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ENDOMORPHISM RINGS OF REFLEXIVE MODULES OVER KRULL ORDERS

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Let R be an order in a semisimple ring Q, and let M be a finite dimensional torsionless right R-module. Zelmanowitz[15] has shown that $k=\text{End } M_R$ is also an order in a semisimple ring End MQ_Q. Subsequently Cozzens[3] has shown that k is a maximal order whenever R is a maximal order and M_R is finite dimensional, reflexive and faithful.

On the other hand, Marubayashi[10] has defined Krull orders in simple Artinian rings as a Krull type generalization of non-commutative Dedekind rings, and a number of results on Krull orders have been obtained in [6], [7], [8], [9], [10] and [11].

In this paper, we shall prove the following:

Theorem. *Let $R=\bigcap_{P\in\boldsymbol{P}} R_P\cap S$ be a Krull order in Q, and let M be a finite dimensional reflexive right R-module. Then, $k=\text{End } M_R$ is a Krull order if and only if MS is projective as a right S-module.*

As an easy consequence, we have

Corollary. *Let $R=\bigcap_{P\in\boldsymbol{P}} R_P\cap S$ be a Krull order in Q. Then* gl.dim. $S\leqq 2$ *if and only if, for each finite dimensional reflexive right R-module M, $k=\text{End } M_R$ is a Krull order.*

If R is bounded and if M is a right v-ideal of R, then the theorem is due to Marubayashi. However, in his proof, it is essential that R is bounded (see §2 of [7]).

Throughout this paper, all rings are associative with identity and all modules are unital. We always write homomorphisms on the opposite side of the scalars. Conditions are assumed to hold on the left and right sides unless otherwise stated. R is an order in a simple Artinian ring Q. M_R(resp. ${}_RM$) signifies that M is a right (resp. left) R-module.

Let n be a positive integer. Then, ${}^nQ$(resp. Q^n) and Q_n denote the set of column(resp. row) vectors and the full $n\times n$ matrix ring over Q. Then, we can assume that ${}^nQ$(resp. Q^n) is a right(resp. left) Q-module, and that $Q^n=\mathrm{Hom}_Q({}^nQ, Q)$, ${}^nQ=\mathrm{Hom}_Q(Q^n, Q)$, $Q_n=\mathrm{End}_Q Q^n=\mathrm{End}\,{}^nQ_Q$ and $Q=\mathrm{End}\,Q^n{}_{Q_n}=\mathrm{End}_{Q_n}{}^nQ$. Now, we prepare a Morita context $(Q_n, Q, {}^nQ, Q^n, (\ ,\ ), [\ ,\ ])$, where the mappings $(\ ,\ ): {}^nQ\times Q^n\to Q_n$ and $[\ ,\ ]: Q^n\times{}^nQ\to Q$ are defined as follows: $(x, f)x'=x(fx')$ and $[f, x]=fx$, where x, $x'\in{}^nQ$ and $f\in Q^n=\mathrm{Hom}_Q({}^nQ, Q)$.

Since R is an order in Q, ${}^nQ_R$(resp. ${}_RQ^n$) is the injective hull of ${}^nR_R$(resp. ${}_RR^n$) and $\mathrm{End}\,{}^nQ_Q=\mathrm{End}\,{}^nQ_R$(resp. $\mathrm{End}_Q Q^n=\mathrm{End}_R Q^n$). Therefore we can assume that $R_n=\mathrm{End}_R R^n=\mathrm{End}\,{}^nR_R\subset Q_n=\mathrm{End}_Q Q^n=\mathrm{End}\,{}^nQ_Q$, and R_n is an order in Q_n.

We write $F_r({}^nR)$ (resp. $F_l(R^n)$) for the set of essential right(resp. left) R-submodules M of ${}^nQ_R$(resp. ${}_RQ^n$) with $M\subset b\cdot{}^nR$(resp. $M\subset R^n\cdot b$) for some regular element $b\in Q_n$. If $n=1$, then $F_r(R)$ (resp. $F_l(R)$) is the set of right(resp. left) R-ideals.

