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ON THE IRREDUCIBILITY OF 2-FOLD BRANCHED COVERS OF S^3

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0. Introduction. Montesinos [8] and Hilden [2] showed that every closed, orientable 3-manifold is a 3-fold irregular covering space of S^3 branched over a link ℓ. And Waldhausen [10] showed that two homotopy equivalent closed orientable, sufficiently large 3-manifolds are homeomorphic. So we study that what kind of 3-manifold is irreducible i.e. an embedded 2-sphere in the 3-manifold bounds a 3-ball. Using the result of Montesinos [8] and the surgery technique, we obtain the following.

Theorem. *Let $\ell=k_1\cup\cdots\cup k_\mu$ be a link in S^3 such that every component k_i $(i=1, 2, \cdots, \mu)$ of ℓ is a trivial knot. If $M(\ell)$ is a 2-fold covering space of S^3 branched over ℓ and if $\pi_2(M(\ell))=0$, $M(\ell)$ is irreducible.*

And in section 2 we study a method of determining whether $\pi_2(M(l))=0$ or not for a given link ℓ whose components are all trivial knots.

1. Proof of Theorem

Lemma 1. *Let k be a trivial knot in S^3. If B^3 is a 3-ball in S^3 such that the intersection $\mathcal{D}^1$ of B^3 with k is homeomorphic to the 1-ball, the pair $(B^3, \mathcal{D}^1)$ is a standard pair (i.e. there is an orientation preserving homeomorphism $h:(B^3, \mathcal{D}_1)\to(D^1\times D^2, D^1\times\{0\})$ where D^n is the standard n-ball.*

Proof. Since k is a trivial knot, there is an embedded 2-ball B^2 in S^3 with $\partial B^2=k$. We may assume that B^2 meets ∂B^3 transversally and so $B^2\cap\partial B^3=$ $\{a\ simple\ arc\}\cup\{simple\ closed\ curves\}$. Let α be a simple closed curve in $B^2\cap\partial B^3$ which is innermost in $B^?$ with respect to $B^2\cap\partial B^3$. α splits ∂B^3 into two 2-balls. Let B_α be one of the two 2-balls such that B_α does not contain the simple arc in $B^2\cap\partial B^3$. Since α is innermost in B^2 with respect to $B^2\cap\partial B^3$, there is a 2-ball B'_α in B^2 with $B'_\alpha\cap B^3=\partial B'_\alpha=\alpha$. Then $B_\alpha\cup_\partial B'_\alpha=S^2$ and so $B_\alpha\cup_\partial B'_\alpha$ bounds a 3-ball. Hence there is an ambient isotopy $\{\phi_t\}: S^3\to S^3$ $(0\leqq t\leqq 1)$ keeping α fixed such that $\phi_0=id.$, $\phi_1(B'_\alpha)=B_\alpha$. We may assume that the support of $\{\phi_t\}$ is a small neighborhood of "one of" 3-balls bounded by the 2-sphere $B_\alpha \cup_\partial B'_\alpha$ in S^3. Then $\partial B^3 \cap \phi_1(B^2) \subsetneqq \partial B^3 \cap B^2$. In fact the components of $B^2 \cap \partial B^3$ contained in B_α can be eliminated. Repeating this process, we obtain an ambient isotopy $\{\Phi_t\}: S^3 \to S^3$ $(0 \leqq t \leqq 1)$ such that $\Phi_0 = id.$ and $\partial B^3 \cap \Phi_1(B^2) =$ a simple arc γ. Then the 1-sphere $\gamma \cup_\partial \mathcal{D}^1$ bounds a non-singular sub-2-ball in B^2. Hence $(B^3, \mathcal{D}^1)$ is the standard pair.

DEFINITION (Equivariant surgery). Let M be a closed orientable 3-manifold and F be a 2-sided closed surface in M. Let τ be an involution of M. We may assume that F meets $\tau(F)$ transversally and so $F \cap \tau(F)$ is the disjoint union of simple closed curves. If there is a 2-ball D in $\tau(F)$ such that $D \cap F = \partial D$ and ∂D splits F into two components, we choose a small product neighborhood $D \times [-1,1]$ of D in M with $D \times \{0\} = D$, $\partial D \times [-1,1] \subset F$. Let $D_+ = D \times \{1\}$, $D_- = D \times \{-1\}$ and $F - (\partial D \times (0,1)) = F'_+ \cup F'_-$ where $F'_+ \cap D_+ = \partial D_+$ and $F'_- \cap D_- = \partial D_-$. Define $F_+ = F'_+ \cup_\partial D_+$ and $F_- = F'_- \cup_\partial D_-$. We say that F_+, F_- have been obtained by *equivariant surgery from F using D*. If τ is a free involution or $Fix(\tau) \cap \partial D = \phi$, it is known that $\#(F_i \cap \tau(F_i)) < \#(F \cap \tau(F))$ $(i=1,2)$ (see Hempel [1. p. 94]) where $\#(F \cap \tau(F))$, $\#(F_i \cap \tau(F_i))$ are the number of component of $F \cap \tau(F)$, $F_i \cap \tau(F_i)$ respectively.

Theorem. *Let $\ell = k_1 \cup \cdots \cup k_\alpha$ be a link in S^3 such that every component k_i $(i=1, 2, \cdots, \mu)$ of ℓ is a trivial knot. If $M(\ell)$ is a 2-fold covering space of S^3 branched over ℓ and if $\pi_2(M(\ell))=0$, $M(\ell)$ is irreducible.*

REMARK. If $M(\ell)$ is a homology 3-sphere, the theorem follows immediately as follows. If τ is a non-trivial covering translation of $M(\ell)$, τ is a periodic map of period 2. So the fixed points set of τ is Z_2-homology sphere by P.A. Smith [9]. Hence the fixed points set of τ is the 1-sphere k and so $\ell = p(k)$ is a knot where $p: M(\ell) \to S^3$ is the 2-fold covering space branched over ℓ. Since $\ell = p(k)$ is a trivial knot by the assumption, $M(\ell) \cong S^3$ and is irreducible.

