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FINITE GROUPS ADMITTING AN AUTOMORPHISM OF PRIME ORDER FIXING A CYCLIC 2-GROUP

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1. Introduction

In this paper, we shall give a proof of the following Theorem, which is a conjecture of B. Rickman [9]; in special case, $C_G(\phi)$ has order 2, M.J. Collins and B. Rickman proved in [2].

Theorem. *Let G be a finite group which admits an automorphism ϕ of odd prime order r whose fixed-point-subgroup $C_G(\phi)$ is a cyclic 2-group. Then G is solvable.*

All groups considered in this paper are assumed finite. Our notation corresponds to that of Gorenstein [7].

An important tool that is brought to attack the problem is B. Baumann's classification of finite simple groups whose Sylow 2-subgroups are maximal [1], and in analogy with Matsuyama [8] that used the results of [1], we shall prove that $\mathcal{U}_G(S;2)\neq 1$, where S is a ϕ-invariant Sylow 3-subgroup of G.

C.A. Rowley has obtained a proof of the theorem under the additional hypothesis that G does not involve S_4, the symmetric group on 4 letters.

The Theorem is a contribution to the continuing problem of showing that finite groups which admit an automorphism ϕ of odd prime order such that $C_G(\phi)$ is a 2-group are solvable.

2. Preliminaries

We first quote some frequently used results.

2.1. (Thompson [12])

Let G be a group which admits a fixed-point-free automorphism of prime order. Then G is nilpotent.

2.2. (Rowley [10])

Let G be a solvable group admitting an automorphism of odd prime order p such that $C_G(\phi)$, the fixed-point-subgroup of ϕ in G, is a cyclic q-group, $q\neq p$. Then, for any prime r, G is either r-nilpotent or r-closed.

2.3. (Glauberman [4])

Let G be a group with a Sylow p-subgroup P, either p odd or $p=2$ and S_4 is not involved in G, in which $C_G(Z(P))$ and $N_G(J(P))$ both have normal p-complements. Then G possesses a normal p-complement.

2.4. (Gilman and Gorenstein [3])

If G is a simple group with Sylow 2-subgroups of class 2, then $G\cong L_2(9)$, $q\equiv 7$, 9 (mod 16), A_7, $Sz(2^n)$, n odd, $n>1$, $U_3(2^n)$, $n\geqq 2$, $L_3(2^n)$, $n\geqq 2$, or Psp $(4, 2^n)$, $n\geqq 2$.

2.5. (Gorenstein [7])

Let P be a Sylow p-subgroup of G, where p is the smallest prime in $\pi(G)$. If $p>2$, assume $d_n(P)\leqq 2$, while if $p=2$, assume P is cyclic. Then G has a normal P-complement.

2.6. (Matsuyama [8])

Let Q be a 2-group admitting an automorphism ϕ of odd order $\neq 1$. If $d_c(Q)=1$, then $Q=E*R$, where E is ϕ-invariant, extra-special or 1, and R is ϕ-invariant, and R is cyclic, D_m, Q_m, or S_m, $m\geqq 4$

2.7. (Collins-Rickman [2])

Let T be an extra-special 2-group admitting an automorphism ϕ of odd prime order r acting fixed-point-freely on T/T'. Let S be the natural semi-direct product $T\langle\phi\rangle$ and let K be a field of nonzero characteristic different from 2 and r. Assume that there exists a KS-module M for which $C_M(\phi)=C_M(T')=0$.

Then (i) $r=2^n+1$ is a Fermat prime,

(ii) $|T|=2^{2n+1}$, and

(iii) $T\cong Q*(\overset{n-1}{\underset{1}{*}}D)$,

where Q and D denote the quaternion and dihedral groups of order 8, respectively, and $*$ denote the central product.