Suppose that M_R is essential in ${}^nQ_R$. Let $e_i=i)\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix}\in{}^nQ$, and let $I_i=\{r\in R\,|\,e_ir\in M\}$. Then I_i contains a regular element $a_i\in R$, because I_i is an essential right ideal of R. Then $a=\begin{pmatrix}a_1 & & 0\\ & \ddots & \\ 0 & & a_n\end{pmatrix}$ is a regular element of Q_n and $a\cdot{}^nR\subset M$. Therefore, $M\in F_r({}^nR)$ if and only if $a\cdot{}^nR\subset M\subset b\cdot{}^nR$ for some regular elements a, $b\in Q_n$.

If $M\in F_r({}^nR)$ and $N\in F_l(R^n)$, we put $(M_R)^*=\{f\in Q^n\,|\,[f, M]\subset R\}$ and $({}_RN)^*=\{x\in{}^nQ\,|\,[N, x]\subset R\}$. Then $(M_R)^*\cong\mathrm{Hom}_R(M, R)$ and $({}_RN)^*\cong\mathrm{Hom}_R(N, R)$.

The following lemma is well known if $n=1$ (see [6]).

Lemma 1. *Let M, $N\in F_r({}^nR)$, and let a be a regular element of Q_n. Then*
(1) $M+N\in F_r({}^nR)$.
(2) $(M_R)^{**}\in F_r({}^nR)$, *and* $M\subset N$ *implies* $(M_R)^{**}\subset(N_R)^{**}$.
(3) $M=\Sigma a_\lambda\cdot{}^nR$, *where a_λ is a regular element of Q_n with* $a_\lambda\cdot{}^nR\subset M$.
(4) $(M_R)^{**}=\cap b_\lambda\cdot{}^nR$, *where b_λ is a regular element of Q_n with* $M\subset b_\lambda\cdot{}^nR$.
(5) $(aM_R)^{**}=a(M_R)^{**}$.

Proof. (1) Since R_n is an order in Q_n, this follows from the same proof of the case when $n=1$.

(2) Since $M\in F_r({}^nR)$, there exist regular elements a, $b\in Q_n$ such that $a\cdot{}^nR\subset M\subset b\cdot{}^nR$. Therefore $R^n\cdot b^{-1}\subset(M_R)^*\subset R^n\cdot a^{-1}$, hence $(M_R)^*\in F_l(R^n)$ and $(M_R)^{**}\in F_r({}^nR)$. It follows from the definition that $M\subset N$ implies $(M_R)^{**}\subset(N_R)^{**}$.

(3) Since R_n is an order in Q_n, any right R_n-ideal is generated by regular elements which it contains, by Lemma 2.2 of [6]. Let $\rho(M)=\{q\in Q_n \mid q\cdot {}^nR\subset M\}$. Then $\rho(M)$ is a right R_n-ideal, so that $\rho(M)=\Sigma a_\lambda\cdot R_n$, where a_λ is a regular element of Q_n with $a_\lambda\cdot{}^nR\subset M$. Since $\rho(M){}^nR=M$, $M=\Sigma a_\lambda\cdot{}^nR$.

(4) This follows from (3).

(5) This follows from (4).

We now extend a definition and a part of Lemma 1.1 of [11] which are concerned with right R-sets (i.e. essential right R-submodules of Q_R) to those concerned with essential right R-submodules of ${}^nQ_R$. Let X be an essential right R-submodule of ${}^nQ_R$. Then we put $\bar{X}_R=\cup\{(M_R)^{**} \mid M\in F_r({}^nR)$ and $M\subset X\}$. It follows from (1), (2) of Lemma 1 that $\bar{X}_R$ is a right R-submodule of ${}^nQ_R$. Now, we note the following:

Lemma 2. *Let X, Y be essential right R-submodules of ${}^nQ_R$. Then*

(1) $X\subset\bar{X}_R$.

(2) *If* $X\subset Y$, *then* $\bar{X}_R\subset\bar{Y}_R$.

(3) *If* $M\in F_r({}^nR)$, *then* $\bar{M}_R=(M_R)^{**}$.

Proof. (1) Since X_R is essential in ${}^nQ_R$, there is a regular element $a\in Q_n$ with $a\cdot{}^nR\subset X$. If $x\in X$, then $xR+a\cdot{}^nR\in F_r({}^nR)$, $xR+a\cdot{}^nR\subset X$ and $x\in(xR+a\cdot{}^nR)^{**}\subset\bar{X}_R$.

(2) and (3) are immediate from (2) of Lemma 1.

