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A REMARK ON ASYMPTOTIC SUFFICIENCY UP TO HIGHER ORDERS IN MULTI-DIMENSIONAL PARAMETER CASE

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1. Introduction. Suppose that n-dimensional random vector $z_n=(x_1, x_2, \cdots, x_n)$ is distributed according to a probability measure $P_{\theta,n}$ parameterized by$\theta\in\Theta\subset \boldsymbol{R}^p$, and each component x_i is independently and identically distributed. In Suzuki [3] it was shown that when $p=1$ a statistic $t_n^*=(\hat{\theta}_n, \Phi_n^{(1)}(z_n, \hat{\theta}_n), \cdots, \Phi_n^{(k)}(z_n, \hat{\theta}_n))$ is asymptotically sufficient up to order $o(n^{-(k-1)/2})$ in the following sense: For each n t_n^* is sufficient for a family $\{Q_{\theta,n}; \theta\in\Theta\}$ of probability measures and that

$$\lim_{n\to\infty} n^{(k-1)/2}||P_{\theta,n}-Q_{\theta,n}|| = 0$$

uniformly on any compact subset of Θ (where $||\cdot||$ means the total variation norm of a signed measure). Here $\hat{\theta}_n$ is some reasonable estimator of θ and $\Phi_n^{(i)}(z_n, \theta)$ means the *i-th* logarithmic derivative relative to θ of the density of $P_{\theta,n}$. In this paper we show that the result can be extended to the case where underlying distribution $P_{\theta,n}$ has multi-dimensional parameter θ. Exact form of t_n^* would be found in the statement of the theorem in Section 3. In Michel [2] a similar result was obtained with order of sufficiency $o(n^{-(k-2/)2})$, and hence ours is more accurate one.

2. Notations and assumptions. Let $\Theta(\neq\phi)$ be an open subset of p-dimensional Euclidean space $\boldsymbol{R}^p$. Suppose that for each $\theta\in\Theta$ there corresponds a probability measure P_θ defined on a measurable space $(\boldsymbol{X}, \boldsymbol{A})$. For each $n\in N=\{1, 2, \cdots\}$ let $(\boldsymbol{X}^{(n)}, \boldsymbol{A}^{(n)})$ be the cartesian product of n copies of $(\boldsymbol{X}, \boldsymbol{A})$, and $P_{\theta,n}$ the product measure of n copies of P_θ. For a signed measure λ on $(\boldsymbol{X}^{(n)}, \boldsymbol{A}^{(n)})$, $||\lambda||$ means the total variation norm of λ over $\boldsymbol{A}^{(n)}$. For a function h and a probability P, $E[h; P]$ stands for the expectation of h under P. In the following it will be assumed that the map: $\theta\to P_\theta$ is one to one, and that for each $\theta\in\Theta$ P_θ has a densiy $f(x, \theta)$ relative to a sigma-finite measure μ on $(\boldsymbol{X}, \boldsymbol{A})$. We assume that $f(x, \theta)>0$ for every $x\in\boldsymbol{X}$ and every $\theta\in\Theta$. We denote by μ_n the product measure of n copies of the same component μ. We define $\Phi(x, \theta)=\log f(x, \theta)$ for each $x\in \boldsymbol{X}$ and $\theta\in\Theta$, and $\Phi_n(z_n, \theta)=\sum_{\nu=1}^{n}\Phi(x_\nu, \theta)$ for eadch $n\in N$, each $z_n=(x_1, \cdots, x_n)\in \boldsymbol{X}^{(n)}$ and $\theta\in\Theta$. For a vector $u\in \boldsymbol{R}^p$, $||u||$ denotes the usual Eculidean norm of u. For $\varepsilon>0$ and $a\in \boldsymbol{R}^p$ we define $U(a: \varepsilon)=\{u\in \boldsymbol{R}^p; ||u-a||<\varepsilon\}$ and $V(a: \varepsilon)=\{u\in \boldsymbol{R}^p; ||u-a||\leqq \varepsilon\}$. Let k be a fixed positive integer.