Proof of Theorem. Let $p: M(\ell) \to S^3$ be a 2-fold covering of S^3 branched over the link ℓ and let $\tau: M(\ell) \to M(\ell)$ be the non-trivial covering translation (so $\tau^2 = id.$). Since $\pi_2(M(\ell))=0$, any embedded 2-sphere in $M(\ell)$ bounds a homotopy 3-ball [3] i.e. a compact contractible 3-manifold. So it is sufficiently to show that a homotopy 3-ball $\mathcal{B}$ in $M(\ell)$ is a 3-ball.

Case (A): $\mathcal{B} \cap p^{-1}(\ell) = \phi$.

Case (Aa): $\mathcal{B} \cap \tau(\mathcal{B}) = \phi$. Then $p|\mathcal{B}$ is a homeomorphism and $p(\mathcal{B})$ is a 3-ball since $p(\mathcal{B}) \subset S^3$. So $\mathcal{B}$ is a 3-ball.

Case (Ab): $\mathcal{B} \cap \tau(\mathcal{B}) \neq \phi$. (i) *If $\partial\mathcal{B} \cap \tau(\partial\mathcal{B}) = \phi$*, then we have that (1) $\mathcal{B} \subset \tau(\mathcal{B})$, (1)′ $\tau(\mathcal{B}) \subset \mathcal{B}$ or (2) $\mathcal{B} \cup \tau(\mathcal{B}) = M(\ell)$. If (2) holds $M(\ell)$ is a homotopy 3-sphere and so is a 3-sphere by Remark and so it is irreducible. If (1) or (1)′ hold, $\tau|\mathcal{B}$ must have fixed points. But the fixed points set of τ is $p^{-1}(\ell)$. It is a contradiction that $\mathscr{B}\cap p^{-1}(\ell)=\phi$. So the case (Ab) (i) (1) and (1)′ can not happen. In this case (Ab) (i) the proof is completed.

Case (ii): $\partial\mathscr{B}\cap\tau(\partial\mathscr{B})\neq\phi$. We assume that $\partial\mathscr{B}$ meets $\tau(\partial\mathscr{B})$ transversally and so $\partial\mathscr{B}\cap\tau(\partial\mathscr{B})$ is the disjoint union of simple closed curves. Let D be a 2-ball in $\tau(\partial\mathscr{B})$ with $D\cap\partial\mathscr{B}=\partial D$.

If $\tau(\partial D)=\partial D$, $p(D)$ is a projective 2-space in S^3. It is a contradiction. So $\tau(\partial D)\cap\partial D=\phi$. Then we obtain two 2-spheres S_1, S_2 by the equivariant surgery from $\partial\mathscr{B}$ using D and $\#(S_i\cap\tau(S_i))<\#(\partial\mathscr{B}\cap\tau(\partial\mathscr{B}))$ $(i=1,2)$. And since $\pi_2(M(\ell))=0$, S_i bounds a fake 3-ball $\mathscr{B}_i$. Thus in this case we can reduce $\#(\partial\mathscr{B}\cap\tau(\partial\mathscr{B}))$.

Case (B): $\mathscr{B}\cap p^{-1}(\ell)\neq\phi$ and $\partial\mathscr{B}\cap p^{-1}(\ell)=\phi$.

Case (Ba): $\partial\mathscr{B}\cap\tau(\partial\mathscr{B})=\phi$. Since $\tau\,|\,p^{-1}(\ell)=id.$, $\mathring{\mathscr{B}}\supset\tau(\mathscr{B})$ or vice versa. So we may assume $\mathring{\mathscr{B}}\supset\tau(\mathscr{B})$. Put $\mathscr{A}=\mathscr{B}-\tau(\mathring{\mathscr{B}})$. Then $\partial\mathscr{A}=\partial\mathscr{B}\cup\tau(\partial\mathscr{B})$ and $\tau(\partial\mathscr{A})=\partial\mathscr{A}$. On the other hand $\tau(\mathscr{A})=\tau(\mathscr{B}-\tau(\mathring{\mathscr{B}}))=\tau(\mathscr{B})-\mathring{\mathscr{B}}=\phi$. It is a contradiction. So the case (Ba) can not happen.

Case (Bb): $\partial\mathscr{B}\cap\tau(\partial\mathscr{B})\neq\phi$. If $\partial\mathscr{B}=\tau(\partial\mathscr{B})$, $p(\partial\mathscr{B})$ is the projective 2-space in S^3. So $\partial\mathscr{B}\neq\tau(\partial\mathscr{B})$. Since we may assume that $\partial\mathscr{B}$ meets $\tau(\partial\mathscr{B})$ transversally, $\partial\mathscr{B}\cap\tau(\partial\mathscr{B})$ is the disjoint union of simple closed curves. By the same way of Case (Ab) (ii), we can eliminate the components of $\partial\mathscr{B}\cap\tau(\partial\mathscr{B})$.

Case (C): $\partial\mathscr{B}\cap p^{-1}(\ell)\neq\phi$.

Denote $\partial\mathscr{B}\cap\tau(\partial\mathscr{B})=S\cup T=S_1\cup S_2\cup T_1\cup T_2$ where $S=S_1\cup S_2=(\partial\mathscr{B}\cup\tau(\partial\mathscr{B}))-p^{-1}(\ell)$, $T=T_1\cup T_2=\partial\mathscr{B}\cap\tau(\partial\mathscr{B})\cap p^{-1}(\ell)$,

$S_1=\{$*simple closed curves*$\}$,

$S_2=\{$*simple open arcs*$\}$; $(\bar{S}_2-S_2=T_1)$,

$T_1=\{t_1\,|\,t_1$ *is a boundary point of some elements of* $S_2\}$; a set of finite points,

$T_2=\{t_2\,|\,t_2$ *has a small neighborhood* $U(t_2)$ *in* $\partial\mathscr{B}\cap\tau(\partial\mathscr{B})$ *such that* $U(t_2)\cap(S\cup T)=t_2\}$; a set of finite points.

Sub-lemma. $T_2=\phi$.