Two orders R_1 and R_2 in Q are *equivalent*, denoted by $R_1\sim R_2$, if there are regular elements a, b, c, $d\in Q$ such that $aR_1b\subset R_2$ and $cR_2d\subset R_1$. An order R in Q is a *maximal order* provided that $R\subset R'\subset Q$ and $R\sim R'$ imply that $R=R'$. Let X, Y be submodules of Q. Then we put $(X:Y)_l=\{q\in Q \mid qY\subset X\}$, $(X:Y)_r=\{q\in Q \mid Yq\subset X\}$, $O_l(X)=\{q\in Q \mid qX\subset X\}$, $O_r(X)=\{q\in Q \mid Xq\subset X\}$, $X^{-1}=\{q\in Q \mid XqX\subset X\}$ and $X_v=X^{-1-1}$. A right (left) R-ideal I is a *right(left) v-ideal of R* if $I=I_v$. A right (left) R-module M is *torsionless* (resp. *reflexive*) if the natural homomorphism $M\rightarrow\mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R)$ is a monomorphism(resp. isomorphism). If R is a maximal order and if $I\in F_r(R)$, then I is a right v-ideal of R if and only if I is reflexive as a right R-module. A subring of Q which contains R is called an *overring* of R. Let R' be an overring of R. Then, ${}_RR'$ is flat and $R\hookrightarrow R'$ is an epimorphism in the category of rings if and only if $F=\{I \mid I$ is a right ideal of R and $IR'=R'\}$ is a right additive topology on R and $R'=R_F=\bigcup_{I\in F}(R:I)_l$ (see §13 of [14]). In this case, $R\hookrightarrow R'$ is said to be a right flat eipmorphism.

An overring R' of R is *right essential* over R if it satisfies the following conditions:

(1) $R\hookrightarrow R'$ is a right flat epimorphism.

(2) If I is a right ideal of R with $IR'=R'$, then $R'I=R'$ (see [7]).

A *left essential* overring is defined in the symmetric way.

An order R in Q is a *Krull order* if there is a family $\{R_i\}_{i\in\mathcal{J}}$ of overrings of R satisfying the following conditions:

(K1) $R=\bigcap_{i\in\mathcal{J}} R_i\cap S(R)$, where $S(R)=\{q\in Q\,|\,qA\subset R$ and $A'q\subset R$ for some non-zero ideals A, A' of $R\}$.

(K2) For each $i\in\mathcal{J}$, R_i is an essential overring of R and it is a Neotherian local Asano order, and $S(R)$ is an essential overring of R and it is a Noetherian simple ring.

(K3) Each regular element of R is invertible in R_i for almost all $i\in\mathcal{J}$.

REMARK. (i) We call $S(R)$ the *Asano overring* of R. Since a Krull order is a maximal order from Proposition 2.1 of [10], $(R:A)_l=A^{-1}=(R:A)_r$ for any non-zero ideal A of R. Therefore $S(R)=\cup\,\{A^{-1}\,|\,A$ is a non-zero ideal of $R\}$.

(ii) Let $R=\bigcap_{i\in\mathcal{J}} R_i\cap S(R)$ be a Krull order in Q, and let P_i' be the unique maximal ideal of R_i for each $i\in\mathcal{J}$. Then $P_i=P_i'\cap R$ is a prime v-ideal of R, and R satisfies the Ore condition with respect to $C(P_i)=\{c\in R\,|\,c+P_i$ is a regular element of $R/P_i\}$ and $R_i=R_{P_i}$ (see Proposition 1.1 of [6] and Proposition 2.1 of [10]). Therefore we write $R=\bigcap_{P\in\boldsymbol{P}} R_P\cap S$ for a Krull order in Q with the overrings $\{R_P\}_{P\in\boldsymbol{P}}$ and $S=S(R)$, and P' for the unique maximal ideal of R_P for each $P\in\boldsymbol{P}$.

In what follows, $R=\bigcap_{P\in\boldsymbol{P}} R_P\cap S$ is a Krull order in Q.

We now prove the next lemma that is well-known in the case of maximal orders, Asano orders and so on (see e.g. [4], [6], [12] and [13]).

Lemma 3. *If $R_n\ni e=e^2\neq 0$, then eR_ne is a Krull order in eQ_ne.*

Proof. It follows from Lemma 1.7 of [4] that eR_ne is a maximal order in eQ_ne.

