambda_0$ $-4\lambda_0$	$I_0(r, s)$ $r\geqslant 2$	
$2\lambda_0+2\Lambda_1+2\Lambda_2$ $-2\lambda_0+2\Lambda_1+2\Lambda_2$	$I_0(r, s)$ $r+s\geqslant 2$	$\mu=2$ if $r\geqslant 2$, $s=1$ or $r=1$, $s\geqslant 2$
$2\lambda_0$ $-2\lambda_0$	$I_0(r, s)$ $r\geqslant 1$	
$4\Lambda_1$ $4\Lambda_2$	$I_0(r, s)$ $s\geqslant 2$	
	$I_1(r, s)$ $s\geqslant 1$	
	$I_2(r, s)$ $s\geqslant 1$	
	$I_3(r, s)$ $r\geqslant 1$	
	$I_4(r, s)$ $r\geqslant 1$	

Our last example is the case $G=Sp(3)$ and $K=Sp(1)\times Sp(2)$ $(n=2)$. Notice that in the case $G=\mathrm{Sp}(2)$ and $K=Sp(1)\times Sp(1)$ $(n=1)$, $M=G/K$ is homothetic to the standard sphere S^4 and therefore $\mathrm{Spec}^p(M)$ has already given in [3]. We set $\Lambda_1=\lambda_1$, $\Lambda_2=\lambda_1+\lambda_2$, $\bar{\Lambda}_0=\lambda_0$, $\bar{\Lambda}_1=\lambda_0+\lambda_1$ and $\bar{\Lambda}_2=\lambda_0+\lambda_1+\lambda_2$. We denote by $I(r,s,t)$ for non-negative integers r, s and t the irreducible G-module with the highest weight $r\bar{\Lambda}_0+s\bar{\Lambda}_1+t\bar{\Lambda}_2$. The eigenvalue of $(-16)\times$ Casimir operator on $I(r,s,t)$ is $2s(s+2t+r+5)+r(r+2t+6)+3t(t+2)$ and the dimension of $I(r,s,t)$ is $(2s+r+2t+5)(s+r+2t+4)(s+r+t+3)(s+r+2)(s+2t+3)(s+t+2)(s+1)\times(r+1)(t+1)/720$. The meaning of each column of Table C is similar to that of Table A. Integers k may take any non-negative value and multiplicity $\mu=1$ unless otherwise denoted.

Table C.

p	H.W.	G-module
0	0	$I(0, k, 0)$.
1	$\lambda_0+\Lambda_1$	$I(0, k, 0)$ $k\geqslant 1$, $I(1, k, 1)$, $I(2, k, 0)$.
2	$2\lambda_0+\Lambda_2$	$I(1, k, 1)$, $I(2, k, 0)$ $k\geqslant 1$, $I(3, k, 1)$.
	$2\lambda_0$	$I(2, k, 0)$.
	$2\Lambda_1$	$I(0, k, 2)$, $I(1, k, 1)$, $I(2, k, 0)$.
3	$3\lambda_0+\Lambda_1$	$I(2, k, 0)$ $k\geqslant 1$, $I(3, k, 1)$, $I(4, k, 0)$.
	$\lambda_0+\Lambda_1+\Lambda_2$	$I(0, k, 0)$ $k\geqslant 2$, $I(0, k, 2)$, $I(1, k, 1)$ $\mu=2$ if $k\geqslant 1$, $I(2, k, 0)$ $k\geqslant 1$, $I(2, k, 2)$, $I(3, k, 1)$ $k\geqslant 1$.
	$\lambda_0+\Lambda_1$	see above
4	$4\lambda_0$	$I(4, k, 0)$.
	$2\lambda_0+2\Lambda_1$	$I(0, k, 0)$ $k\geqslant 2$, $I(1, k, 1)$ $k\geqslant 1$, $I(2, k, 0)$ $k\geqslant 1$, $I(2, k, 2)$, $I(3, k, 1)$, $I(4, k, 0)$.
	$2\lambda_0+\Lambda_2$	see above
	$2\Lambda_2$	$I(0, k, 0)$ $k\geqslant 2$, $I(1, k, 1)$ $k\geqslant 1$, $I(2, k, 2)$.
	Λ_2	$I(0, k, 0)$ $k\geqslant 1$, $I(1, k, 1)$.
	0	see above

References

[1] B.L. Beers and R.S. Millman: *The spectra of the Laplace-Beltrami operator on compact, semisimple Lie groups*, Amer. J. Math. **99** (1977), 801–807.

[2] J.E. Humphreys: Introduction to Lie algebras and representation theory, Graduate Texts in Mathematics 9, Springer-Verlag, New York-Heidelberg-Berlin (1972).

[3] A. Ikeda and Y. Taniguchi: *Spectra and eigenforms of the Laplacian on S^n and P^n (C)*, Osaka J. Math. **15** (1978), 515–546.
[4] J. Lepowsky: *Multiplicity formula for certain semisimple Lie groups,* Bull. Amer. Math. Soc. **77** (1971), 601–605.
[5] A. Weil: Introduction à l'étude des variétés Kähleriennes, Hermann, Paris (1958).

Department of Mathematics
Kyoto University
Kyoto 606
Japan