We may assume that $\partial\mathscr{B}$ meets $p^{-1}(\ell)$ transversally.

If $T_2\neq\phi$, let ω be a point in T_2. Take a small neighborhood $U(\omega)$ of ω in $M(\ell)$ with $U(\omega)\cong B^3$ (*3-ball*). Put $D_0=U(\omega)\cap\partial\mathscr{B}$, $D_1=U(\omega)\cap\tau(\partial\mathscr{B})$, then D_0, D_1 are both 2-balls. We may assume $D_1=\tau(D_0)$ since $\tau(\omega)=\omega$. Since ω is an isolated point with respect to $\partial\mathscr{B}\cap\tau(\partial\mathscr{B})$, D_0 meets D_1 non-transversally at ω. At $U(\omega)$ two cases (i.e. $D_0\subset\tau(\mathscr{B})$ or $D_0\cap\tau(\mathscr{B})=\phi$) will happen. In both cases, there is a point η in $p^{-1}(\ell)\cap U(\omega)$ with the property "$\eta\in\mathscr{B}$ and $\eta\notin\tau(\mathscr{B})$" or "$\eta\notin\mathscr{B}$ *and* $\eta\in\tau(\mathscr{B})$". Since $\tau\,|\,p^{-1}(\ell)=id.$, it is a contradiction. So $T_2=\phi$.

In the following we will prove the theorem by induction for the number of components of S. Since $T_2=\phi$, there is a 2-ball D in $\tau(\partial\mathscr{B})$ such that $D\cap\partial\mathscr{B}=\partial D$.

Case (Ca): $\partial D\cap p^{-1}(\ell)=\phi$(*i.e.* $\partial D\subset S_1$). Then $\tau(\partial D)=\partial D$ or $\tau(\partial D)\cap\partial D=\phi$. If $\tau(\partial D)=\partial D$, $p(D)$ is the projective 2-space in S^3. If $\tau(\partial D)\cap\partial D=\phi$, by the same way of Case (Ab) (ii) we can reduce the number of components of S. That is, we obtain two 2-spheres S_1, S_2 by equivariant surgery from $\partial\mathcal{B}$ using D such that $\#(S_i \cap \tau(S_i)) < \#(\partial\mathcal{B} \cap \tau(\partial\mathcal{B}))$ $(i=1, 2)$ and S_i bounds a homotopy 3-ball in $M(\ell)$.

Case (Cb): $\partial D \cap p^{-1}(\ell) \neq \phi$.

Case (Cba): $\partial D = \tau(\partial D)$. Since $\tau | p^{-1}(\ell) = id.$, $\partial D \cap p^{-1}(\ell)$ is exactly two points ω_1, ω_2 by Smith's Theorem [9]. We suppose that ω_i $(i=1,2)$ are the boundary of two arcs respectively in the set of the intersection of type S_2. (For the case that ω_i $(i=1,2)$ are the boundary of r arcs more than two arcs, see Case (Cbb)).) Then $D \cup_\partial \tau(D)$ is a 2-shpere with only two fixed points ω_1, ω_2 of τ. And $p(D \cup_\partial \tau(D))$ is a 2-sphere in S^3. Since $p^{-1}(\ell)$ meets both $\partial\mathcal{B}$ and $\tau(\partial\mathcal{B})$ transversally at the two points, ω_1, ω_2 belong to the same component, say $p^{-1}(k_i)$, of ℓ. Suppose that S_1^2, S_2^2 have been obtained by equivariant surgery from $\partial\mathcal{B}$ with $D \cup_\partial \tau(D) = S_1^2$ and $(\partial\mathcal{B} - \tau(\mathring{D})) \cup D = S_2^2$. Since $\pi_2(M(\ell))=0$, S_i^2 bounds a homotopy 3-ball $\mathcal{B}_i$ in $M(\ell)$. If $\mathcal{B}_i \cup \tau(\mathcal{B}_i) = M(\ell)$, $M(\ell)$ is a homotopy 3-sphere and so $M(\ell) \cong S^3$ is irreducible by Remark. So we assume $\mathcal{B}_i \cup \tau(\mathcal{B}_i) \subsetneqq M(\ell)$. If $D \subset \mathcal{B}$, by the same way of (Ab) (ii), S_1^2 and S_2^2 bound homotopy 3-balls $\mathcal{B}_1$, $\mathcal{B}_2$ respectively such that $\mathcal{B} = \mathcal{B}_1 \cup_D \mathcal{B}_2$, $\tau(\mathcal{B}_1) = \mathcal{B}_1$ and $\#(\partial\mathcal{B}_2 \cap \tau(\partial\mathcal{B}_2)) < \#(\partial\mathcal{B} \cap \tau(\partial\mathcal{B}))$. If $D \cap \mathring{\mathcal{B}} = \phi$, S_1^2 and S_2^2 bound homotopy 3-balls $\mathcal{B}_1$, $\mathcal{B}_2$ respectively such that $\mathcal{B}_2 = \mathcal{B} \cup \mathcal{B}_1$ or $\mathcal{B}_2 = \mathcal{B}_1 - \mathcal{B}$ and such that $\tau(\mathcal{B}_1) = \mathcal{B}_1$ and $\#(\partial\mathcal{B}_2 \cap \tau(\partial\mathcal{B}_2)) < \#(\partial\mathcal{B} \cap \tau(\partial\mathcal{B}))$. We show that $\mathcal{B}_1$ is homeomorphic to a 3-ball. Because if $\mathcal{B}_1$ does not contain any component of $p^{-1}(\ell)$ except $p^{-1}(k_i) \cap \mathcal{B}_1$, $\mathcal{B}_1$ is a 2-fold covering of a 3-ball D^3 branched over $\mathcal{D}^1 = k_i \cap p(\mathcal{B}_i)$ where $(D^3, \mathcal{D}^1)$ is the standard ball pair by lemma 1 and where D^3 is the 3-ball in S^3 containing $k_i \cap p(\mathcal{B}_1)$ and bounded by $p(S_1^2)$. So $\mathcal{B}_1$ is a 3-ball. If $\mathcal{B}_1$ contains some components of $p^{-1}(\ell)$, we take a 3-ball B^3 and identify ∂B^3 with $\partial\mathcal{B}_1^3$ by the natural identification. Then $\Sigma = B^3 \underset{\partial}{\cup} \mathcal{B}_1^3$ is a homotopy 3-sphere. We extend $\tau | \mathcal{B}_1$ to B^3 naturally. Then the extended involution, say τ', has a μ-component link $(\mu \geqq 2)$ as the set of fixed points. It contradicts to Smith's Theorem [9]. So $\mathcal{B}_1$ does not contain any other component of $p^{-1}(\ell)$ except $p^{-1}(k_i) \cap \mathcal{B}_1$.