First, we shall prove that R_n is a Krull order in Q_n. Clearly, $R_n=\bigcap_{P\in\boldsymbol{P}}(R_P)_n\cap S_n$. Let B be a non-zero ideal of R_n. Then $B=A_n$ for some non-zero ideal A of R, and $B^{-1}=(A^{-1})_n$. Therefore $S(R_n)=S_n$. By Proposition 2.1 of [10], $BS(R_n)=A_nS_n=(AS)_n=S_n=S(R_n)=S(R_n)B$. It follows from Lemma 2.2 of [10] that $S(R_n)$ is an essential overring of R_n. It is clear from well known facts (see e.g. p. 37 of [2]) that $S(R_n)=S_n$ is a Noetherian simple ring and $(R_P)_n$ is a Noetherian, local Asano order in Q_n. Since R_P is the partial quotient ring of R with respect to $C(P)$, $(R_P)_n$ is also the partial quotient ring of R_n with respect to $\left\{\begin{pmatrix} c & & 0\\ & \ddots & \\ 0 & & c\end{pmatrix}\in R_n\,\middle|\,c\in C(P)\right\}$. Therefore $(R_P)_n$ is an essential overring of R_n.

Let (c_{ij}) be a regular element of R_n. Then there is $(q_{ij})=(a_{ij}d^{-1})\in Q_n$ such that $(c_{ij})(q_{ij})=\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}$. Then $(c_{ij})(R_P)_n\supset(c_{ij})(a_{ij})(R_P)_n=(dR_P)_n$. Therefore R_n satisfies the condition (K3), so that R_n is a Krull order in Q_n.

It is sufficient, now, to consider the case when $n=1$. Clearly, $eRe=\bigcap_{P\in\boldsymbol{P}} eR_Pe\cap eSe$. Let B be a non-zero ideal of eRe. Then RBR is a non-zero ideal of R, and $B^{-1}=e(RBR)^{-1}e$. Therefore $S(eRe)=eSe$. It is easy to check that $S(eRe)=eSe$ is a Noetherian simple ring and eR_Pe is a Noetherian ring with a unique maximal ideal $eP'e$. It follows from Lemma 4.2 of [12] that eR_Pe is an Asano order in eQe. Since $eR_Pe/eP'e\cong(e+P')(R_P/P')(e+P')$ is simple Artinian, eR_Pe is a local ring.

It follows from Theorem 3 of [13] that $eR_Pe/eP'e\cong(e+P)(R_P/P')(e+P)$ is the quotient ring of $eRe/ePe\cong(e+P)(R/P)(e+P)$. Hence any element $C(ePe)$ is invertible in eR_Pe. Let $F=\{I\mid I$ is a right ideal of R and $IR_P=R_P\}$. Then, by the proof of Proposition 1.1,(4) of [6], $I\in F$ if and only if $I+P/P$ is an essential right ideal of R/P. Therefore if $I\in F$, then $eIe+ePe/ePe$ is an essential right ideal of eRe/ePe, by the similar proof of Lemma 4 of [13]. Thus, $eIe\cap C(ePe)\neq\phi$ if $I\in F$.

Let $r\in eRe$, $c\in C(ePe)$. Then $c^{-1}r\in eR_Pe\subset R_P$, hence $c^{-1}rI\subset R$ for some $I\in F$. Then $c^{-1}reIe\subset eRe$. Since there is $d\in eIe\cap C(ePe)$, $c^{-1}rd=s\in eRe$. Thus eRe satisfies the Ore condition with respect to $C(ePe)$.

It follows from the similar proof of Lemma 3 of [13] that $ece+1-e\in C(P)$ whenever $ece\in C(ePe)$. This implies that $eRe_{ePe}\subset eR_Pe$. Conversely if $q\in eR_Pe$, then there is $I\in F$ such that $qI\subset R$. Then $qeIe\subset eRe$. Since there is $c\in eIe\cap C(ePe)$, $qc=r\in eRe$. Then $q=rc^{-1}\in eRe_{ePe}$, so that $eR_Pe=eRe_{ePe}$ is a right essential overring of eRe. In the symmetric way, we obtain that eR_Pe is a left essential overring of eRe.

If ece is a regular element of eRe, then $ece+1-e$ is a regular element of R. Therefore eRe satisfies (K3), so that eRe is a Krull order in eQe. This completes the proof.

