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CORRECTIONS AND SUPPLEMENTS TO "INDEX OF THE EXPONENTIAL MAP ON A COMPLEX SIMPLE LIE GROUP"

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In this note, we shall fix up some gaps in my previous papers [2] and [3], and will discuss related topics. Since [2] is a special case of [3], we may restrict our attension to [3] and will retain the notation adopted there throughout the paper.

In [3], the proofs of Lemma 1 and Proposition in Section 3 were incomplete. After some preparations, we will give a detailed proof for each of them in §3. On the way, we can see some relations between the index of a connected complex semisimple Lie group and the index of its Borel subgroup, which will be stated in the last part of this note.

I am indebted to Professor Morikuni Goto who pointed out my mistake and gave me great help to correct it.

1. On a theorem of Kostant concerning three dimensional subalgebras

Let G be a complex semisimple Lie algebra, A a nilpotent element ($\neq 0$) in G. According to Kostant [1], we can find a semisimple h and a nilpotent B in G so that

$$(*) \qquad [h,A]=A, \quad [h,B]=-B, \quad [A,B]=h\,.$$

Furthermore, the three dimensional subalgebra $S=Ch+CA+CB$ is uniquely determined by A up to conjugacy, i.e. if A is conjugate with A', then a three dimensional subalgebra $S'=Ch'+CA'+CB'$ corresponding to A' is conjugate with S, and so is h' to h.

Proposition B. *If A is a regular nilpotent element in G* (c.f. [4]), *then the h satisfying* (*) *must be a regular semisimple element.*

Proof. By [4], we can suppose that $A=\sum_{j=1}^{l} e_{\alpha_j}$. We shall choose $B\in G$

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such that $h=\sum_{j=1}^{k} h_j$, A and B satisfy (*). Notice that $\alpha_j(h)=1$ for $j=1,\cdots,l$, and h is regular.

Let $c_{kj}=-2\langle\alpha_k,\alpha_j\rangle/\langle\alpha_j,\alpha_j\rangle$ $(k,j=1,\cdots,l)$ be Cartan integers. The Cartan matrix $(-c_{kj})$ is known to be non-singular and so the system of linear equations

$$\sum_{k=1}^{l} y_k c_{jk} = 1 \quad (j=1,\cdots,l)$$

has a unique solution $y_1,\cdots,y_l$. We put

$$B=\sum_{k=1}^{l} \frac{2}{\langle\alpha_k,\alpha_k\rangle} y_k e_{-\alpha_k}.$$

Then

$$\begin{aligned}[A,B] &= \sum_{k=1}^{l} \frac{2}{\langle\alpha_k,\alpha_k\rangle} y_k [e_{\alpha_k}, e_{-\alpha_k}] = -\sum_{k=1}^{l} \frac{2}{\langle\alpha_k,\alpha_k\rangle} y_k h_{\alpha_k} \\ &= -\sum_{k=1}^{l} y_k h^*_{\alpha_k} = -\sum_{k=1}^{l} y_k \left(-\sum_{j=1}^{l} c_{jk} h_j\right) \\ &= \sum_{j=1}^{l} h_j = h,\end{aligned}$$

and it is easy to see that

$$[h,A]=A, \quad [h,B]=-B. \qquad \text{Q.E.D.}$$

2. A key lemma

G still denotes a complex semisimple Lie algebra of rank l.

Lemma C. *Let G_1 be a semisimple subalgebra of G with the same rank l, A a nilpotent element which is regular in G_1. Then $z_G(A)$ (the centralizer of A in $G=\{x\in G;\ [x,A]=0\}$) is composed of nilpotent elements.*

Proof. It suffices to prove: for any nonzero semisimple element $x_0\in G$, A is not conjugate with any (nilpotent) element in $z_G(x_0)$.

Any semisimple element $x_0\in G$ is conjugate to some element h_0 in the (fixed) Cartan subalgebra H. Moreover, if $x_0=Adg\cdot h_0$, then $z_G(x_0)=Adg\cdot z_G(h_0)$. So we may assume that $x_0\neq 0$ lies in H.