2.8. (Glauberman [5] [6])

Let G be a solvable group with a Sylow 2-subgroup Q with $G\neq C(Z(Q))N(J(Q))$, and $O(X)=1$. Put

$Z=\langle Z^*\,|\,G\rhd Z^*$: 2-subgroup and $O_2(G/C(Z^*))=1\rangle$

and $J=\langle x\in G\,|\,x$: 2-element, $|Z/C_Z(x)|=2\rangle$

and $H=\langle J, C(Z)\rangle$. Then the following hold;

(i) there exists a normal subgroup G_i of H containing $C(Z)$, $1\leqq i\leqq m$, such that, for $i=1,\cdots,m, G_i/C(Z)\cong S_3$, and $H/C(Z)=G_1/C(Z)\times\cdots\times G_m/C(Z)$.

(ii) let $V_i=[G_i, Z]$, $1\leqq i\leqq m$, and let $V=V_1\oplus\cdots\oplus V_m$, then $Z=V\oplus C_Z(H)$ and $V_i\cong Z_2\times Z_2$, $1\leqq i\leqq m$.

(iii) there is a 3-element x_0 of H such that, for each $g \in H$, $H = \langle Q \cap H, x_0^g, C(Z) \rangle$ and $G/C(Z) = H/C(Z)\, C_{G/C(Z)}(x_0^g C(Z))$.

2.9. (Matsuyama [8])
Let G be a group with a Hall π-subgroup H, and let $1 \neq P \in \mathrm{Syl}_3(H)$, Q 2-group. If $N_G(H) = HQ$, $d_c(Q) = 1$, $\Omega_1(Z(Q)) = \langle w \rangle$, $C_H(w) = 1$, and $\mathsf{M}_G(P;\pi') = 1$, then, for each $P^g \neq P$, $g \in G$, $m(P \cap P^g) \leqq 1$.

2.10. (Burnside's theorem [7])
If a Sylow p-subgroup of G lies in the center of its normalizer in G, then G has a normal p-complement.

2.11. (Burnside's theorem [7])
If P is a Sylow p-subgroup of G, then two normal subsets of P are conjugate in G if and only if they are conjugate in $N_G(P)$. In paticular, two elements of $Z(P)$ are conjugate in G if and only if they are conjugate in $N_G(P)$.

2.12. (Smith-Tyrer [11])
Let G be a group with an Abelian Sylow p-subgroup P for some odd prime p. If $[N(P):C(P)] = 2$ and $P \cap N(P)'$ is noncyclic, then G is p-solvable.

2.13. (Thompson Transitivity theorem [7])
Let G be a group in which the normalizer of every nonidentity p-subgroup is p-constrained. Then if $A \in SCN_3(P)$, $C_G(A)$ permutes transitively under conjugation the set of all maximal A-invariant q-subgroups of G for any prime $q \neq p$.

2.14. (Collins-Rickman [2])
Let G be a group, and let p and q be distinct prime divisors of G. Assume that G has an Abelian Sylow p-subgroup P for which $m(P) \geqq 3$ and that, whenever P_0 is a subgroup of P with $m(P/P_0) \leqq 2$, $N_G(P_0)$ is p-constrained. Then $C_G(P)$ permutes the elements of $\mathsf{M}_G^*(P;q)$ transitively under conjugation.

2.15. (Frobenius theorem [7])
G is p-nilpotent if and only if $N_G(H)/C_G(H)$ is a p-group for every nonidentity p-subgroup H of G.

3. The proof of the Theorem

Let G be a minimal counterexample to the Theorem, for the remainder of this paper.

Lemma 3.1. *G is simple.*

Proof. By Lemma 5.1. of [2].

Lemma 3.2. *Let p be a prime divisor of G and $P\in Syl_p(G)$. If $N_G(P)$ has a normal p-complement, then $p=2$ and the symmetric group S_4 is involved in G.*

Proof. By Lemma 5.2. of [2], (2.2) and (3.1).

For the remainder of this paper, Q denotes the ϕ-invariant Sylow 2-subgroup of G, and let $C_G(\phi)=\langle x\rangle$ and $\Omega_1(C_G(\phi))=\langle w\rangle$.

Then Q is a unique ϕ-invariant Sylow 2-subgroup, and let p be an odd prime in $\pi(G)$ and $P\in \mathrm{Sly}_p(G)$, then, by (3.2), $N_G(P)\ni w$.