Let M be a finite dimensional torsionless right R-module, and let $(\operatorname{End} MQ_Q, Q, MQ, \operatorname{Hom}_Q(MQ,Q), (\ ,\ ), [\ ,\ ])$ be the Morita context derived from MQ_Q, i.e. the mappings $(\ ,\ ): MQ\times\operatorname{Hom}_Q(MQ,Q)\to\operatorname{End} MQ_Q$ and $[\ ,\ ]: \operatorname{Hom}_Q(MQ, Q)\times MQ\to Q$ are defined as follows:

$(x,f)x'=x(fx')$ and $[f,x]=fx$, where $x,x'\in MQ$ and $f\in\operatorname{Hom}_Q(MQ,Q)$. Then we put $(M_R)^*=\{f\in\operatorname{Hom}_Q(MQ,Q)\mid[f,M]\subset R\}$. Through the natural isomorphism we identify MQ with $\operatorname{Hom}_Q(\operatorname{Hom}_Q(MQ,Q),Q)$. Then $(M_R)^{**}=\{x\in MQ\mid[(M_R)^*,x]\subset R\}$, and M_R is reflexive if and only if $M=(M_R)^{**}$. If $M\in F_r({}^nR)$, then we identify $(\operatorname{End} MQ_Q, Q, MQ, \operatorname{Hom}_Q(MQ,Q), (\ ,\ ), [\ ,\ ])$ with

$(Q_n, Q, {}^nQ, Q^n, (\ ,\ ), [\ ,\ ])$.

Let R' be a flat epimorphic overring of R. If M' is a right R'-submodule of ${}^nR'_{R'}$, $(M' \cap {}^nR)R' = M'$. If M, N are right R-submodules of ${}^nR_R$ with $M \cap N = 0$, $(M \oplus N)R' = MR' \oplus NR'$. We now prove the following:

Lemma 4. *Let R' be an overring, and let $R \hookrightarrow R'$ be a flat epimorphism. Suppose that M is a finite dimensional torsionless right R-module. Then*

$$(MR'_{R'})^* = R'(M_R)^* \textit{ and } (MR'_{R'})^{**} = (M_R)^{**}R' .$$

Proof. Since $[R'(M_R)^*, MR'] = R'[M^*, M]R' \subset R'$, $R'(M_R)^* \subset (MR'_{R'})^*$. In order to show the converse inclusion, we take four cases.

Case 1. $M \in F_r({}^nR)$ and M_R is finitely generated.

Let $g \in (MR'_{R'})^*$. Since M_R is finitely generated, there exists $J \in F_l$ such that $J[g, M] \subset R$, where $F_l = \{J \mid J$ is a left ideal of R and $R'J = R'\}$. Then $Jg \subset M^*$, so that $g \in R'g = R'Jg \subset R'M^*$.

Case 2. $M \in F_r({}^nR)$ and $M = M^{**}$.

From Lemma 3, R_n is a Krull order in Q_n. For $N \in F_r({}^nR)$, let $\rho(N) = \{q \in Q_n \mid q \cdot {}^nR \subset N\}$. Then ρ induces a lattice isomorphism between integeral right v-ideals of R_n and essential reflexive right R-submodules of ${}^nR_R$, by Lemma 1. Therefore ${}^nR_R$ satisfies the asceding chain condition on essential reflexive right R-submodules. Hence there exists a finitely generated right R-submodule $M_0 \subset M$ such that $M_0 \in F_r({}^nR)$ and $M^* = M_0^*$. Then $R'M^* = R'M_0^* = (M_0R'_{R'})^* \subset (MR'_{R'})^*$, by case 1.

Case 3. $M \in F_r({}^nR)$.

By Lemma 2, $M^{**} = \bar{M}_R \subset \overline{MR'}_R$. Let $x \in \overline{MR'}_R$. Then there exists $M_0 \in F_r({}^nR)$ such that $M_0 \subset MR'$ and $x \in (M_0)^{**}$. Since ${}^nR_R$ satisfies the ascending chain condition on essential reflexive right R-submodules, we can choose M_0 to be finitely generated. By cases 1 and 2, $x \in M_0^{**} \subset M_0^{**}R' = ({}_{R'}R'M_0^*)^* = (M_0R'_{R'})^{**} \subset (MR'_{R'})^{**}$. Thus, $\overline{MR'}_R \subset (MR'_{R'})^{**}$ and hence $M^{**}R' \subset (MR'_{R'})^{**}$. Consequently, $(MR'_{R'})^* \subset (M^{**}R'_{R'})^* = R'M^*$, by case 2.