Next, any $x_0\in H$ can be expressed as $x_0=\sum_{j=1}^{l}(c_j+id_j)h_j (c_j,d_j\in \boldsymbol{R})$. Denote $x_1=\sum_{j=1}^{l} c_j h_j$, $x_2=\sum_{j=1}^{l} d_j h_j$, so that $x_0=x_1+ix_2$ with $x_1, x_2\in H_0(=\sum_{\alpha\in\Delta}\boldsymbol{R}h_\alpha=\sum_{j=1}^{l}\boldsymbol{R}h_i)$. Notice that $z_G(x_0)$ is generated by H and those e_α's with $\alpha\in\Delta$ satisfying $0=\alpha(x_0)=\alpha(x_1)+i\alpha(x_2)$, which implies that the two real numbers $\alpha(x_1)$ and $\alpha(x_2)$ must be zero, i.e. $z_G(x_0)=z_G(x_1)\cap z_G(x_2)$. By assumption, $x_0\neq 0$, so $x_1\neq 0$ or $x_2\neq 0$. Without loss of generality, we may assume that $x_1\neq 0$. Since $z_G(x_0)\subset z_G(x_1)$, to prove the Lemma, we may replace x_0 by x_1, i.e. we assume that $x_0 \in H_0$.

Denote $W_0 = \{x \in H_0;\ \alpha_j(x) > 0\ \ j = 1, \cdots, l\}$. Since the Weyl group $Ad(\Delta)$ acts transitively on the set of Weyl chambers, any element x_0 in H_0 is conjugate with an element in $\bar{W}_0$. So we reduce our problem to the case that $x_0 \in \bar{W}_0$, i.e. $x_0 = y_1 h_1 + \cdots + y_l h_l$ with $y_j \geq 0$ $(j = 1, \cdots, l)$. It is easy to see that $z_G(x_0)$ is generated by H and those e_α's with $\alpha \in \Delta$ satisfying the following condition: If $\alpha = \sum_{j=1}^{l} n_j \alpha_j$, then $n_j = 0$ whenever $y_j \neq 0$; i.e. $z_G(x_0) = H + \sum_{\beta \in \Delta_1} \boldsymbol{C} e_\beta$ where $\Delta_1 = \Delta \cap \sum_{i \in I} \boldsymbol{Z} \alpha_i$ with $I = \{i;\ 1 \leq i \leq l \text{ and } y_i = 0\}$. The assumption $x_0 \neq 0$ implies that y_j cannot be all zero, say $y_k \neq 0$. Thus we have $z_G(x_0) \subset z_G(h_k)$. Let us prove that a regular nilpotent element A is not conjugate with any nilpotent element in $z_G(h_k)$.

Any nilpotent element $A' \in z_G(h_k)$ lies in $G_2 = [z_G(h_k), z_G(h_k)]$. Let h and B (resp. h' and B') be chosen to satisfy relation (*) for the element A (resp. A') in the last section. We may choose $h, B \in G_1$ and $h', B' \in G_2$ (because G_1 and G_2 are semisimple subalgebras). It is easy to see that G_2 has $\sum_{j \neq k} \boldsymbol{C} h_j$ as its Cartan subalgebra, and any element $x \in \sum_{j \neq k} \boldsymbol{C} h_j$ satisfies $\alpha_k(x) = 0$, so x is non-regular when considered as an element in G. Therefore, the semisimple element $h' \in G_2$ is non-regular in G. On the other hand, Proposition B shows that h is regular in G_1, so h is regular in G because rank G = rank G_1. Hence h cannot be conjugate with h'. By Kostant's theorem, A cannot be conjugate with A'. This finishes our proof.

3. Corrections to [3]

Throughout this section, we shall follow the notation used in [3].

First, we give a correct proof for Lemma 1 [3].

Let h_0 and $\beta_1, \cdots, \beta_l$ be the same as in p. 563 [3].

The argument in p. 563 [3] proves that we can find a positive integer d and some element $h \in \Omega'$ such that $\beta_j(dh_0 + h) = 0$ for $j = 1, \cdots, r$, i.e. $\alpha(dh_0 + h) = 0$ for all $\alpha \in \Delta(h_0)$. Let d be the smallest positive integer for this to be true, then ind$(\exp h_0 \cdot \exp N)$ is a factor of d.

Assume that $\beta_i = \sum_{j=1}^{l} q_{ij} \alpha_j$. Consider the following system of linear equations in the unknowns $y_1, \cdots, y_l$:

$$q_{i1} y_1 + \cdots + q_{il} y_l = 2\pi i k_i \qquad i = 1, \cdots, r;$$
$$q_{i1} y_1 + \cdots + q_{il} y_l = 0 \qquad i = r+1, \cdots, l.$$

Since (q_{ij}) is a nonsingular matrix (because $\beta_1, \cdots, \beta_l$ is linearly independent), this has a (unique) solution which is nontrivial because some $k_i \neq 0$ by our assumption on h_0.