Lemma 3.3. $d_c(Q)\geqq 2$.

Proof. If $d_c(Q)=1$, by (2.6) and hypothesis $Q=E*R$ where E is ϕ-invariant, extra-special, and R is ϕ-invariant, cyclic. If $E=1$, by (2.5) G is 2-nilpotent, contrary to (3.1). So $E\neq 1$. Since $cl(Q)=2$, by (2.4) this is a contradiction.

Lemma 3.4. *Every ϕ-invariant proper subgroup of G is 2-nilpotent.*

Proof. Assume otherwise. Let M be a non-nilpotent maximal ϕ-invariant subgroup of G without a normal Sylow 2-subgroup. If $N(O_2(M))$ is 2-nilpotent, M is nilpotent, a contradiction. By (2.2), $N(O_2(M))$ is 2-closed. Hence $M=N(O_2(M))$, $O_2(M)=Q$, and $M=N_G(Q)$. Thus there is an odd prime p dividing the index $[N_G(Q):C_G(Q)]$.

By (3.3), there is a characteristic subgroup C of Q such that $C\cong Z_2\times\cdots\times Z_2$, C contains $\Omega_1(Z(Q))$, and $[C, \phi]=1$. Let P_0 be a ϕ-invariant Sylow p-subgroup of $N_G(Q)$ and P be a ϕ-invariant Sylow p-subgroup containing P_0.

We now claim that $[C, P_0]=1$. We may assume that $w\in C$. $[w, P_0]\subseteqq Q\cap P=1$. Since P_0 centralizes $C/C_C(P_0)$, $[P_0, C]=1$. Thus $C\subseteqq N_G(P_0)$.

Let M_0 be a maximal ϕ-invariant subgroup containing $N_G(P_0)$. If M_0 is 2-closed, $M_0=N_G(Q)$. Since $N_P(P_0)=P_0$, $P=P_0$. Let Q_0 be a ϕ-invariant Sylow 2-subgroup of $N_G(P)$. Then $[P, Q_0]\subseteqq P\cap Q=1$, so $N_G(P)$ is P-nilpotent, and by (3.2), $p=2$, a contradiction. Thus M_0 is 2-nilpotent. Hence $M_0=N_G(P)$. Since $C\subseteqq N_G(P)$, $1\neq[C, \phi]\subseteqq C_G(P)$.

Now put $Z_0=[\Omega_1(Z(Q)), \phi]$. If $Z_0=1$, $P, Q\subseteqq C_G(Z_0)$. When $C_G(Z_0)$ is 2-closed, $P\subseteqq N_G(Q)$, and $[Q_0, P_0]\subseteqq Q\cap P=1$, a contradiction. Hence $C_G(Z_0)$ is 2-nilpoent. Therefore as $Q\subseteqq N_G(P)$, $[Q, P_0]\subseteqq Q\cap P=1$, a contradiction. Thus we may assume that $Z_0=1$, hence that $\Omega_1(Z(Q))=\langle w\rangle$.

Put $\bar{Q}=Q/\langle w\rangle$ and let C_1 be the inverse image of $Z(\bar{Q})\cap\bar{C}$ in Q. As $[C_1, x]\subseteqq\langle w\rangle$, $C_1\subseteqq N_G(\langle x\rangle)$. On the other hand, let $y\in C_1$. Then $[y, \phi]\in C_G(\langle x\rangle)$, since $(y^{-1}xy)^{\phi}=y^{-1}xy$. Put $C_0=[C_1, \phi]$, so that $1\neq C_0\subseteqq N_G(P)$, hence $C_G(P_0)$ contains P_0 and x.

Now let M_1 be a maximal ϕ-invariant subgroup of G containing $C_G(C_0)$. If M_1 is 2-closed, $M_1=N_G(Q)$, and $[Q_0, P]=1$, contradiction. Thus M_1 is 2-nilpotent, i.e. $M_1=N_G(P)$.