Case (Cbb): $\partial D \neq \tau(\partial D)$. Put $r = \#(\partial D \cap p^{-1}(\ell))$. We take three processes $r=1$, 2 or $r \geqq 3$ as follows.

When $r=1$, $\tau(\partial D)$ splits $\tau(\partial\mathcal{B})$ into closed 2-balls. Let E be one of the two 2-balls where E does not contain D. If ∂E is the innermost curve in $\tau(\partial\mathcal{B})$ with respect to $\partial\mathcal{B} \cap \tau(\partial\mathcal{B})$ (*i.e.* $\mathring{E} \cap \partial\mathcal{B} = \phi$), this process is the finish$\cdots$(1). If $\mathring{E} \cap \partial\mathcal{B} \neq \phi$, there is an innermost curve in $\mathring{E}$ for $\partial\mathcal{B} \cap \tau(\partial\mathcal{B})$ i.e. there is a 2-ball D_1 in $\mathring{E}$ such that $\mathring{D}_1 \cap \partial\mathcal{B} = \phi$. We consider D_1 instead of D and repeat the processes.

When $r=2$, we denote two 2-balls E, E' in $\tau(\partial\mathcal{B})$ bounded by ∂D and $\tau(\partial D)$. If $\mathring{E} \cap \partial\mathcal{B} = \phi$ and $\mathring{E}' \cap \partial\mathcal{B} = \phi$, this process is the finish $\cdots$(2). If $\mathring{E} \cap \partial\mathcal{B} \neq \phi$, there is a 2-ball D_1 in $\mathring{E}$ such that $D_1 \cap \partial\mathcal{B} = \phi$. We consider D_1 instead of D

and repeat the processes. It is the same in the case $\mathring{E}' \cap \partial\mathcal{B}=\phi$.

When $r \geqq 3$, the component of $\partial D \cap S$, say α, is an open arc. Since $\bar{\alpha}-\alpha$ is contained in T_1 where $\bar{\alpha}$ is the closure of α, $\tau|\bar{\alpha}-\alpha=id$. Let E be a region in $\tau(\partial\mathcal{B})$ bounded by $\bar{\alpha}$ and $\tau(\bar{\alpha})$. If $E \cap \partial\mathcal{B}=\phi$ for any such E, this process is the finish $\cdots$ (3). If $\mathring{E} \cap \partial\mathcal{B} \neq \phi$, there is a 2-ball D_1 in E with $\mathring{D}_1 \cap \partial\mathcal{B}=\phi$. Consider D_1 instead of D and repeat the processes.

Since $\partial\mathcal{B} \cap \tau(\partial\mathcal{B})=S \cup T=S_1 \cup S_2 \cup T_1$ and S,T contain finite components, either (1), (2) or (3) of the above happen by repeating the above processes finite times. If the cases from (A) to (Cba) happened in the processes, the way of dealing has been done. So we may denote the way of dealing with the following cases ①$'$, ②$'$ or ③$'$

Case ①$'$: There is a 2-ball D in $\tau(\partial\mathcal{B})$ satisfying

(a) $\sharp(\partial D \cap p^{-1}(\ell))=2$, $\partial D \neq \tau(\partial D)$ and

(b) if E,E' are 2-balls in $\tau(\partial\mathcal{B})$ bounded by ∂D and $\tau(\partial D)$, $\mathring{E} \cap \partial\mathcal{B}=\phi$ and $\mathring{E}' \cap \partial\mathcal{B}=\phi$.

Case ②$'$: There is a 2-ball D in $\tau(\partial\mathcal{B})$ satisfying

(c) $\sharp(\partial D \cap p^{-1}(\ell))=1$, $\partial D \neq \tau(\partial D)$ and

(d) if E is one of two 2-balls in $\tau(\partial\mathcal{B})$ bounded by $\tau(\partial D)$ such that E does not contain D, $\mathring{E} \cap \partial\mathcal{B}=\phi$.

Case ③$'$: There is a 2-ball D in $\tau(\partial\mathcal{B})$ satisying

(e) $\sharp(\partial D \cap p^{-1}(\ell)) \geqq 3$, $\partial D \neq \tau(\partial D)$ and

(f) if E is any one of 2-balls in $\tau(\partial\mathcal{B})$ bounded by ∂D and $\tau(\partial D)$, $\mathring{E} \cap \partial\mathcal{B}=\phi$.

When the case ①$'$ happened, put $D \cap E=\alpha$, $D \cap E'=\beta$, $\partial E-\mathring{\alpha}=\gamma$, $\partial E'-\mathring{\beta}=\delta$ and $\partial D \cap p^{-1}(\ell)=\omega_1 \cup \omega_2$. We may assume that $\partial\mathcal{B}$ meets $\tau(\partial\mathcal{B})$ transversally. If $\tau(\alpha)=\gamma$ and $\tau(\beta)=\delta$, $\tau(\partial E)=\partial E$ and $\tau(\partial E')=\partial E'$. So if $\mathring{E} \subset \mathcal{B}$, $\mathring{E}' \cap \mathcal{B}=\phi$. But then $\mathring{D} \cap \mathcal{B}=\phi$ from $\mathring{E} \subset \mathcal{B}$ and $\mathring{D} \subset \mathcal{B}$ from $\mathring{E}' \cap \mathcal{B}=\phi$. It is a contradiction. It is the same for $\mathring{E} \cap \mathcal{B}=\phi$. Hence the case can not occur. So $\tau(\alpha)=\delta$ and $\tau(\beta)=\gamma$.

