Ninomiya, N.
Osaka J. Math.
17 (1980), 187–198

ON THE KERNEL OF POSITIVE DEFINITE TYPE

NOBUYUKI NINOMIYA

(Received March 31, 1979)

In a locally compact Hausdorff space, let $k(P, Q)$ be a real-valued function, continuous for any points P and Q, may be ∞ for $P=Q$ and always finite for $P\neq Q$, and $n(P, Q)$ a real-valued function finite and continuous for any points P and Q. A complex-valued function

$$K(P, Q) = k(P, Q)+in(P, Q)$$

is said to be of positive definite type if the double integral (called energy integral)

$$\iint K(P, Q)d\sigma(Q)d\bar{\sigma}(P)$$

of any complex-valued measure σ supported by a relatively compact Borelian set, whenever it is finitely determined, is non-negative. As is well-known, any function $K(P, Q)$ of positive definite type is symmetric:

$$K(P, Q) = \overline{K(Q, P)} \quad i.e. \quad k(P, Q) = k(Q, P) \quad \text{and} \quad n(P, Q) = -n(Q, P),$$

and

$$K(P, P) \geqq 0 \quad \text{and} \quad |K(P, Q)| \leqq \sup K(P, P)$$

for any points P and Q. In the real function theory, we see some results which characterize functions of positive definite type. In the present paper, we shall try to characterize functions of positive definite type on the point of view of the potential theory. We shall advance the argument adopting an idea and a method in the previous paper [2].

For any measure α and β (real-valued or complex-valued) supported by a relatively compact Borelian set, consider the potential taken with respect to a kernel $K(P, Q)$

$$K(P, \alpha) = \int K(P, Q)d\alpha(Q), \qquad K(\alpha, P) = \int K(Q, P)d\alpha(Q)$$

and the double integral (called mutual energy integral)

$$K(\alpha, \beta) = \int d\alpha(P)\int K(P, Q)d\beta(Q).$$

Similarly, we shall consider $k(P, \alpha)$, $k(\alpha, P)$, $k(\alpha, \beta)$, $n(P, \alpha)$, $n(\alpha, P)$ and $n(\alpha, \beta)$ with respect to kernels $k(P, Q)$ and $n(P, Q)$. We are going to prove following theorems.

Theorem 1. *Suppose that a kernel $K(P, Q)$ is symmetric. A necessary and sufficient condition that the kernel $K(P, Q)$ is of positive definite type is that the following property is satisfied:*

[P_1] *Let E_1, E_2, F_1 and F_2 be compact sets, E_1 and E_2 being disjoint, F_1 and F_2 being disjoint. Let μ_1, μ_2, ν_1 and ν_2 be positive measures supported by E_1, E_2, F_1 and F_2 respectively and with total mass a_1, a_2, b_1 and $b_2(=1)$ respectively. For certain constants A and B, if there hold inequalities*

$$k(P, \mu_1) \leqq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A$$

on the support of μ_1 and

$$k(P, \nu_1) \leqq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B$$

on the support of ν_1, then there holds the inequality

$$\begin{aligned} k(\mu_2, \mu_1) + k(\nu_2, \nu_1) \leqq & \; k(\mu_2, \mu_2) - n(\nu_1 - \nu_2, \mu_2) \\ & + k(\nu_2, \nu_2) - n(\nu_2, \mu_1 - \mu_2) + a_1 A + b_1 B\,. \end{aligned}$$

Furthermore, consider the kernel of positive definite type in a stronger form.

Definition. A kernel $K(P, Q)$ of positive definite type is said to satisfy the energy principle, if the double integral (called energy integral)

$$K(\bar{\sigma}, \sigma) = \iint K(P, Q) d\sigma(Q) d\bar{\sigma}(P)$$

of any complex-valued measure σ supported by a relatively compact Borelian set, whenever it is finitely determined, is non-negative and vanishes only when $\sigma \equiv 0$.