Condition R. (1) For every $x\in \boldsymbol{X}$ $\Phi(x, \theta)$ is $(k+2)$-times continuously differentiable with respect to θ in Θ. For $m\in N$ define $J_m=\{(i_1, \cdots, i_m); i_j=1, \cdots, p(j=1, \cdots, m)\}$. For each $m(1\leqq m\leqq k+2)$ and each $(i_1, i_2, \cdots, i_m)\in J_m$ define

$$\Phi^{i_1\cdots i_m}(x, \theta) = \partial^m\Phi(x, \theta)/\partial\theta_{i_1}\cdots\partial\theta_{i_m},$$

and

$$\Phi_n^{i_1\cdots i_m}(z_n, \theta) = \sum_{\nu=1}^{n}\Phi^{i_1\cdots i_m}(x_\nu, \theta).$$

(2) For every $a=(a_1, \cdots, a_p)\in \boldsymbol{R}^p(a\neq 0)$ and every $\theta\in\Theta$ we have

$$P_\theta(\sum_{i=1}^{p} a_i\Phi^i(x, \theta)\neq 0)>0.$$

(3) For every $\theta\in\Theta$, there exists a positive number ε such that

a. $\sup_{\tau\in V(\theta:\varepsilon)} E[\sup_{\sigma\in V(\theta:\varepsilon)}\{\Phi^{i_1\cdots i_{k+2}}(x, \sigma)\}^2; P_\tau]<\infty$ for every $(i_1, \cdots, i_{k+2})\in J_{k+2}$

b. $\sup_{\tau\in V(\theta:\varepsilon)} E[|\Phi^{i_1\cdots i_{k+1}}(x, \tau)|\cdot u_\varepsilon^i(x, \tau); P_\tau]<\infty$ and $E[u_\varepsilon^i(x, \theta)]<\infty$for every $(i_1, \cdots, i_{k+1})\in J_{k+1}$ and every $i(1\leqq i\leqq p)$, where $u_\varepsilon^i(x, \tau)=\sup_{\sigma\in V(\tau:\varepsilon)}[|(\partial f(x, \theta)/\partial\theta_i)_{\theta=\sigma}|/f(x, \tau)]$.

c. $0<\inf_{\tau\in V(\theta:\varepsilon)}\mathrm{Var}(\Phi^{i_1\cdots i_{k+1}}(x, \tau); P_\tau)\leqq \sup_{\tau\in V(\theta:\varepsilon)}\mathrm{Var}(\Phi^{i_1\cdots i_{k+1}}(x, \tau; P_\tau)<\infty$ for every $(i_1, \cdots, i_{k+1})\in J_{k+1}$.

We define for each $\varepsilon'>0$, $\sigma\in\Theta$ and $(i_1, \cdots, i_{k+1})\in J_{k+1}$

$$\bar{Z}^{i_1\cdots i_{k+1}}(x; \varepsilon', \sigma) = \sup\{\Phi^{i_1\cdots i_{k+1}}(x, \tau)-E[\Phi^{i_1\cdots i_{k+1}}(x, \tau); P_\tau];\ \tau\in V(\sigma: \varepsilon')\cap\Theta\}$$

and

$$\underline{Z}^{i_1\cdots i_{k+1}}(x; \varepsilon', \sigma) = -\inf\{\Phi^{i_1\cdots i_{k+1}}(x, \tau) -E[(\Phi^{i_1\cdots i_{k+1}}(x, \tau); P_\tau];\ \tau\in V(\sigma: \varepsilon')\cap\Theta\}$$

(4) For each $\theta\in\Theta$ there exist positive numbers η and ρ such that for every $(i_1, \cdots, i_{k+1})\in J_{k+1}$ and every $(t, \varepsilon')\in(-\rho, \rho)\times(0, \eta]$ the moment generating functions (m.g.f'.s) of $\bar{Z}^{i_1\cdots i_{k+1}}(x; \varepsilon', \sigma)$ and $\underline{Z}^{i_1\cdots i_{k+1}}(x; \varepsilon', \sigma)$ converge uniformly in $\sigma\in U(\theta: \eta)$.