Then $\tau(\partial E)=\partial E'$ and $\tau(\partial E')=\partial E$. Let F be a region in $\partial\mathcal{B}$ bounded by α and γ such that F does not contain $\tau(\mathring{D})$ and F' be a region in $\partial\mathcal{B}$ bounded by β and δ such that F' does not contain $\tau(\mathring{D})$. Then F,F' are both 2-balls and $F=\tau(E')$, $F'=\tau(E)$. Put $\Sigma_1=E \cup \tau(E')=E \cup F$, then Σ_1 is a 2-sphere and $\tau(\Sigma_1)=\tau(E) \cup E'=E' \cup F'$. Since $\mathring{E} \cap \partial\mathcal{B}=\mathring{E}' \cap \partial\mathcal{B}=\phi$, $\Sigma_1 \cap \tau(\Sigma_1)=\omega_1 \cup \omega_2$ and so $p|\Sigma_1\colon \Sigma_1 \to p(\Sigma_1)$ is a homeomorphism. Hence $p(\Sigma_1)$ is a 2-sphere. $p(\Sigma_1)$ bounds a 3-ball B_0^3 in S^3 and so Σ_1 bounds a 3-manifold $W_1=p^{-1}(B_0^3)$ in $M(\ell)$. If $\mathring{W}_1 \cap p^{-1}(\ell) \neq \phi$, $\mathring{W}_1 \cap \tau(W_1) \neq \phi$. So $W_1 \subsetneqq \tau(W_1)$ or $\tau(W_1) \subsetneqq W_1$ since $\partial W_1 \cap \tau(\partial W_1)=\Sigma_1 \cap \tau(\Sigma_1)=\omega_1 \cup \omega_2$. If $W_1 \subsetneqq \tau(W_1)$, put $W'=W_1-\tau(W_1)$. Then $\tau(W')=\tau(W_1)-W_1=\phi$. It is a contradiction. It is the same for $W_1 \subsetneqq \tau(W_1)$. Hence $\mathring{W}_1 \cap p^{-1}(\ell)=\phi$ and $W_1 \cong B^3$, the 3-ball.

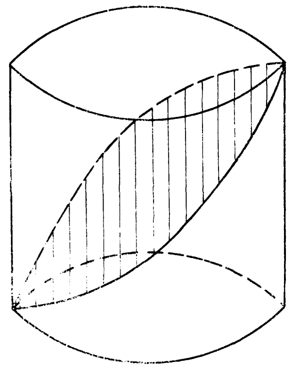
Case (2–1): When $D \subset \mathcal{B}$, $(E \cup E') \cap \mathring{\mathcal{B}}=\phi\ (*)$. And $(F \cup F') \cap \tau(\mathring{\mathcal{B}})=\phi\ (**)$.

Let $\mathscr{B}=\mathscr{B}_1\cup\mathscr{B}_2$ ($\mathscr{B}_i$: homotopy 3-ball) and $\overline{\partial\mathscr{B}-(\tau(D)\cup F\cup F')}\subset\mathscr{B}_1$, then $(\mathring{\gamma}\cup\mathring{\delta})\cap\partial\mathscr{B}_1=\phi$ and $(\gamma\cup\delta)\cap\partial\mathscr{B}_1=\omega_1\cup\omega_2$. So the region $\overline{\tau(\partial\mathscr{B})-(D\cup E\cup E')}$ bounded by $\gamma\cup\delta$ in $\tau(\partial\mathscr{B})$ which does not contain D is contained in $\mathscr{B}_2$. Put $\Sigma_2=D\cup F\cup F'\cup\tau(D)$, then Σ_2 is a 2-sphere and it is contained in $\mathscr{B}$. Furthermore $\Sigma_2=\partial\mathscr{B}_2$ and $\Sigma_2=\cap\tau(\Sigma_2)=D\cup\tau(D)$. So $\overline{\tau(\partial\mathscr{B})-(D\cup E\cup E')}\cap\partial\mathscr{B}\subset\mathscr{B}_2\cap\partial\mathscr{B}=\tau(D)\cup F\cup F'$. On the other hand $\overline{\tau(\partial\mathscr{B})-(D\cup E\cup E')}\cap(\tau(D)\cup F\cup F')=\gamma\cup\delta$, so $(\tau(\partial\mathscr{B})-(D\cup E\cup E'))\cap\partial\mathscr{B}=\phi$ (***). From (*), (**) and (***), the region $\mathscr{B}'$ in $\mathscr{B}$ bounded by $D\cup F\cup F'\cup(\overline{\tau(\partial\mathscr{B})-(D\cup E\cup E')})$ satisfies $\mathring{\mathscr{B}}'\cap\tau(\mathscr{B})=\phi$. So $\partial\mathscr{B}$ intersects $p^{-1}(\ell)$ transversally at ω_1 and ω_2. It is a contradiction the same as the proof of $T_2=\phi$. So this case cannot happen.