Then, we have:

Theorem 2. *Suppose that a kernel $K(P, Q)$ is of positive definite type. A necessary and sufficient condition that the kernel satisfies the energy principle is that the following property is satisfied:*

[P_2] *Let E_1, E_2, F_1 and F_2 be relatively compact Borelian sets, E_1 and E_2 being disjoint, F_1 and F_2 being disjoint. Let μ_1, μ_2, ν_1 and ν_2 be positive measures supported by E_1, E_2, F_1 and F_2 respectively and with total mass a_1, a_2, b_1 and b_2 $(=1)$ respectively. For certain constants A and B, if there hold inequalities*

$$k(P, \mu_1) \leqq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A$$

almost everywhere with respect to μ_1 *and*

$$k(P, \nu_1) \leqq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B$$

almost everywhere with respect to ν_1, *then there holds the inequality*

$$k(\mu_2, \mu_1) + k(\nu_2, \nu_1) < k(\mu_2, \mu_2) - n(\nu_1 - \nu_2, \mu_2) \\ + k(\nu_2, \nu_2) - n(\nu_2, \mu_1 - \mu_2) + a_1 A + b_1 B .$$

It is the following lemma that is important to prove the theorems. For any complex-valued measure supported by a relatively compact Borelian set

$$\sigma = \mu + i\nu ,$$

we have

$$K(\bar{\sigma}, \sigma) = \iint \{k(P, Q) + in(P, Q)\} \{d\mu(Q) + id\nu(Q)\} \{d\mu(P) - id\nu(P)\} .$$

$K(P, Q)$ being symmetric, *i.e.*

$$k(P, Q) = k(Q, P) \quad \text{and} \quad n(P, Q) = -n(Q, P) ,$$

we have further

$$K(\bar{\sigma}, \sigma) = k(\mu, \mu) + k(\nu, \nu) + 2n(\nu, \mu) .$$

When μ is a real-valued measure of variable sign, it's positive part μ^+ and negative part μ^- in Hahn's decomposition are positive measures supported by disjoint relatively compact Borelian sets respectively. This is similar with respect to ν, too. Hence, for relatively compact Borelian sets E_1, E_2, F_1 and F_2, E_1 and E_2 being disjoint, F_1 and F_2 being disjoint, each of E_1 and E_2 possibly intersecting with F_1 and F_2, each of F_1 and F_2 possibly intersecting with E_1 and E_2, and for positive measures μ_1, μ_2, ν_1 and ν_2 supported by E_1, E_2, F_1 and F_2 respectively, consider the quantity

$$\begin{aligned} I(\mu_1, \mu_2; \nu_1, \nu_2) &= k(\mu_1 - \mu_2, \mu_1 - \mu_2) + k(\nu_1 - \nu_2, \nu_1 - \nu_2) + 2n(\nu_1 - \nu_2, \mu_1 - \mu_2) \\ &= k(\mu_1, \mu_1) - 2k(\mu_1, \mu_2) + k(\mu_2, \mu_2) + k(\nu_1, \nu_1) - 2k(\nu_1, \nu_2) \\ &\quad + k(\nu_2, \nu_2) + 2n(\nu_1 - \nu_2, \mu_1 - \mu_2) \end{aligned}$$

We are going to study the minimum of this quantity. We should like to suppose that each of E_1, E_2, F_1 and F_2 is of k-transfinite diameter positive[1)].

Lemma. *Let* E_1, E_2, F_1 *and* F_2 *be relatively compact Borelian sets as stated*

1) For a symmetric kernel $k(P, Q)$, a compact set is said to be of k-transfinite diameter positive if it supports a positive measure with total mass 1 of k-energy integral finite. A Borelian set is said to be of k-transfinte diameter positive when it contains a compact set of k-transfinite diameter positive. Concerning the relation between the transfinite diameter and energy integrals, see for instance [1] (p. 45).

above, and μ_1', μ_2', ν_1' *and* ν_2' *positive measures supported by* E_1, E_2, F_1 *and* F_2 *respectively and with total mass* a_1, a_2, b_1 *and* b_2 $(=1)$ *respectively. Suppose that, among all the pairs* $(\mu_1', \mu_2', \nu_1', \nu_2')$ *of such measures, a pair* $(\mu_1, \mu_2, \nu_1, \nu_2)$ *makes minimum of* $I(\mu_1', \mu_2'; \nu_1', \nu_2')$:

$$-\infty < I(\mu_1, \mu_2; \nu_1, \nu_2) \leqq I(\mu_1', \mu_2'; \nu_1', \nu_2') .$$