Put $\tilde{Q}=Q/\Phi(Q)$. $[x, P_0]\subseteq P\cap Q=1$. Since P_0 centralizes $\tilde{Q}/C_{\tilde{Q}}(P_0)$, P_0 centralizes Q. Hence $[P_0, Q]=1$, a contradiction. Hence the lemma is proved.

For the remainder of this paper, in analogy with Matsuyama [8], we shall prove the following result;

(i) $3/|G|$;

(ii) $|C_G(S)|$ is odd, where S is a ϕ-invariant Sylow 3-subgroup of G;

(iii) $\mathit{И}_G(S;2)\neq 1$; and

(iv) $m(S)\geqq 4$.

On the other hand, in analogy with Collins-Rickman [2], we shall prove that $\mathit{И}_G(S;2)=1$. Hence this contradicts above.

For the remainder of this paper, we shall write down the results which can be similarly proved as [8].

(3.5) $C_G(w)\subseteq Q$.

(3.6) If p is an odd prime in $\pi(G)$ and $P\in \mathrm{Syl}_p(G)$, then P is Abelian.

(3.7) If p is an odd prime in $\pi(G)$ and A is any p-subgroup of G, then $\mathrm{Aut}_G(A)=N_G(A)/C_G(A)$ is a 2-group.

(3.8) If $\Omega_1(Z(Q))\neq\langle w\rangle$, then $N_G(T)$ is a 2-group for any nontrivial ϕ-invariant 2-subgroup T of G.

Now put P be a ϕ-invariant Sylow p-subgroup of G for any odd prime p in $\pi(G)$. Let K_p be a normal 2-complement of $N_G(P)$ and $Q_p=Q\cap N_G(P)$. Then $N_G(P)=Q_pK_p$, $Q_p\subsetneqq Q$. Furthermore let $Q_p^*=C_{Q_p}(K_p)$, and then $Q_p^*=[Q_p^*, \phi]$, since $w\notin Q_p^*$.

Hence, for any $s\in\pi(K_s)$, $K_p=K_s$, $Q_p=Q_s$, and $Q_p^*=Q_s^*$. In particular, K_p is a nilpotent Hall subgroup of G.

(3.9) $C_{Q_p}(P)=Q_p^*$.

(3.10) $d_c(Q_p/Q_p^*)=1$.

Furthermore let $M_p=N_G(P)$ and $\bar{M}_p=M_p/Q_p^*K_p$. Then by (2.6) and hypothesis, $\bar{M}_p=\bar{E}_p*\bar{E}_p$, where either $\bar{E}_p=1$ or $\bar{E}_p$ is ϕ-invariant, extra-special and $\bar{R}_p$ is ϕ-invariant, cyclic.

On the other hand, by (3.4), $N_G(Q)$ is nilpotent, and then $N_G(Q)=Q$ by (3.5). Hence by (3.2), S_4 is involved in G, yields $3/|G|$. Furthermore let $S\in\mathrm{Syl}_3(G)$, and then $m(S)\geqq 3$.

Lemma 3.11. *Let p be an odd prime in $\pi(G)$. We can write $\bar{M}_p=\bar{E}_p*\bar{R}_p$, where either $\bar{E}_p=1$ or $\bar{E}_p$ is ϕ-invariant, extra-special, and $\bar{R}_p$ is ϕ-invariant,* *cyclic.*

If $\bar{E}_p \neq 1$, then $r=2^n+1$ is a Fermat prime.

Proof. By (2.7), it is immediate that $C_{\Omega_1(P)}(\phi)=C_{\Omega_1(P)}(\bar{E}_p')=0$. By (2.7), it suffices to prove that ϕ acts on $\bar{E}_p/\bar{E}_p'$ fixed-point-freely. First we may assume that $|\bar{R}_p|=2$. Then, since we can suppose that ϕ centralizes an element of $\bar{E}_p$ of order 4, it is not necessarily trivial.