3. Asymptotic sufficient statistics up to higher orders. An estimator of θ depending on $z_n=(z_1, \cdots, x_n)\in \boldsymbol{X}^{(n)}$ is an $\boldsymbol{A}^{(n)}$-measurable function from $\boldsymbol{X}^{(n)}$ to $\boldsymbol{R}^p$. Such estimator will be called strict if its range is a subset of Θ. For each $\delta(0<\delta<1/2)$ we denote by $\boldsymbol{C}_k(\delta)$ the class of all sequences of strict estimators $\hat{\theta}_n$ of θ such that for every compact subset K of Θ

$$\sup_{\theta\in K} P_{\theta,n}(n^{1/2}||\hat{\theta}_n(z_n)-\theta||>n^{\delta}) = o(n^{-(k-1)/2}) .$$

Here the notation $o(a_n)$ means that $\lim_{n\to\infty} o(a_n)/a_n=0$. In Pfanzagl [1] it was shown that under Condition R for any δ satisfying $0<\delta<1/2$ $\boldsymbol{C}_k(\delta)$ does not empty. Let $\delta_0=1/[2(k+2)]$ and $\boldsymbol{C}_k=\bigcup_{0<\delta<\delta_0} \boldsymbol{C}_k(\delta)$. We have the following result which is an extension of Tehorem 2 in Suzuki [3] to a multi-dimensional parameter case. Since the proof is much analogous to the one in [3] we shall only sketch the outlines and details will be omitted (see [3] for precise arguments).

Theorem. *Suppose that Condition R is satisfied, and that* $\{\hat{\theta}_n\}\in \boldsymbol{C}_k$ *then there exists a sequence* $\{Q_{\theta,n};\theta\in\Theta\}$, $n\in N$, *of families of probability measures on* $(\boldsymbol{X}^{(n)}, \boldsymbol{A}^{(n)})$ *with the following properties*: (1) *For each* $n\in N$, *the statistic* $t_n^*=(\hat{\theta}_n, \Phi_n^i(z_n, \hat{\theta}_n)\,(i=1, \cdots, p), \Phi_n^{ij}(z_n, \hat{\theta}_n)((i,j)\in J_2), \cdots, \Phi_n^{i_1\cdots i_k}(z_n, \hat{\theta}_n)\,((i_1, \cdots, i_k)\in J_k))$ *is sufficient for* $\{Q_{\theta,n};\theta\in\Theta\}$. (2) *For every compact subset K of* Θ

$$\sup_{\theta\in K} ||P_{\theta,n}-Q_{\theta,n}|| = o(n^{-(k-1)/2})$$

Proof. Suppose that Condition R is satisfied, and that $\{\hat{\theta}_n\}\in \boldsymbol{C}_k(\delta_1)$ where δ_1 satisfies $0<\delta_1<\delta_0$. Let δ and γ be two numbers satisfying $\delta_1<\delta<\delta_0$ and $\delta<\gamma<(1/2)-(k+1)\delta$, and let $\varepsilon_n=n^{\delta-(1/2)}$ $\varepsilon_n'=n^{\gamma-(1/2)}$. Define

$$W_n^1 = \{z_n\in \boldsymbol{X}^{(n)};\ ||\theta-\hat{\theta}_n(z_n)||\leqq\varepsilon_n \quad \text{and} \quad [\theta:\hat{\theta}_n]\subset\Theta\}$$
$$W_n^2 = \{z_n\in \boldsymbol{X}^{(n)};\ \gamma_n(z_n)\leqq\varepsilon_n'\}$$

where

$$[\theta:\hat{\theta}_n] = \{t\theta+(1-t)\hat{\theta}_n;\ 0\leqq t\leqq 1\}$$

and

$$\gamma_n(z_n) = \max_{(i_1,\cdots,i_{k+1})\in J_{k+1}} \sup \{|\Phi_n^{i_1\cdots i_{k+1}}(z_n, \tau)/n-E[\Phi^{i_1\cdots i_{k+1}}(x, \tau); P_\tau]|;\ \tau\in V(\hat{\theta}_n: 2\varepsilon_n)\cap\Theta\} .$$