Then, we have

(E_{11}) $A_1 \leqq e_1(P)$ *on* E_1 *except for a set of k-capacity zero*[2], *where*

$$e_1(P) = k(P, \mu_1) - k(P, \mu_2) + n(\nu_1 - \nu_2, P)$$

and

$$a_1 A_1 = k(\mu_1, \mu_1) - k(\mu_1, \mu_2) + n(\nu_1 - \nu_2, \mu_1) ,$$

(E_{12}) $e_1(P) \leqq A_1$ *on* E_1 *almost everywhere with respect to* μ_1. *When* $\bar{E}_1$ *and* $\bar{E}_2$ *are disjoint, the exceptional set of k-capacity zero might be replaced by of k-transfinite diameter zero. Similary, putting*

$$e_2(P) = k(P, \mu_2) - k(P, \mu_1) - n(\nu_1 - \nu_2, P) ,$$

$$a_2 A_2 = \int e_2(P) d\mu_2(P) ,$$

$$f_1(P) = k(P, \nu_1) - k(P, \nu_2) + n(P, \mu_1 - \mu_2) ,$$

$$b_1 B_1 = \int f_1(P) d\nu_1(P) ,$$

$$f_2(P) = k(P, \nu_2) - k(P, \nu_1) - n(P, \mu_1 - \mu_2) ,$$

and

$$b_2 B_2 = B_2 = \int f_2(P) d\nu_2(P) ,$$

we have

(E_{21}) $A_2 \leqq e_2(P)$ *on* E_2 *except for a set of k-capacity zero,*
(E_{22}) $e_2(P) \leqq A_2$ *on* E_2 *almost everywhere with respect to* μ_2,
(F_{11}) $B_1 \leqq f_1(P)$ *on* F_1 *except for a set of k-capacity zero,*
(F_{12}) $f_1(P) \leqq B_1$ *on* F_1 *almost everywhere with respect to* ν_1,
(F_{21}) $B_2 \leqq f_2(P)$ *on* F_2 *except for a set of k-capacity zero,*
(F_{22}) $f_2(P) \leqq B_2$ *on* F_2 *almost everywhere with respect to* ν_2.

When $\bar{E}_1$ *and* $\bar{E}_2$ *are disjoint and* $\bar{F}_1$ *and* $\bar{F}_2$ *disjoint, the exceptional sets of k-capacity zero might be replaced by of k-transfinite diameter zero.*

We are going to prove (E_{11}) and (E_{12}) only. Take any positive measure

2) A compact set is said to be of k-capacity positive if it supports a positive measure with total mass 1 whose k-potential is bounded from above on any compact set. A Borelian set is said to be of k-capacity positive when it contains a compact set of k-capacity positive. Evidently, if a set is of k-capacity positive, it is of k-transfinite diameter positive.

α_1 supported by E_1 with total mass a_1 such that $k(\alpha_1, \alpha_1)<\infty$ and $k(\alpha_1, \mu_1)<\infty$. Then, we have for any positive number ε smaller than 1 an inequality

$$I(\mu_1, \mu_2; \nu_1, \nu_2) \leqq I((1-\varepsilon)\mu_1+\varepsilon\alpha_1, \mu_2; \nu_1, \nu_2),$$

which, in consideration of the symmetricity of $k(P, Q)$, induces an inequality

$$\begin{aligned}&(2-\varepsilon)k(\mu_1, \mu_1)-2k(\mu_1, \mu_2)+2n(\nu_1-\nu_2, \mu_1)\\ &\quad\leqq 2(1-\varepsilon)k(\alpha_1, \mu_1)-2k(\alpha_1, \mu_2)+2n(\nu_1-\nu_2, \alpha_1)+\varepsilon k(\alpha_1, \alpha_1).\end{aligned}$$

Making $\varepsilon\to 0$, we have

$$a_1A_1 \leqq \int e_1(P)d\alpha_1(P),$$

hence

$$A_1 \leqq e_1(P)$$

almost everywhere with respect to any positive measures α_1 supported by E_1 with total mass a_1 such that $k(\alpha_1, \alpha_1)<\infty$ and $k(\alpha_1, \mu_2)<\infty$. The inequality naturally holds for $\alpha_1=\mu_1$ and there is an evident equality

$$a_1A_1 = \int e_1(P)d\mu_1(P),$$

so we have

$$e_1(P) \leqq A_1$$

almost everywhere with respect to μ_1. When $\bar{E}_1$ and $\bar{E}_2$ are disjoint, the inequality (E_{11}) holds with respect to any positive measure α_1 of k-energy integral finite supported by E_1 with total mass a_1. Then, the exceptional set might be considered as of k-transfinite diameter zero.