Let $h_0' = \sum_{j=1}^{l} y_j h_j$, then $\beta_1(h_0') = \beta_1(h_0), \cdots, \beta_r(h_0') = \beta_r(h_0)$ and $\beta_1, \cdots, \beta_l \in$

$\Delta(h_0')$. Suppose that d' is the smallest positive integer for which we can find $h'\in\Omega'$ satisfying $\beta_j(d'h_0'+h')=0$ for $j=1,\cdots,l$, then $\alpha(d'h_0'+h')=0$ for any $\alpha\in\Delta$ because α can be written as a rational linear combination of $\beta_1,\cdots,\beta_l$ (they are linearly independent and $l=\text{rank } G$), this implies that $d'h_0'+h'=0$, or $d'h_0'\in\Omega'$, and d' is the smallest positive integer for this to hold.

On the other hand, $\beta_j(d'h_0+h')=\beta_j(d'h_0)+\beta_j(h')=\beta_j(d'h_0')+\beta_j(h')=\beta_j(d'h_0'+h')=0$ for $j=1,\cdots,r$, so that d' must be a multiple of d, and hence a multiple of $\text{ind}(\exp h_0\cdot\exp N)$.

The proof of Lemma 1[3] will be complete after we prove the following lemma.

Lemma 1 A. *There exists a nilpotent element* $N'\in\sum_{\beta\in\Delta(h_0')^+} Ce_\beta$ *so that*

$$ind(\exp h_0'\cdot\exp N') = d' .$$

Also, we need a more detailed discussion in the proof of section 3 [3] for that element we chose to have index exactly equal to p_jm_j.

Proof of Lemma 1 A. Recall that $h_0'\in H$ was chosen so that $\pi(h_0')$ has cardinality $l=\text{rank } G$, i.e. $G_1=G(1, Ad \exp h_0')$ is a semisimple subalgebra of G with rank l. If $\mathfrak{G}_1$ is the connected Lie subgroup of $\mathfrak{G}$ with G_1 as its Lie algebra, then $\exp h_0'$ is a central element in $\mathfrak{G}_1$. Note that d' as we have chosen is exactly equal to the order of this central element.

Let $N'\in\sum_{\beta\in\Delta(h_0')^+} Ce_\beta$ be a regular nilpotent element in G_1, and $g=\exp h_0'\cdot\exp N'$. We claim that $\text{ind}(g)=d'$.

Let q be a positive integer so that $g^q=\exp x$ for some $x\in G$. Consider the Jordan decomposition of x: $x=x_0+N_0$, where x_0 is semisimple, N_0 is nilpotent and $[x_0,N_0]=0$. The equality $\exp x_0\cdot\exp N_0=\exp x=g^q=\exp qh_0'\cdot\exp qN'$ and the uniqueness of decomposition imply that $\exp N_0=\exp qN'$. But the exponential map is one-one on the nilpotent part, so $N_0=qN'$. Therefore $x=x_0+qN'$ with x_0 semisimple and $[x_0,N']=0$. Since N' is a regular nilpotent element in the semisimple subalgebra G_1, which has rank $l=\text{rank } G$, we conclude from Lemma C that $x_0=0$, i.e. $(\exp h_0')^q=\exp x_0=1$. This implies that q must be a multiple of d' and proves that $\text{ind}(g)=d'$. Q.E.D.

A similar argument can prove the assertion we made in section 3 [3]. Let $h_0=2\pi ih_j/m_j$, $N=\sum_{0\le i\le l, i\ne j}e_{\alpha_i}$, N is a regular nilpotent element in the semisimple subalgebra $G_1=G(1,Ad \exp h_0)$, and rank $G_1=l=\text{rank } G$.

Let $g=\exp h_0\cdot\exp N$. If q is a positive integer so that $g^q=\exp x$ for some $x\in G$, then $x=x_0+qN$ with x_0 semisimple satisfying $[x_0,N]=0$. Apply Lemma C again, we conclude that $x_0=0$, so $(\exp h_0)^q=\exp x_0=1$. This implies that $qh_0\in\Omega^*$, the smallest q for this to hold is p_jm_j, so that $\text{ind}(g)=p_jm_j$.