Now suppose that there exists an element $\bar{y}$ of $\bar{E}_p$ of order 4 such that $[\bar{y}, \phi]=1$. As $\bar{E}_p$ is extra-special, the conjugate class of $\bar{y}$ is $\{\bar{y}, \bar{y}\bar{w}\}$. Hence $[\bar{E}_p: C_{\bar{E}_p}(\bar{y})]=2$. Then ϕ acts on the set, $\bar{E}_p-C_{\bar{E}_p}(\bar{y})$, fixed-point-freely. It is impossible.

Lemma 3.12. *Let S be a ϕ-invariant Sylow 3-subgroup of G. If $[Q_3/Q_3^*, \phi]=1$, then S is a T.I.-set.*

Proof. If not, there exists an element g of G such that $S^g \neq S$ and $S^g \cap S \neq 1$. First we shall show that $C_{Q_3}(z)=Q_3^*$ for any $z \in S^\#$. It is immediate that $C_{Q_3}(z) \supseteq Q_3^*$. If $C_{Q_3}(z) \supsetneqq Q_3^*$ for some $z \in S^\#$, $w \in C_{Q_3}(z)$, by hypothesis. But this is impossible. Next we will prove that, for any $z \in S^\#$, $C_G(z)$ is 3-nilpotent.

Now put $C_G(z)=C$ and let S_1 be a nontrivial subgroup of S. By (3.7), $\mathrm{Aut}_C(S_1)$ is a 2-group. Put $\mathrm{Aut}_C(S_1) \ni t \neq 1$. Then t is a 2-element. Furthermore there exists an element y of S_1 such that $y^t \neq y$, i.e. y and y^t are conjugate in $C_G(z)$. By (2.11), y and y^t are conjugate in $N_C(S)$. Thus we may assume that $t \in N_C(S)$, and $t \in Q_3$. Then $t \in C_{Q_3}(z)=Q_3^*=C_{Q_3}(S)$, a contradiction. Hence $C_G(z)$ is 3-nilpotent by (2.15), especially $C_G(z)$ is 3-constrained.

Furthermore put $3 \neq p \in \pi(K_3)$, and let P be a ϕ-invariant Sylow p-subgroup of G. $N_G(S)=N_G(P)$. Thus $C_G(z)$ is $\pi(K_3)$-nilpotent.

Next put $1 \neq y \in S^g \cap S$, and let M be a $\pi(K_3)$-complement of $C_G(y)$, and then we will prove that M is a 2-group.

S normalizes M and $(|S|, |M|)=1$. Now suppose that M is not a 2-group. There exists an odd prime q in $\pi(M)$ such that $q \notin \pi(K_3)$. Furthermore there exists a Sylow q-subgroup Q_1 of M normalized by S. Since $\mathrm{Aut}_G(Q_1)$ is a 2-group, $S \subseteq C_G(Q_1)$, and hence $Q_1 \subseteq K_3$. It is impossible. Thus M is a 2-group.

On the other hand, it is easy to show that $M \supseteq Q_3^*$. Now suppose that $M=Q_3^*$. Then $C_G(y)=Q_3^* K_3$, and since S, $S^g \subseteq C_G(y)$, $S=S^g$, a contradiction. Hence $M \supsetneqq Q_3^*$, $C_G(S) \subseteq N_G(M)$.

Let $\bar{M}$ be the intersection of all elements of $\mathit{И}_G^*(S;2)$. By (2.14), $\bar{M} \supseteq M$. On the other hand, as $\bar{M}$ is ϕ-invariant, $S\bar{M}$ is ϕ-invariant. By (3.4), $S\bar{M}$ is 2-nilpotent. Thus $[\bar{M}, S] \subseteq \bar{M} \cap S=1$. Hence $\bar{M} \subseteq C_G(S)=Q_3^*$, a contradiction. This completes the proof of Lemma 3.12.

Now if $\bar{E}_3 \neq 1$, $r=2^n+1$ is a Fermat prime by (3.11), where $r=|\phi|$.

On the other hand, when $\bar{E}_3=1$, by (3.12), S is a T.I.-set, where S is a ϕ-invariant Sylow 3-subgroup of G.

By B. Baumann [1], Q is not a maximal subgroup of G, and thus there exists a proper subgroup X of G containing Q such that Q is a maximal subgroup of X.