REMARK. The proof of Lemma never takes need of the anti-symmetricity of $n(P, Q)$.

Proof of Theorem 1. First, we shall prove that if a kernel $K(P, Q)$ is of positive definite type, it satisfies the property [P_1]. Let E_1, E_2, F_1 and F_2 be compact sets in the property [P_1] and μ_1, μ_2, ν_1 and ν_2 positive measures in the same manner. For certain constants A and B, suppose that there hold inequalities

$$k(P, \mu_1) \leqq k(P, \mu_2)-n(\nu_1-\nu_2, P)+A$$

on the support of μ_1 and

$$k(P, \nu_1) \leqq k(P, \nu_2)-n(P, \mu_1-\mu_2)+B$$

on the support of ν_1. We have naturally inequalities

$$k(\mu_1, \mu_1)-k(\mu_1, \mu_2)+n(\nu_1-\nu_2, \mu_1)\leqq a_1A$$

and

$$k(\nu_1, \nu_2)-k(\nu_1, \nu_2)+n(\nu_1, \mu_1-\mu_2)\leqq b_1B\,.$$

Unless we have the result of the property [P_1], we have the inequality

$$a_1A+b_1B<k(\mu_2, \mu_1)+k(\nu_2, \nu_1)-k(\mu_2, \mu_2)-k(\nu_2, \nu_2)+n(\nu_1-\nu_2, \mu_2)+n(\nu_2, \mu_1-\mu_2)\,.$$

Then, we have from those three inequalities

$$I(\mu_1, \mu_2; \nu_1, \nu_2)<0\,,$$

which is a contradiction. Next, we shall prove that if a kernel $K(P, Q)$ satisfies the property [P_1], it is of positive definite type. Let E_1, E_2, F_1 and F_2 be relatively compact Borelian sets, E_1 and E_2 being disjoint, F_1 and F_2 being disjoint. We suppose that each of E_1, E_2, F_1 and F_2 contains a compact set of k-transfinite diameter positive. We are going to prove

$$I(\mu_1', \mu_2'; \nu_1', \nu_2')\geqq 0\,,$$

whenever it is finitely determined, for any pair $(\mu_1', \mu_2', \nu_1', \nu_2')$ of positive measures supported by E_1, E_2, F_1 and F_2 respectively and with total mass a_1, a_2, b_1 and $b_2(=1)$ respectively. Consider the case when E_1, E_2, F_1 and F_2 all are compact sets. Then, we have a pair that gives minimum of $I(\mu_1', \mu_2'; \nu_1', \nu_2')$ among all the pairs of four positive measures stated above. Since, putting

$$I=\inf I(\mu_1', \mu_2'; \nu_1', \nu_2')\,,$$

I is finite and there exists a sequence of pairs of positive measures μ_{1n}, μ_{2n}, ν_{1n} and ν_{2n} supported by E_1, E_2, F_1 and F_2 respectively and with total mass a_1, a_2, b_1 and $b_2(=1)$ respectively such that

$$I(\mu_{1n}, \mu_{2n}; \nu_{1n}, \nu_{2n})\downarrow I\,.$$

We may suppose that the sequence $\{\mu_{1n}\}$, $\{\mu_{2n}\}$, $\{\nu_{1n}\}$ and $\{\nu_{2n}\}$ all are vaguely convergent by means of taking suitable sub-sequences out of them. Let μ_1, μ_2, ν_1 and ν_2 be their limiting measures respectively. They are positive measures supported by E_1, E_2, F_1 and F_2 respectively and with total mass a_1, a_2, b_1 and $b_2(=1)$ respectively. The k-potential of positive measures with compact supports being lower semi-continuous in the whole space, finite and continuous outside of their supports and the n-potential being finite and continuous everywhere, we have inequalities

$$k(\mu_1, \mu_1)\leqq \varliminf_{n\to\infty} k(\mu_{1n}, \mu_{1n})\,,$$

$$k(\mu_2, \mu_2)\leqq \varliminf_{n\to\infty} k(\mu_{2n}, \mu_{2n})\,,$$

$$k(\mu_1, \mu_2) = \lim_{n\to\infty} k(\mu_{1n}, \mu_{2n}),$$

those similar with respect to ν_1 and ν_2 and further an equality

$$n(\nu_1 - \nu_2, \mu_1 - \mu_2) = \lim_{n\to\infty} n(\nu_{1n} - \nu_{2n}, \mu_{1n} - \mu_{2n}).$$