Case (2–2): When $D\cap\mathring{\mathscr{B}}=\phi$, $E\cup E'\subset\mathscr{B}$ (*)$'$. So $\tau(D)\cap\tau(\mathring{\mathscr{B}})=\phi$ and $F\cup F'\subset\tau(\mathscr{B})$ (**)$'$. Put $\Sigma_3=D\bigcup_{\alpha\cup\beta}(F\cup F'\cup\tau(D))$, then Σ_3 is a 2-sphere. And $\Sigma_4=(D\cup E\cup E')\bigcup_{\gamma\cup\delta}\tau(D)$ is also a 2-sphere. $E\cup_\partial F$ bounds a 3-ball W_1 and $E'\cup_\partial F'=\tau(E)\cup_\partial\tau(F)$ bounds a 3-ball $\tau(W_1)$. Denote $\Sigma_3\dot{\times}[-1,1]$, $\Sigma_4\dot{\times}[-1,1]$ small product neighborhood mod $\omega_1\cup\omega_2$, i.e. $\Sigma_3\dot{\times}[-1,1]=\Sigma_3\times[-1,1]/\omega_i\times\{t\}\sim\omega_i\times\{0\}$ $(i=1,2)$, $t\in[-11]$ $\Sigma_4\dot{\times}[-11]=\Sigma_4\times[-11]/\omega_i\times\{t\}\sim\omega_i\times\{0\}$ where $\Sigma_3\times\{0\}=\Sigma_3$, $\Sigma_4\times\{0\}=\Sigma_4$, $(\Sigma_3\times\{1\})\cap(W_1\cup\tau(W_1))=\phi$ and $(\Sigma_4\times\{-1\})\cap(W_1\cup\tau(W_1))=\phi$. By (*)$'$ and (**)$'$, $\tau(\Sigma_3\times\{1\})=\Sigma_4\times\{-1\}$. So $\tilde{\Sigma}\cap\tau(\tilde{\Sigma})=\omega_1\cup\omega_2$ where $\tilde{\Sigma}=\Sigma_3\times\{1\}$. Hence $p\,|\,\tilde{\Sigma}\colon\tilde{\Sigma}\to p(\tilde{\Sigma})$ is a homeomorphism and $\tilde{\Sigma}$ bounds a 3-manifold, say W_2, since $p(\tilde{\Sigma})$ bounds a 3-ball, say B_0. We may suppose $W\supset\mathscr{B}$. Then $(\tau(\partial\mathscr{B})-(D\cup E\cup E'))\cap\mathring{W}=\phi$ and $(\tau(\partial\mathscr{B})-(D\cup E\cup E'))\cap\partial\mathscr{B}=\phi$(***)$'$. By (*)$'$, (**) and (**)$'$, for a region $\mathscr{B}''$ in $\mathscr{B}$ bounded by $\tau(D)\cup E\cup E'\cup\overline{\partial\mathscr{B}-(\tau(D)\cup F\cup F')}$, $\tau(\mathring{\mathscr{B}})\cap\mathring{\mathscr{B}}=\phi$ and $\partial\mathscr{B}''\to\omega_1\cup\omega_2$. So $\partial\mathscr{B}$ intersect transversally $p^{-1}(\ell)$ at ω_1, ω_2. It is a contradiction. This case can not also happen. Therefore the case 2 of ①$'$ can not happen.

The case ②$'$. Let $D\times[-1,1]$ be a small product neighborhood of D in $M(\ell)$ such that $D\times\{0\}=D$, $\partial D\times\{1\}\subset\tau(E)$ and $\partial D\times\{-1\}\subset\partial\mathscr{B}-\tau(E)$. Let k_i be a component of $p^{-1}(\ell)$ which intersects ∂D with only one point and $\omega=\partial D\cap p^{-1}(\ell)=\partial D\cap k_i$. Put $\Sigma=D\cup_\partial\tau(E)$, then Σ is a 2-shpere and $\Sigma\cap\tau(\Sigma)=(D\cup\tau(E))\cup(\tau(D)\cap E)$. On the other hand since $\mathring{D}\cap\partial\mathscr{B}=\phi$ and $\mathring{E}\cap\partial\mathscr{B}=\phi$, it follows that $D\cap\tau(D)=\partial D\cap k_i=\omega$, $E\cap\tau(E)=\partial E\cap\tau(\partial E)=\tau(\partial D)\cap\partial D=\omega$, $D\cap E=\partial D\cap\tau(\partial D)=\omega$ and $\tau(E)\cap\tau(D)=\tau(D\cap E)=\tau(\omega)=\omega$. So $\Sigma\cap\tau(\Sigma)=\omega$. Hence $p\,|\,\Sigma\colon\Sigma\to p(\Sigma)$ is a homeomorphism and $p(\Sigma)$ bounds a 3-ball B_0^3 in S^3. We take B_0^3 so that B_0^3 contains $p(D\times[0,1])$. Then Σ bounds a 3-manifold $W=p^{-1}(B_0^3)$ in $M(\ell)$ and $\mathring{W}\cap k_i=\phi$ because $\Sigma\cap k_i=\omega$. If $\mathring{W}\cap p^{-1}(\ell)=\phi$, $p\,|\,W$ is a homeomorphism and W is a 3-ball since $W\cap\tau(W)=\omega$. If $\mathring{W}\cap p^{-1}(\ell)\neq\phi$ (i.e. $\mathring{W}$ contains some other components of $p^{-1}(\ell)$), $\mathring{W}\cap\tau(\mathring{W})\neq\phi$. So $W\subsetneqq\tau(W)$ or $\tau(W)\subsetneqq W$ since $\partial W\cap\tau(\partial W)=\Sigma\cap\tau(\Sigma)=\omega$. If $\tau(W)\subsetneqq W$, let $W'=W-\tau(W)$.

Then $\tau(W')=\tau(W)-W=\phi$. It is a contradiction. The case $W\subsetneqq\tau(W)$ is also the same as the above. So $\mathring{W}\cap p^{-1}(l)=\phi$. Put $\Sigma_1=E_1\cup_{\partial}D\times\{1\}$ and $\Sigma_2=(\partial\mathscr{B}-\tau(\mathring{E}))\cup D$. (Be careful of $\Sigma_2\neq(\partial\mathscr{B}-(\tau(E)\cup\partial D\times[-1,0])\cup(\partial D\times\{-1\}))$. Since $\tau(\Sigma_1)\cap\Sigma_1=\phi$, $p|\Sigma_1$ is a homeomorphism. And since $\mathring{W}\cap p^{-1}(l)=\phi$, Σ_1 bounds a 3-ball W_1 where W_1 is contained in W. Σ_2 bounds the homotopy 3-ball $\overline{\mathscr{B}-W}$ provided $D\subset\mathscr{B}$ and it bounds the homotopy 3-ball $\mathscr{B}\cup W$ provided $D\cap\mathscr{B}=\phi$. Take a 2-ball D' near D containing in $D\times[0,1]$ (as Fig.) and let E'



(Fig.)