In analogy with Matsuyama [8], we can say the following.

X is a solvable $\{2, 3\}$-subgroup with $O(X)=1$, and X satisfies the hypothesis of (2.8). Thus the structure of X is one of the following two type.

⟨Type I⟩

$X/O_2(X)$ is isomorphic to S_3, the symmetric group on 4 letters. $Z(O_2(X))$ contains $Z(Q)$ and $Z(O_2(X))=[Z(O_2(X)), X]\oplus C_{Z(O_2(X))}(X)$, where $[Z(O_2(X)),X]$ is isomorphic to $Z_2\times Z_2$.

⟨Type II⟩

X has a subgroup H containing $O_2(X)$ such that $[X:H]=2$. $H/O_2(X) =X_1/O_2(X)\times X_2(O_2(X))$, $X_i/O_2(X)$ is isomorphic to S_3, $i=1,2$. $Z(O_2(X))$ contains $Z(Q)$ and $Z(O_2(X))=[Z(O_2(X)), X_1]\oplus[Z(O_2(X)), X_2]\oplus C_{Z(O_2(X))}(H)$, where $[Z(O_2(X)), X_i]$ is isomorphic to $Z_2\times Z_2$, $i=1,2$.

On the other hand, considering the structure of X, $Z(Q)$ is noncyclic, by (3.8), $Q_3{}^*=1$.

Now we will show that $\mathit{V}_G(K_3;2)\neq 1$. For the remainder of this paper, let S be a ϕ-invariant Sylow 3-subgroup of G.

Lemma 3.13. $\mathit{V}_G(S;\pi(K_3)')=\mathit{V}_G(S;2)$.

Proof. It is easy that $\mathit{V}_G(S;\pi(K_3)')\supseteqq\mathit{V}_G(S;2)$. If there exists an element A of $\mathit{V}_G(S;\pi(K_3)')$ that is not a 2-group, by [7; 6.2.2], S normalizes some Sylow p-subgroup S^* of A. As $\mathrm{Aut}_G(S^*)$ is a 2-group, $[S,S^*]=1$. But it contradicts $C_G(S)=K_3$.

By (3.13), it suffices to prove that $\mathit{V}_G(S;\pi(K_3)')\neq 1$.

Now we suppose that $\mathit{V}_G(S;\pi(K_3)')=1$. By Matsuyama [8], we can say the following.

(3.14) If $S^g\neq S$, $g\in G$, then $m(S\cap S^g)\leqq 1$.

(3.15) There exists a nontrivial proper subgroup Z_1 of $Z(Q)$ such that $3/|C_G(Z_1)|$ and $[Z(Q):Z_1]=2$.

Furthermore, in analogy with Matsuyama [8], we can show the next lemma.

Lemma 3.16. *There exists a nontrivial element a of* $\Omega_1(Z(Q))$ *such that* $|a^{\langle\phi\rangle}\cap Z_1|>\frac{1}{2}|a^{\langle\phi\rangle}|$ *or* $\Omega_1(Z(Q))^{\#}=\{a^{\langle\phi\rangle}\}$.

Proof. Put $a_1 \in \Omega_1(Z(Q))^\#$, $a_1 \neq w$. Let $A_1 = \{a_1^{\langle\phi\rangle}\}$. If there exists an element of $\Omega_1(Z(Q))^\# - A_1$ that does not equal w, let a_2 denote this element. So let $A_2 = \{a_2^{\langle\phi\rangle}\}$, and then $A_1 \cap A_2 = \phi$. Inductively, if there exists an element of $\Omega_1(Z(Q))^\# - \bigcup_{k=1}^{i-1} A_i$ that does not equal w, we let a_i denote this element. Then we can write the following,

$$\Omega_1(Z(Q))^\# - \langle w \rangle = \bigcup_{i=1}^{m} A_i\,,$$

where $A_i \cap A_j = \phi$ if $i \neq j$, $1 \leqq i, j \leqq m$.