Case 4. M is a finite dimensional torsionless right R-module.

By Proposition 2.4 of [5], we can assume that $M \subset {}^nR$ for some integer $n \geqq 1$. Then there is a right R-submodule N of ${}^nR_R$ such that $(M \oplus N)_R$ is essential in ${}^nR_R$. Then $M \oplus N \in F_r({}^nR)$. Since $(MR'_{R'})^* \oplus (NR'_{R'})^* \cong (MR' \oplus NR')^* = ((M \oplus N)R'_{R'})^* = R'(M \oplus N)^* \cong R'M^* \oplus R'N^*$, $(MR'_{R'})^* = R'(M_R)^*$. This completes the proof.

The next lemma plays a key role in the discussion of the Asano overring of $k = \operatorname{End} M_R$.

Lemma 5. *Let M be a finite dimensional reflexive right R-module, and let $k = \operatorname{End} M_R$. Then*

End $MS_S \supset S(k) \supset (MS, (MS_S)^*) \neq 0$.

Proof. Since MS_S is torsionless and faithful, $(MS, (MS_S)^*)MS = MS[(MS_S)^*, MS] \neq 0$. Hence $(MS, (MS_S)^*) \neq 0$.

It follows from Lemma 4 that $(MS, (MS_S)^*) = (MS, M^*)$. Let A be a non-zero ideal of R. Then $(MA^{-1}, M^*)(MA, M^*) = ((MA^{-1}, M^*)MA, M^*) = (MA^{-1}[M^*, M]A, M^*) \subset (M, M^*) \subset k$. Since k is a maximal order from Theorem 2.8 of [3], $(MA^{-1}, M^*) \subset (MA, M^*)^{-1} \subset S(k)$.

Let B be a non-zero ideal of k. Since $[[M^*, BM]M^*, B^{-1}M] = [M^*B(M, M^*), B^{-1}M] \subset [M^*, M] \subset R$, $B^{-1}M \subset ({}_R[M^*, BM]M^*)^*$. By Lemma 2.1 of [10] and Lemma 4, $({}_R[M^*, BM]M^*)^*S = ({}_SS[M^*, BM]M^*)^* = ({}_SSM^*)^* = MS$, for M_R is reflexive. Therefore $B^{-1}MS \subset MS$, and hence $B^{-1} \subset \text{End } MS_S$. Thus, $S(k) \subset \text{End } MS_S$. This completes the proof.

The following lemma is modeled on and generalizes Lemmas in §2 of [7] so that we exclude the hypothesis that R is bounded.

Lemma 6. *Let I be a right v-ideal of R. Then*

(1) $O_l(I) = \bigcap_{P \in \boldsymbol{P}} O_l(IR_P) \cap O_l(IS)$ *is a maximal order in Q.*

(2) $O_l(IR_P) = IR_PI^{-1}$ *is a Noetherian local Asano order in Q, and it is an essential overring of $O_l(I)$, where $P \in \boldsymbol{P}$.*

Moreover, suppose that IS_S is projective. Then

(3) $O_l(IS) = ISI^{-1} = S(O_l(I))$ *is a Noetherian simple ring, and it is an essential overring of $O_l(I)$.*

(4) $O_l(I)$ *is a Krull order in Q.*

Proof. (1) Since I is a right v-ideal, $O_l(I)$ is a maximal order from Satz. 1.3 of [1] (or Theorem 2.8 of [3]), and $(\bigcap_{P \in \boldsymbol{P}} O_l(IR_P) \cap O_l(IS))I \subset \bigcap_{P \in \boldsymbol{P}} IR_P \cap IS = I$. Hence $O_l(I) = \bigcap_{P \in \boldsymbol{P}} O_l(IR_P) \cap O_l(IS)$.

(2) Since R_P is hereditary, $O_l(IR_P) = (IR_P)(IR_P)^{-1} = IR_PI^{-1}$ by Theorem 1.5 of [12] and Lemma 4. It follows from Lemma 2.3 of [12] that $O_l(IR_P)$ is a Noetherian local Asano order.