be a region bounded by $\tau(\partial D')$ which contains E. Put $\Sigma_2'=(\partial\mathscr{B}-\tau(\mathring{E}'))\cup D'$. Then by the same way of Σ_2, Σ_2' bounds the homotopy 3-ball $\overline{\mathscr{B}-W'}$ provided $D'\subset\mathscr{B}$ and it bounds the homotopy 3-ball $\mathscr{B}\cup W'$ provided $\mathring{D}'\cap\mathscr{B}=\phi$ where W' is a 3-ball bounded by $\tau(E')\cup D'$. (Existence of W' and $W'\cong B^3$ are the same as W.)
And

$$\Sigma_2'\cap\tau(\Sigma_2')=((\partial\mathscr{B}-\tau(\mathring{E}'))\cup D')\cap((\tau(\partial\mathscr{B})-\mathring{E}')\cup\tau(D'))$$
$$=\omega\cup((\partial\mathscr{B}-\tau(\mathring{E}'))\cap(\tau(\partial\mathscr{B})-\mathring{E}')\,.$$

So

$$\#(\Sigma_2'\cap\tau(\Sigma_2')\cap S)<\#(\partial\mathscr{B}\cap\tau(\partial\mathscr{B})\cap S)\ \text{ and}$$
$$\#(\Sigma_2'\cap\tau(\Sigma_2')\cap T)=\#(\partial\mathscr{B}\cap\tau(\partial\mathscr{B})\cap T)\,.$$

Hence $\#(\Sigma_2'\cap\tau(\Sigma_2'))<\#(\partial\mathscr{B}\cap\tau(\partial\mathscr{B}))$ and the induction for the number of the components of S proceeds.

The case ③$'$. Let $E_1,\cdots,E_r$ ($r\geqq3$) be 2-balls in $\tau(\partial\mathscr{B})$ bounded by ∂D and $\tau(\partial D)$. Then $\partial E_i=\bar{\alpha}_i\cup\tau(\bar{\alpha}_i)$ and $\tau|\partial\bar{\alpha}_i=id.$ where α_i is an open arc in the intersection of type S_2. We may assume that $\partial\mathscr{B}$ meets $\tau(\partial\mathscr{B})$ transversally. So if $\mathring{E}_1\subset\mathscr{B}$, $\mathring{E}_2\cap\mathscr{B}=\phi$. But then $\mathring{D}\cap\mathscr{B}=\phi$ *from* $\mathring{E}_1\subset\mathscr{B}$ and $\mathring{D}\subset\mathscr{B}$ from $\mathring{E}_2\cap\mathscr{B}=\phi$. It is a contradiction. It is the same for $\mathring{E}_1\cap\mathscr{B}=\phi$. Hence the case ③$'$ can not occur.

The proof of Theorem is completed.

Remark.
$$\begin{array}{c} \tau(E)\cap\tau(\mathring{\mathcal{B}})=\phi \Rightarrow D\times\{-1\}\subset\tau(\mathcal{B}) \\ \Downarrow \\ E\cap\mathcal{B}=\phi \Rightarrow \tau(D)\times\{-1\}\subset\mathcal{B}\,. \\ \tau(E)\subset\tau(\mathcal{B}) \Rightarrow (D\times\{-1\})\cap\tau(\mathring{\mathcal{B}})=\phi \\ \Downarrow \\ E\subset\mathcal{B} \Rightarrow (\tau(D)\times\{-1\})\cap\mathring{\mathcal{B}}=\phi\,. \end{array}$$

By the above facts and that $\partial\mathcal{B}-T$ meets $\tau(\partial\mathcal{B})-T$ transversally, there are odd components of S through ω other than ∂D and $\tau(\partial D)$. So after doing the surgery above, ω is not the isolated point although the intersection ∂D and $\tau(\partial D)$ can be eliminated.

2. Deciding of $\pi_2(M(\ell))$. In this section we study a method of determining whether $\pi_2(M(\ell))=0$ or not for a given link ℓ whose components are all trivial knots.

Lemma 2. *Let $p: M(\ell)\to S^3$ be a 2-fold covering of S^3 branched over a link ℓ in S^3. If Σ^2 is a 2-sphere embedded in S^3 such that $\Sigma^2\cap\ell$ is exactly two points. Then $p^{-1}(\Sigma^2)$ is homeomorphic to the 2-sphere.*

Proof. Since p is a 2-fold covering and $\Sigma^2\cap\ell$ is two points, $p\,|\,p^{-1}(\Sigma^2)$ is also a 2-fold covering i.e. $p^{-1}(\Sigma^2)$ is connected. So the Euler characteristic $\chi(p^{-1}(\Sigma^2))=2$ and hence $p^{-1}(\Sigma^2)$ is homeomorphic to the 2-sphere.

Proposition 1. *Let $p: M(\ell)\to S^3$ be a 2-fold covering of S^3 branched over a link ℓ. If there is a 2-sphere Σ^2 in S^3 satisfying that*

(1) *$\Sigma^2\cap\ell$ is exactly two points and*

(2) *$B_i^3\cap\ell$ is not homeomorphic to the 1-ball for $i=1,2$ where $S^3=B_1^3\cup_{\Sigma}B_2^3$, then $p^{-1}(\Sigma^2)$ is not homotopic to 0 in $M(\ell)$.*