Now suppose that $m \geqq 2$. Let $|\Omega_1(Z(Q))| = 2^n$, and as $[\Omega_1(Z(Q)) : \Omega_1(Z_1)] = 2$, $|\Omega_1(Z_1)| = 2^{n-1}$. If, any i, $1 \leqq i \leqq m$, $|a_i^{\langle\phi\rangle} \cap Z_1| \leqq \frac{1}{2} |a_i^{\langle\phi\rangle}|$, then since $|a_i^{\langle\phi\rangle}| = r$ is odd. $|\bigcup_{i=1}^{m} (a_i^{\langle\phi\rangle} \cap Z_1)| \leqq |\Omega_1(Z(Q)) - \Omega_1(Z_1)| - 2$.

But, on the other hand, $|\Omega_1(Z(Q)) - \Omega_1(Z_1)| = 2^{n-1}$, and $|\Omega_1(Z_1)^\#| = 2^{n-1} - 1$. It is impossible. Hence $m = 1$. $\Omega_1(Z(Q))^\# = \{a_1^{\langle\phi\rangle}\}$. This lemma is proved.

(3.17) a^{ϕ^i} normalizes some Sylow 3-subgroup of G, $0 \leqq i \leqq r-1$.

Now put $\Delta_i = (a^{\phi^i})^G \cap Q_3$, and then $\Delta_i \neq \phi$, and $\Delta_i^\phi = \Delta_{i+1}$, $0 \leqq i \leqq r-1$. Furthermore, as $Q_3{}^* = 1$, $Q_3 = E_3 * R_3$.

If $E_3 = 1$, then S is a T.I.-set, by (3.13). In analogy with the above argument, we can show that $\Delta_i \neq \phi$, $0 < i < r-1$.

But, in this time, w is an only involution in Q_3. This is a contradiction.

Hence, for the remainder of this paper, we may assume that $E_3 \neq 1$, i.e. $r = 2^n + 1$ is a Fermat prime. Then (3.16) is reduced that there exists a nontrivial element a of $\Omega_1(Z(Q))$ such that $|a^{\langle\phi\rangle} \cap Z_1| > \frac{1}{2} |a^{\langle\phi\rangle}|$.

On the other hand, $m(S) \geqq 4$.

(3.18) There exists an element b_i, b_j of Δ_i, Δ_j, respectively, $0 \leqq i, j \leqq r-1$, $i \neq j$, $[b_i, b_j] = 1$.

Next Δ_i is determined as the following.

Lemma 3.19. $\Delta_i = \{b_i, b_i w\}$, $0 \leqq i \leqq r-1$, $b_i \neq w$.

Proof. If $w \in \Delta_i$, then w centralizes some element of order 3, a contradiction. Thus $w \notin \Delta_i$.

For the remainder, we set $b = b_i$.

Suppose that b, $b^g \in \Delta_i$, $g \in G$, $b \neq b^g$. Then b, $b^g \in Q_3$. Since $S = C_S(w) \oplus C_S(bw)$, $\frac{1}{2} m(S) = m(C_S(b)) = m(C_S(bw)) \geqq 2$.

Let S^* be a Sylow 3-subgroup of $C_G(b^g)$ containing $C_S(b^g)$. There exists an element h of $C_G(b^g)$ such that $(C_S(b))^{gh} \subseteq S^*$. On the other hand, let S_0 be a

Sylow 3-subgroup of G containing S^*, and then $S=S_0$ as $C_S(b^g)\subseteq S\cap S_0$. Since $(C_S(b))^{gh}\subseteq S\cap S^{gh}$, $gh\in N_G(S)$. Since $b^g=b^{gh}$, b and b^g are conjugate in $N_G(S)$. As $N_G(S)=Q_3K_3$, b and b^g are conjugate in Q_3. Hence $\Delta_i=\{b_i, b_iw\}$.