Since R_P is a principal ideal ring, $IR_P = aR_P$ for some regular element $a \in I$. Then $IR_PI^{-1} = aR_Pa^{-1}$. Let $F = \{X \mid X \text{ is a right ideal of } R \text{ and } XR_P = R_P\}$, and let Y be a right ideal of $O_l(I)$. If $YIR_PI^{-1} = IR_PI^{-1}$, then $a^{-1}YIR_P = R_P$. Hence $a^{-1}YI \cap C(P) \neq \phi$, and hence $a^{-1}YI \cap R \in F$ and $a(a^{-1}YI \cap R)I^{-1} \subset Y$. Conversely if $X \in F$, $aXI^{-1}IR_PI^{-1} = aXR_PI^{-1} = aR_PI^{-1} = IR_PI^{-1}$. Therefore, $YIR_PI^{-1} = IR_PI^{-1}$ if and only if $Y \supset aXI^{-1}$ for some $X \in F$.

Let $F_I = \{Y \mid Y \text{ is a right ideal of } O_l(I) \text{ and } YIR_PI^{-1} = IR_PI^{-1}\}$. Then F_I is a right additive topology on $O_l(I)$. In fact, (i) if $x \in O_l(I)$ and if $Y \in F_I$, then $Y \supset aXI^{-1}$ for some $X \in F$. Since $a^{-1}xa \in R_P$, $(a^{-1}xa)^{-1}X \in F$. Since

$a((a^{-1}xa)^{-1}X)I^{-1}\subset x^{-1}Y$, $x^{-1}Y\in F_I$. (ii) If $Y\in F_I$ and if Z is a right ideal of $O_l(I)$ such that $x^{-1}Z\in F_I$ for each $x\in Y$, then $Z\in F_I$ (see e.g. the proof of Lemma 1.1 of [7]).

If $q\in O_l(I)_{F_I}=\bigcap_{Y\in F_I}(O_l(I):Y)_l$, then $qaXI^{-1}\subset O_l(I)$ for some $X\in F$. Therefore $q\in qIR_PI^{-1}=qaXI^{-1}IR_PI^{-1}\subset IR_PI^{-1}$. Conversely if $q\in IR_PI^{-1}$, then $q=ata^{-1}$ for some $t\in R_P$. Then $tX\subset R$ for some $X\in F$. Hence $qaXI^{-1}=atXI^{-1}\subset O_l(I)$. Thus $O_l(I)_{F_I}=IR_PI^{-1}$. Furthermore, for each $X\in F$, $IR_PI^{-1}\supset IR_PI^{-1}aXI^{-1}=IR_PXI^{-1}=IR_PI^{-1}$. Consequently, $O_l(IR_P)=IR_PI^{-1}$ is a right essential overring of $O_l(I)$. Since I is a right v-ideal, $O_l(I)=O_r(I^{-1})$. Therefore we obtain that $O_l(IR_P)=O_r(R_PI^{-1})$ is a left essential overring of $O_l(I)=O_r(I^{-1})$, in the symmetric way as the above.

(3) Since IS_S is projective, $O_l(IS)=(IS)(IS)^{-1}=ISI^{-1}=S(O_l(I))$ by Theorem 1.5 of [12] and Lemmas 4 and 5. Since S is a simple ring, IS_S is a progenerator. Hence $S(O_l(I))=O_l(IS)$ is a Noetherian simple ring. Let B be a non-zero ideal of $O_l(I)$. Since $ISI^{-1}\supset BISI^{-1}\supset I(I^{-1}BI)SI^{-1}=ISI^{-1}$, $S(O_l(I))=BS(O_l(I))$. In the symmetric way, we get $S(O_l(I))=S(O_l(I))B$. Therefore $S(O_l(I))$ is an essential overring of $O_l(I)$, by Lemma 2.2 of [10].

(4) Since a is a regular element, $O_l(IR_P)=aR_Pa^{-1}=R_P$ for almost all $P\in \boldsymbol{P}$. If x is a regular element of $O_l(I)$, x is invertible in R_P for almost all $P\in\boldsymbol{P}$. Thus $O_l(I)$ satisfies (K3). Therefore $O_l(I)$ is a Krull order in Q from (1), (2) and (3).