This gives a complete proof for the main theorem in [3].

4. ind ($\mathfrak{B}$) ($\mathfrak{B}$ denotes a Borel subgroup of $\mathfrak{G}$)

In this section, we let $\mathfrak{G}$ be a connected complex semisimple Lie group, $\mathfrak{B}$ a Borel subgroup of $\mathfrak{G}$ (it is well known that $\mathfrak{B}$ is uniquely determined up to conjugacy). We like to study the relation between $\mathrm{ind}_{\mathfrak{B}}(g)$ and $\mathrm{ind}_{\mathfrak{G}}(g)$ for $g\in\mathfrak{B}$, where $\mathrm{ind}_{\mathfrak{B}}(g)$ (resp. $\mathrm{ind}_{\mathfrak{G}}(g)$) denotes the index of g regarded as an element in $\mathfrak{B}$ (resp. in $\mathfrak{G}$). Clearly, $\mathrm{ind}_{\mathfrak{B}}(g)\geq\mathrm{ind}_{\mathfrak{G}}(g)$ for any $g\in\mathfrak{B}$.

We may assume that the Lie subalgebra B corresponding to $\mathfrak{B}$ is given by $B=H+\sum_{\alpha\in\Delta^+}\boldsymbol{C}e_\alpha$.

Any element in $\mathfrak{B}$ can be expressed as $\exp h_0\cdot\exp N$ with $\exp h_0\cdot\exp N=\exp N\cdot\exp h_0$. In the proof of Lemma 1[3], we may choose $\beta_1,\cdots,\beta_l$ from Δ^+, so that the discussion can be restricted on B. We have proved that $\mathrm{ind}_{\mathfrak{G}}(g)$ is a factor of d'. Moreover, for the particular element h_0' we chose and the corresponding regular nilpotent element N' in $\sum_{\beta\in\Delta(h_0')^+}\boldsymbol{C}e_\beta(\subset\sum_{\alpha\in\Delta^+}\boldsymbol{C}e_\alpha)$, $\mathrm{ind}_{\mathfrak{G}}(\exp h_0'\cdot\exp N')=d'$. But it is clear that

$$(\exp h_0'\cdot\exp N')^{d'}=\exp d'N'\in\mathfrak{B}$$

and $(\exp h_0\cdot\exp N)^d=\exp(dh_0+h+dN)\in\mathfrak{B}$ (for some $h\in\Omega'$). Therefore $\mathrm{ind}_{\mathfrak{G}}(\exp h_0\cdot\exp N)$ is also a factor of d' and $\mathrm{ind}_{\mathfrak{B}}(\exp h_0'\cdot\exp N')=d'$. We have proved the following:

For any $g\in\mathfrak{B}$, we can find $g'\in\mathfrak{B}$, so that $\mathrm{ind}_{\mathfrak{B}}(g)$, as well as $\mathrm{ind}_{\mathfrak{G}}(g)$ is a factor of $\mathrm{ind}_{\mathfrak{B}}(g')=\mathrm{ind}_{\mathfrak{G}}(g')$.

If $\mathrm{ind}_{\mathfrak{B}}(g')=\mathrm{ind}_{\mathfrak{G}}(g')=d=qr$, then clearly $\mathrm{ind}_{\mathfrak{B}}(g'^q)=\mathrm{ind}_{\mathfrak{G}}(g'^q)=r$. We conclude that:

Proposition. *If $\mathfrak{G}$ is a connected complex semisimple Lie group, $\mathfrak{B}$ its Borel subgroup, then $\{\mathrm{ind}_{\mathfrak{B}}(g);\ g\in\mathfrak{B}\}=\{\mathrm{ind}_{\mathfrak{G}}(g);\ g\in\mathfrak{B}\}=\{\mathrm{ind}_{\mathfrak{G}}(g);\ g\in\mathfrak{G}\}$. In particular $\mathrm{ind}(\mathfrak{B})=\mathrm{ind}(\mathfrak{G})$.*

This is also true for a connected complex reductive Lie group.

REMARK. It is not clear to me whether $\mathrm{ind}_{\mathfrak{B}}(g)=\mathrm{ind}_{\mathfrak{G}}(g)$ for any $g\in\mathfrak{B}$.

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