Proof. Let $\tau: M(\ell)\to M(\ell)$ be the non-trivial covering translation and $Fix(\tau)$ be the set of fixed points of τ, then $Fix(\tau)=p^{-1}(\ell)$. Then $p^{-1}(\Sigma^2)\cong S^2$ by lemma 2 and $\tilde{\Sigma}^2$ splits $M(\ell)$ where $\tilde{\Sigma}^2=p^{-1}(\Sigma^2)$. So we can denote $M(\ell)=M_1\cup_{\tilde{\Sigma}}M_2$. If neither M_1 nor M_2 is homoemorphic to a homotopy 3-ball, $\tilde{\Sigma}^2\neq 0$ in $M(\ell)$. So we can show the contradiction by assuming M_i ($i=1$ or 2) a homotopy 3-ball. Since $\tau(\tilde{\Sigma})=\tilde{\Sigma}$, it happens that $\tau(M_i)=M_i (i=1,2)$ or $\tau(M_1)=M_2$. If $\tau(M_1)=M_2$, $M(\ell)$ is a homotopy 3-sphere. So $p^{-1}(\ell)$ is a 1-sphere by Smith's Theorem [9] and ℓ is a 1-component link (=knot). It contradicts to (1) and (2). And if $\tau(M_i)=M_i$, $p\,|\,M_i: M_i\to B_i^3$ is a 2-fold covering of B_i^3 branched over $B_i^3\cap\ell$. And if M_i is a homotopy 3-ball, $p^{-1}(B_i^3\cap\ell)=Fix(\tau\,|\,M_i)$ is 1-ball by Smith's Theorem [9]. Hence $B_i^3\cap\ell\cong D^1$. It contradicts to (2). So $\tilde{\Sigma}^2=p^{-1}(\Sigma^2)\neq 0$ in $M(\ell)$.

Proposition 2. *In Proposition* 1, *assume that* Σ^2 *satisfies the following conditions* (3), (4) *instead of* (1), (2) *in Proposition* 1;

(3) $\Sigma^2 \cap \ell$ *is exactly two points and*

(4) $(B_i^3, B_i^3 \cap \ell) \cong (D^1 \times D^2, D^1 \times \{0\})$ (*standard ball pair*) *for* $i=1$ *or* 2 *where* $S^3 = B_1^3 \cup_\Sigma B_2^3$.

Then $p^{-1}(\Sigma^2) \simeq 0$ *in* $M(\ell)$.

Proof. By lemma 2, $p^{-1}(\Sigma^2)$ is homeomorphic to a 2-sphere. Since $(B_i^3, B_i^3 \cap \ell)$ is the standard ball pair, $p^{-1}(B_i^3)$ is a 3-ball. Since $\partial(p^{-1}(B_i^3)) = p^{-1}(\partial B_i^3) = p^{-1}(\Sigma^2)$, $p^{-1}(\Sigma^2) \simeq 0$ in $M(\ell)$.

REMARK. Let $p: M(\ell) \to S^3$ be a 2-fold covering of S^3 branched over ℓ and $\tilde{\Sigma}^2$ be a 2-sphere embedded in $M(\ell)$. By doing equivariant surgeries, $\tilde{\Sigma}^2$ splits into some 2-spheres $\{\tilde{\Sigma}_i^2\}$ and each 2-sphere satisfies that $\tilde{\Sigma}_i^2 \cap \tau(\tilde{\Sigma}_i^2) = \phi$ or $\tilde{\Sigma}_i^2 = \tau(\tilde{\Sigma}_i^2)$. And $p(\tilde{\Sigma}_i^2) \cong S^2$. So we denote again $\tilde{\Sigma}^2$ a 2-sphere embedded in $M(\ell)$ such that $\tilde{\Sigma}^2 \cap \tau(\tilde{\Sigma}^2) = \phi$ or $\tilde{\Sigma}^2 = \tau(\tilde{\Sigma}^2)$. Put $\Sigma^2 = p(\tilde{\Sigma}^2)$. Now if $p^{-1}(\Sigma^2) = \tilde{\Sigma}^2 \cup \tau(\tilde{\Sigma}^2)$ and $\tilde{\Sigma}^2 \cap \tau(\tilde{\Sigma}^2) = \phi$, $\Sigma^2 \cap \ell = \phi$. If $\ell \cap B_1^3 = \phi$, $p^{-1}(B_1^3) = B_{11}^3 \cup B_{12}^3$ (disjoint union of 3-balls) and $\partial B_{11}^3 = \tilde{\Sigma}^2$, $\partial B_{12}^3 = \tau(\tilde{\Sigma}^2)$. So $\tilde{\Sigma}^2 \simeq 0$ in $M(\ell)$. It is the same for the case $\ell \cap B_2^3 = \phi$. If $B_i^3 \supset \ell_i (i=1,2)$ where $\ell_i (i=1,2)$ are non-empty sublinks of ℓ with $\ell = \ell_1 \cup \ell_2$, $p^{-1}(B_i^3)$ are both connected 3-manifolds with $\partial p^{-1}(B_i^3) = \tilde{\Sigma}^2 \cup \tau(\tilde{\Sigma}^2)$. So $\tilde{\Sigma}^2 \not\simeq 0$ and $\tau(\tilde{\Sigma}^2) \not\simeq 0$ in $M(\ell)$. Because if $\tilde{\Sigma}^2 \simeq 0$ in $M(\ell)$, $\tilde{\Sigma}^2$ bounds a homotopy 3-ball in $M(\ell)$ [3]. Hence $\partial p^{-1}(B_1^3) \cong S^2$ or $\partial p^{-1}(B_2^3) \cong S^3$. It is a contradiction. Now the case $p^{-1}(\Sigma^2) = \tilde{\Sigma}^2$ and $\tilde{\Sigma}^2 = \tau(\tilde{\Sigma}^2)$ hold. In general $\Sigma^2 \cap \ell = \phi$ or even points. But it does not happen that $\Sigma^2 \cap \ell = \phi$ under the above conditions. And if $\#(\Sigma^2 \cap \ell) \geqq 4$, $p^{-1}(\Sigma^2) \not\cong S^2$. So we may consider the case $\#(\Sigma^2 \cap \ell) = 2$. (In the case $p^{-1}(\Sigma^2) \cong S^2$ by lemma 2.) So we can decide whether $\tilde{\Sigma}^2 = p^{-1}(\Sigma^2)$ is homotopic to 0 or not except the next case by using Proposition 1 and 2;

i.e. (5) $\#(\Sigma^2) \cap \ell = 2$ and

(6) $(B_i^3, B_i^3 \cap \ell)$ is a non-standard ball pair.

So if ℓ is a link whose components are all trivial knot, we can easily decide $\pi_2(M(\ell)) = 0$ or not by observing ℓ and by lemma 1.

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