By a Taylor expansion of $\Phi_n(z_n, \theta)$ around $\theta=\hat{\theta}_n$ we have

$$\Phi_n(z_n, \theta) = \Phi_n(z_n, \hat{\theta}_n)+\Psi_n(t_n^*, \theta)+R_n(z_n, \theta) \tag{3.1}$$

where denoting by $\hat{\theta}_{n,i}$ the i-th comonents of $\hat{\theta}_n$

$$\Psi_n(t_n^*, \theta) = \sum_{m=1}^{k}\sum_{i_1=1}^{p}\cdots\sum_{i_m=1}^{p}\prod_{j=1}^{m}(\theta_{i_j}-,\hat{\theta}_{n,i_j})\cdot\Phi_n^{i_1\cdots i_m}(z_n, \hat{\theta}_n)/m!+s_n'(\theta_n, \theta) ,$$

$$s_n'(\hat{\theta}_n, \theta) = n\sum_{i_1=1}^{p}\cdots\sum_{i_{k+1}=1}^{p}\prod_{j=1}^{k+1}(\theta_{i_j}-\hat{\theta}_{n,i_j})E[\Phi^{i_1\cdots i_{k+1}}(x,\theta); P_\theta]/(k+1)!$$

$$R_n(Z_n, \theta) = \begin{cases} \Phi_n(z_n,\theta)-\Phi_n(z_n,\hat{\theta}_n)-\Psi_n(t_n^*,\theta) \quad (\text{if } [\theta:\hat{\theta}_n]\not\subset\Theta), \\ n\sum_{i_1=1}^{p}\cdots\sum_{i_{k+1}=1}^{p}\prod_{j=1}^{k+1}(\theta_{i_j}-\hat{\theta}_{n,i_j})[\int_0^1(1-\lambda)^k\{\Phi_n^{i_1\cdots i_{k+1}}(z_n,\hat{\theta}_n+\lambda(\theta-\hat{\theta}_n))/n \\ \quad -E[\Phi^{i_1\cdots i_{k+1}}(x,\theta); P_\theta]\}d\lambda]/k! \quad (\text{if } [\theta:\hat{\theta}_n]\subset\Theta). \end{cases}$$

Define

$$\text{(3.2)} \qquad q_n^*(z_n,\theta) = I_{W_n^1}(z_n)\cdot I_{W_n^2}(z_n)\cdot\exp\{\Phi_n(z_n,\hat{\theta}_n)+\Psi_n(t_n^*,\theta)\} = r_n^*(t_n^*,\theta)\cdot s_n^*(z_n)\geqq 0,$$

where

$$r_n^*(t_n^*,\theta) = I_{W_n^1}(z_n)\exp\{\Psi_n(t_n^*,\theta)\},$$
$$s_n^*(z_n) = I_{W_n^2}(z_n)\cdot\exp\{\Phi_n(z_n,\hat{\theta}_n)\}$$

and $I_{W_n^i}(z_n)$ mean the indicator functions of W_n^i. The integrability of $q_n^*(\cdot,\theta)$ follows from (3.4). Let $Q_{\theta,n}^*$ be a measure on $(\boldsymbol{X}^{(n)}, \boldsymbol{A}^{(n)})$ defined by

$$Q_{\theta,n}^*(A) = \int_A q_n^*(z_n,\theta)d\mu_n (A\in\boldsymbol{A}^{(n)}).$$

By (3.1) and (3.2) we have

$$\text{(3.3)} \qquad \int_{X^{(n)}}|p_n(z_n,\theta)-q_n^*(z_n,\theta)|d\mu_n = T_n^1(\theta)+T_n^2(\theta)+T_n^3(\theta)$$

where

$$p_n(z_n,\theta) = \prod_{\nu=1}^{n} f(x_\nu,\theta),$$
$$T_n^1(\theta) = \int_{W_n^1\cap W_n^2}|1-\exp\{-R_n(z_n,\theta)\}|p_n(z_n,\theta)d\mu_n,$$
$$T_n^2(\theta) = P_{\theta,n}((W_n^1)^c) \quad\text{and}$$
$$T_n^3(\theta) = P_{\theta,n}(W_n^1\cap(W_n^2)^c).$$

Let θ_0 be an arbitrarily fixed point of Θ, and let K be a compact subset of Θ. We assume without loss of generality that K contains θ_0. From Condition R it follows that there exist positive numbers ε^*, ρ^* and η^* depending only on K but not depending on θ in K such that

$$M_1 = \sum_{(i_1,\cdots,i_{k+2})\in J_{k+2}}\sup_{\theta\in K}\sup_{\tau\in V(\theta:\varepsilon^*)}E[\sup_{\sigma\in V(\theta:\varepsilon^*)}\{\Phi^{i_1\cdots i_{k+2}}(x,\sigma)\}^2; P_\tau]<\infty$$