Now put $\Delta=\langle\Delta_i\,|\,0\leqq i\leqq r-1\rangle$, and then, by (3.19), Δ is ϕ-invariant Abelian. Furthermore, as $[\Delta,\phi]\neq 1$, $[\Delta,\phi]K_3$ is nilpotent, and

$$1\neq[\Delta,\phi]\subseteq C_{Q_3}(K_3)=Q_3^*=1\,,$$

this is a contradiction. Hence $\varkappa_G(S;2)\neq 1$.

On the other hand, we will prove the next lemma, and then, in analogy with Collins-Rickman [2], the proof of the main theorem is complete.

Lemma 3.20. *Let S_0 be a proper subgroup of S such that $m(S/S_0)\leqq 2$. Then $N_G(S_0)$ is 3-solvable.*

Proof. First we shall consider the case $m(S_0)>2$. In this case, we will show that $C_G(S_0)$ is 3-nilpotent. Put $C=C_G(S_0)$, and let S_1 be a nontrivial subgroup of S. If there exists a nontrivial element t of $\mathrm{Aut}_C(S_1)$, t is a 2-element as $\mathrm{Aut}_C(S_1)$ is a 2-group. Then there exists an element y of S_1 such that $y^t\neq y$. Thus y and y^t are conjugate in $C_G(S_0)$. By (2.11), y and y^t are conjugate in $N_C(S)$. Hence we may assume that $t\in Q_3\cap C\cong Z_2\times\cdots\times Z_2$. As $t\neq w$, $S=C(t)\oplus C_S(tw)$. Hence

$$\frac{1}{2}m(S)=m(C_S(t))=m(C_S(tw))\,.$$

This is a contradiction. By (2.15), $C_G(S_0)$ is 3-nilpotent. $C_G(S_0)/S_0$ is 3-solvable. Hence $N_G(S_0)$ is 3-solvable.

Now we may assume that $m(S)=4$ and $m(S_0)=2$. In this case, similarly, if $C_{M_3}(S_0)=C(S)$, $M_3=N_G(S)$, then by (2.10), $C_G(S_0)$ is 3-nilpotent. Hence, furthermore, we may assume that $C_{M_3}(S_0)\supsetneqq C(S)$.

If there exists an element x_0 of $C_{M_3}(S_0)$ such that $|x_0|=4$, then $x_0^2=w\in C_{M_3}(S_0)$, a contradiction.

If there exists a four-group $\langle x_1\rangle\times\langle x_2\rangle$ in $C_{M_3}(S_0)$, then $S=\langle C_S(x_1), C_S(x_2), C_S(x_1x_2)\rangle$. On the other hand, S_0 is contained in $C_S(x_1)$, $C_S(x_2)$, and $C_S(x_1x_2)$, and since

$$m(C_S(x_1))=m(C_S(x_2))=m(C_S(x_1x_2))=2\,,$$

$C_S(x_1)=C_S(x_2)=C_S(x_1x_2)=S_0$, a contradiction. Hence we can write the following;

$$C_{M_3}(S_0)=C(S)\langle t\rangle\,,$$

where $t^2\in C(S)$ and $S=S_0\oplus[S,t]$.

Put $\overline{C_G(S_0)}=C_G(S_0)/S_0$. Then $\bar{S}=\bar{S}\cap N_{\overline{(C_G S_0)}}(\bar{S})'$. By (2.12), $\overline{C_G(S_0)}$ is 3-solvable. Hence, in this case, $N_G(S_0)$ is 3-solvable. This lemma is complete.

Now we already proved that $\text{И}_G^*(S;2)\neq 1$. Next we will show that there exists a ϕ-invariant element Q_1 of $\text{И}_G^*(S;2)$. Suppose false. Since $\text{И}_G^*(S;2)$ is ϕ-invariant, r divides $|\text{И}_G^*(S;2)|$. On the other hand, by (2.14), the element of $\text{И}_G^*(S;2)$ permuted by $C(S)$ transitively. This is a contradiction.

Let $N=SQ_1$. By (3.4), N is nilpotent. Hence

$$Q_1\subseteqq C_G(S)=C_G(K_3)\,.$$

On the other hand, as $Q_3^*=1$, $|C_G(S)|$ is odd. This is a contradiction. The main theorem is proved.

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