We are now in a position to prove the following theorem that is the object of this paper.

Theorem 7. *Let $R=\bigcap_{P\in\boldsymbol{P}}R_P\cap S$ be a Krull order in Q, and let M be a finite dimensional reflexive right R-module. Then, $k=\mathrm{End}\,M_R$ is a Krull order if and only if MS is projective as a right S-module.*

Proof. First, assume that $k=\mathrm{End}\,M_R$ is a Krull order. Then $S(k)$ is a simple ring. Since $(MS,(MS_S)^*)$ is a non-zero ideal of $S(k)$ by Lemma 5, $1\in S(k)=(MS,(MS_S)^*)$. Therefore MS_S is projective.

Conversely, assume that MS_S is projective. Since MS_S is finite dimensional torsionless, it is isomorphic with a submodule of a finitely generated free right S-module from Proposition 2.4 of [5]. Since S is Noetherian, MS_S is finitely generated. Hence there exist an integer $n\geqq 1$ and a right S-submodule N' of ${}^nS_S$ such that $MS\oplus N'={}^nS$. Let $N=N'\cap{}^nR$. Then $(N_R)^{**}\subset({}^nR_R)^{**}={}^nR$ and $(N_R)^{**}\subset(N_R)^{**}S=(NS_S)^{**}=(N'_S)^{**}=N'$ from Lemma 4, so that $(N_R)^{**}\subset N'\cap{}^nR=N$. Hence N_R is reflexive. Thus $M\oplus N$ is an essential reflexive right R-submodule of ${}^nQ_R$ and $(M\oplus N)S={}^nS$. Since $(M\oplus N)_R$ is reflexive, there exist an integer $m\geqq 1$ and a right R-submodule X of ${}^mR_R$ such that $M\oplus N\oplus X$ is isomorphic with an essential reflexive submodule of ${}^mR_R$, by

Proposition 2.4 of [5]. As stated in the proof of Lemma 4, ${}^mR_R$ satisfies the ascending chain condition on essential reflexive submodules. Hence, for some $z_1, \cdots, z_l \in M \oplus N \oplus X$, $M \oplus N \oplus X = (z_1R + \cdots + z_lR)^{**}$. Write $z_i = y_i + x_i$ where $y_i \in M \oplus N$ and $x_i \in X$ for $i = 1, \cdots, l$. Then $M \oplus N \oplus X = (z_1R + \cdots + z_lR)^{**} \subset (y_1R + \cdots + y_lR)^{**} \oplus X \subset M \oplus N \oplus X$. Hence $M \oplus N = (y_1R + \cdots + y_lR)^{**}$. Since $y_1R + \cdots + y_lR \in F_r({}^nR)$, $M \oplus N \in F_r({}^nR)$ by Lemma 1. Therefore, it follows from the similar proof when $n=1$ that $\mathrm{End}(M \oplus N)_R \sim R_n$. Since $\mathrm{End}(M \oplus N)_R$ is a maximal order by Theorem 2.8 of [3], there is a right v-ideal I of R_n such that $\mathrm{End}(M \oplus N)_R = O_l(I)$. Since S_n is a simple ring, $S_n = \mathrm{End}(M \oplus N)S_S = S(\mathrm{End}(M \oplus N)_R) = S(O_l(I)) = (IS_n)(IS_n)^{-1}$ by Lemma 5. Therefore IS_n is projective as a right S_n-module. It follows from Lemma 6 that $\mathrm{End}(M \oplus N)_R = O_l(I)$ is a Krull order in Q_n. Consequently, by Lemma 3, $k = e(\mathrm{End}(M \oplus N)_R)e$ is a Krull order in eQ_ne, where e is the projection from $M \oplus N$ onto M. This completes the proof.

Corollary 8. *Let $R = \bigcap_{P \in \boldsymbol{P}} R_P \cap S$ be a Krull order in Q. Then* gl.dim. $S \leqq 2$ *if and only if, for each finite dimensional reflexive right R-module M, $k=$*End M_R *is a Krull order.*

Proof. If M is a finite dimensional reflexive right R-module, then MS is a finitely generated reflexive right S-module. Conversely if M' is a finitely generated right S-module, then there is a finite dimensional reflexive right R-module M such that $M' \cong MS$. It is now easy to complete the proof using Proposition 3.2 of [3] and the theorem.

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