$$M_2 = \max_{(i_1,\cdots,i_{k+1})\in J_{k+1}}\sum_{j=1}^{p}\sup_{\theta\in K}E[|\Phi^{i_1\cdots i_{k+1}}(x,\theta)|\cdot u_{\varepsilon^*}^j(x,\theta); P_\theta]<\infty$$

$$0<\inf_{\tau\in K}\text{Var}(\Phi^{i_1\cdots i_{k+1}}(x,\tau); P_\tau)$$
$$\leqq\sup_{\tau\in K}\text{Var}(\Phi^{i_1\cdots i_{k+1}}(x,\tau); P_\tau)<\infty \quad (\text{for every } (i_1,\cdots,i_{k+1})\in J_{k+1})$$

and that for every $\theta \in K$ and every $(t, \varepsilon', \sigma) \in (-\rho^*, \rho^*) \times (0, \eta^*] \times U(\theta: \eta^*)$ the m.g.f.'s of $\bar{Z}^{i_1 \cdots i_{k+1}}(x; \varepsilon', \sigma)$ and $\underline{Z}^{i_1 \cdots i_{k+1}}(z; \varepsilon', \sigma)$ exist and converge uniformly in $\sigma \in U(\theta: \eta^*)$. Hence there exists a number n_1 such that for every $n \geqq n_1$, $\theta \in K$, $(i_1, \cdots, i_{k+1}) \in J_{k+1}$ and every $z_n \in W_n^1 \cap W_n^2$ we have

$$\begin{aligned}
&\sup_{\tau \in V(\theta: \varepsilon_n)} |\Phi_n^{i_1 \cdots i_{k+1}}(z_n, \tau)/n - E[\Phi^{i_1 \cdots i_{k+1}}(x, \theta); P_\theta]| \\
&\leqq \sup_{\tau \in V(\theta: \varepsilon_n)} |\Phi_n^{i_1 \cdots i_{k+1}}(z_n, \tau)/n - E[\Phi^{i_1 \cdots i_{k+1}}(x, \tau); P_\tau]| \\
&+ \sup_{\tau \in V(\theta: \varepsilon_n)} |E[\Phi^{i_1 \cdots i_{k+1}}(x, \tau); P_\tau] - E[\Phi^{i_1 \cdots i_{k+1}}(x, \theta); P_\theta]| \\
&\leqq \sup_{\tau \in V(\hat{\theta}_n: 2\varepsilon_n)} |\Phi_n^{i_1 \cdots i_{k+1}}(z_n, \tau)/n - E[\Phi^{i_1 \cdots i_{k+1}}(x, \tau); P_\tau]| + [M_1^{1/2} + M_2] \cdot \varepsilon_n \\
&\leqq \gamma_n(z_n) + \varepsilon_n' \\
&\leqq 2\varepsilon_n' .
\end{aligned}$$

Thus we have

$$\begin{aligned}
\sup_{\theta \in K} T_n^1(\theta) &\leqq \sup_{\theta \in K} \int_{X^{(n)}} |R_n(z_n, \theta)| \cdot \exp(|R_n|) dP_{\theta,n} \\
&\leqq 4 \cdot n^{-(k-1)/2} n^{(k+1)\delta + \gamma - (1/2)} / (k+1)!
\end{aligned}$$

for sufficiently large n. Therefore

$$\sup_{\theta \in K} T_n^1(\theta) = o(n^{-(k-1)/2}) . \tag{3.4}$$

By the definition of $\boldsymbol{C}_k(\delta_1)$ we have easily

$$\sup_{\theta \in K} T_n^2(\theta) = o(n^{-(k-1)/2}) . \tag{3.5}$$

Next we evaluate the third term $T_n^3(\theta)$ as follows.

$$\begin{aligned}
\sup_{\theta \in K} T_n^3(\theta) &= \sup_{\theta \in K} P_{\theta,n}(||\theta - \hat{\theta}_n(z_n)|| \leqq \varepsilon_n,\ [\theta: \hat{\theta}_n] \subset \Theta,\ \gamma_n(z_n) > \varepsilon_n') \\
&\leqq \sup_{\theta \in K} P_{\theta,n}(\max_{(i_1, \cdots, i_{k+1}) \in J_{k+1}} \sup_{\tau \in V(\theta: 3\varepsilon_n)} |\Phi_n^{i_1 \cdots i_{k+1}}(z_n, \tau)/n \\
&\qquad - E[\Phi^{i_1 \cdots i_{k+1}}(x, \tau); P_\tau])| > \varepsilon_n') \\
&\leqq \sum_{(i_1 \cdots i_{k+1}) \in J_{k+1}} \sup_{\theta \in K} P_{\theta,n}(\sum_{\nu=1}^n \sup_{\tau \in V(\theta: 3\varepsilon_n)} \{\Phi^{i_1 \cdots i_{k+1}}(x_\nu, \tau) \\
&\qquad - E[\Phi^{i_1 \cdots i_{k+1}}(x, \tau); P_\tau]\} > n\varepsilon_n') \\
&+ \sum_{(i_1, \cdots, i_{k+1}) \in J_{k+1}} \sup_{\theta \in K} P_{\theta,n}(\sum_{\nu=1}^n \inf_{\tau \in V(\theta: 3\varepsilon_n)} \{\Phi^{i_1 \cdots i_{k+1}}(x_\nu, \tau) \\
&\qquad - E[\Phi^{i_1 \cdots i_{k+1}}(x, \tau); P_\tau]\} < -n\varepsilon_n') .
\end{aligned} \tag{3.6}$$

Let $a(\varepsilon) = \varepsilon/[4 \cdot M_1^{1/2} + 2M_2]$ and let $Z_\nu(\varepsilon, \theta) = \bar{Z}^{i_1 \cdots i_{k+1}}(x_\nu; a(\varepsilon), \theta)$ $(\nu = 1, \cdots, n)$. According to the lemma in Suzuki [3] there exist constants $\beta > 0$ and $\varepsilon^{**} > 0$ such that

$$\sup_{\theta\in K} P_{\theta,n}(\sum_{\nu=1}^{n} Z_\nu(\varepsilon, \theta)\geqq n\varepsilon)\geqq(1-\beta\varepsilon^2)^n$$

for every $n\in N$ and every ε satisfying $0<\varepsilon\leqq\varepsilon^{**}$. Hence we have

$$(3.7)\quad \sup_{\theta\in K} P_{\theta,n}(\sum_{\nu=1}^{n} \sup_{\tau\in V(\theta:3\varepsilon_n)} \{\Phi^{i_1\cdots i_{k+1}}(x_\nu, \tau)-E[\Phi^{i_1\cdots i_{k+1}}(x, \tau); P_\tau]\}>n\varepsilon_n')$$
$$\leqq \sup_{\theta\in K} P_{\theta,n}(\sum_{\nu=1}^{n} Z_\nu(\varepsilon_n', \theta)\geqq n\varepsilon_n')$$
$$= o(n^{-(k-1)/2}).$$

Considering the random variable $\underline{Z}^{i_1\cdots i_{k+1}}(x; a(\varepsilon), \theta)$ instead of $\bar{Z}^{i_1\cdots i_{k+1}}(x; a(\varepsilon), \theta)$ we obtain by a similar method to (3.7) that

$$(3.8)\quad \sup_{\theta\in K} P_{\theta,n}(\sum_{\nu=1}^{n} \inf_{\tau\in V(\theta:3\varepsilon_n)} \{\Phi^{i_1\cdots i_{k+1}}(x_\nu, \tau)-E[\Phi^{i_1\cdots i_{k+1}}(x, \tau); P_\tau]\}<-n\varepsilon_n')$$
$$= o(n^{-(k-1)/2}).$$

From (3.6), (3.7) and (3.8) we have

$$(3.9)\quad \sup_{\theta\in K} T_n^3(\theta) = o(n^{-(k-1)/2}).$$

From (3.3), (3.4), (3.5) and (3.9) we have

$$(3.10)\quad \sup_{\theta\in K} \|P_{\theta,n}-Q_{\theta,n}^*\| = o(n^{-(k-1)/2}).$$

Since θ_0 is contained in K it follows from (3.10) that there exists a number n_0^* such that for every n satisfying $n\geqq n_0^*$

$$\|P_{\theta_0,n}-Q_{\theta_0,n}^*\|<1/2.$$

Hence particularly for every $n\geqq n_0^*$ we have

$$(3.11)\quad \int_{X^{(n)}} q_n^*(z_n, \theta_0)d\mu_n>0.$$

Define $\Theta_n = \{\theta\in\Theta; \int_{X^{(n)}} q_n^*(z_n, \theta)d\mu_n>0\}$, $c_n(\theta) = [\int_{X^{(n)}} q_n^*(z_n, \theta)d\mu_n]^{-1}$ for $\theta\in\Theta_n$ and $c_n(\theta)=0$ for $\theta\notin\Theta_n$. From (3.11) $n\geqq n_0^*$ implies $\theta_0\in\Theta_n$. Let $d_n(\theta)$ be the indicator function of Θ_n i.e., $d_n(\theta)=1$ if $\theta\in\Theta_n$ and $d_n(\theta)=0$ if $\theta\notin\Theta_n$. We define a 'sufficient density' $q_n(z_n, \theta)$ as follows: $q_n(z_n, \theta)=[c_n(\theta)r_n^*(t_n^*, \theta)+c_n(\theta_0)(1-d_n(\theta))r_n^*(t_n^*, \theta_0)]s_n^*(z_n)$ for each $n\geqq n_0^*$, $=p_n(z_n, \theta_0)$ for each n satisfying $1\leqq n\leqq n_0^*-1$ where $z_n\in X^{(n)}$ and $\theta\in\Theta$. It can be easily seen that for every $n\in N$ and every $\theta\in\Theta$

$$\int_{X^{(n)}} q_n(z_n, \theta)d\mu_n = 1.$$

Let $Q_{\theta,n}$ be a probability measure on $(X^{(n)}, A^{(n)})$ defined by

$$Q_{\theta,n}(A)=\int_A q_n(z_n,\theta)d\mu_n \quad (A\in\mathcal{A}^{(n)}).$$

We note that the density $q_n(z_n,\theta)$ has the following form:

$$q_n(z_n,\theta)=r_n(t_n^*,\theta)\cdot s_n(z_n)$$

where

$$\begin{aligned} r_n(t_n^*,\theta) &= c_n(\theta)r_n^*(t_n^*,\theta)+c_n(\theta_0)(1-d_n(\theta))r_n^*(t_n^*,\theta_0) && \text{for } n\geqq n_0^* \\ &= 1 && \text{for } n\leqq n_0^*-1 \end{aligned}$$

and

$$\begin{aligned} s_n(z_n) &= s_n^*(z_n) && \text{for } n\geqq n_0^* \\ &= p_n(z_n,\theta_0) && \text{for } n\leqq n_0^*-1\,. \end{aligned}$$

Hence according to the factorization theorem t_n^* is sufficient for the family $\{Q_{\theta,n};\ \theta\in\Theta\}$ for each $n\in N$.

By (3.10) there exists a number n_1^* such that for every $n\geqq n_1^*$ we have

$$\sup_{\theta\in K}\|P_{\theta,n}-Q_{\theta,n}^*\|<1/2\,.$$

Hence $n\geqq n_1^*$ implies $K\subset\Theta_n$. Thus if $n\geqq n_2^*=\max(n_0^*,n_1^*)$ then for every $\theta\in K$

$$q_n(z_n,\theta)=c_n(\theta)\cdot q_n^*(z_n,\theta)\,.$$

From this we have for every $n\geqq n_2^*$

$$\begin{aligned} 2\cdot\|Q_{\theta,n}^*-Q_{\theta,n}\| &= |1-c_n^{-1}(\theta)|=|P_{\theta,n}(\mathcal{X}^{(n)})-Q_{\theta,n}^*(\mathcal{X}^{(n)})| \\ &\leqq \|P_{\theta,n}-Q_{\theta,n}^*\|\,, \end{aligned}$$

and hence

$$\sup\|P_{\theta,n}-Q_{\theta,n}\|\leqq\sup_{\theta\in K}\|P_{\theta,n}-Q_{\theta,n}^*\|+\sup_{\theta\in K}\|Q_{\theta,n}^*-Q_{\theta,n}\|\leqq 2\cdot\sup_{\theta\in K}\|P_{\theta,n}-Q_{\theta,n}^*\|\,.$$

Thus by (3.10) we obtain

$$\sup_{\theta\in K}\|P_{\theta,n}-Q_{\theta,n}\|=o(n^{-(k-1)/2})\,.$$

This completes the proof of the theorem.

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