So, we have

$$I \leqq I(\mu_1, \mu_2; \nu_1, \nu_2) \leqq \varliminf_{n\to\infty} I(\mu_{1n}, \mu_{2n}; \nu_{1n}, \nu_{2n}) = I$$

Then, by Lemma we have inequalities

$$k(P, \mu_1) \leqq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A_1$$

on the support of μ_1 and

$$k(p, \nu_1) \leqq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B_1$$

on the support of ν_1. Therefore, we have an inequality

$$\begin{aligned} &k(\mu_2, \mu_1) + k(\nu_2, \nu_1) - k(\mu_2, \mu_2) + n(\nu_1 - \nu_2, \mu_2) - k(\nu_2, \nu_2) \\ &\quad + n(\nu_2, \mu_1 - \mu_2) \leqq a_1 A_1 + b_1 B_1, \end{aligned}$$

similarly, an inequality

$$\begin{aligned} &k(\mu_1, \mu_2) + k(\nu_1, \nu_2) - k(\mu_1, \mu_1) - n(\nu_1 - \nu_2, \mu_1) - k(\nu_1, \nu_1) \\ &\quad - n(\nu_1, \mu_1 - \mu_2) \leqq a_2 A_2 + b_2 B_2. \end{aligned}$$

Thus, we have the inequality looking for:

$$0 \leqq 2(a_1 A_1 + b_1 B_1 + a_2 A_2 + b_2 B_2) = 2I(\mu_1, \mu_2; \nu_1, \nu_2).$$

Finally, suppose that some of E_1, E_2, F_1 and F_2 are not compacts. Let μ_1, μ_2, ν_1 and ν_2 be positive measures supported by E_1, E_2, F_1 and F_2 respectively and with total mass a_1, a_2, b_1 and $b_2(=1)$ respectively. Then, taking sequences of compact sets $\{E_{1n}\}$, $\{E_{2n}\}$, $\{F_{1n}\}$ and $\{F_{2n}\}$ such that

$$\begin{aligned} &E_{11} \subset E_{12} \subset \cdots \subset E_{1n} \subset \cdots \subset E_1, \qquad \mu_1(E_{1n}) \uparrow a_1, \\ &E_{21} \subset E_{22} \subset \cdots \subset E_{2n} \subset \cdots \subset E_2, \qquad \mu_2(E_{2n}) \uparrow a_2, \\ &F_{11} \subset F_{12} \subset \cdots \subset F_{1n} \subset \cdots \subset F_1, \qquad \nu_1(F_{1n}) \uparrow b_1, \\ &F_{21} \subset F_{22} \subset \cdots \subset F_{2n} \subset \cdots \subset F_2, \qquad \nu_2(F_{2n}) \uparrow b_2(=1), \end{aligned}$$

and taking restrictions μ_{1n}, μ_{2n}, ν_{1n} and ν_{2n} of μ_1, μ_2, ν_1 and ν_2 to E_{1n}, E_{2n}, F_{1n} and F_{2n} respectively, we have

$$I(\mu_{1n}, \mu_{2n}; \nu_{1n}, \nu_{2n}) \geqq 0,$$

making $n \to \infty$

$$I(\mu_1, \mu_2; \nu_1, \nu_2) \geqq 0 .$$

Proof of Theorem 2. First, we shall prove that if a kernel $K(P, Q)$ is of positive definite type and satisfies the energy principle, it satisfies the property [P_2]. Let E_1, E_2, F_1 and F_2 be relatively compact sets in the property [P_2] and μ_1, μ_2, ν_1 and ν_2 positive measures in the same manner. For certain constants A and B, suppose that there hold inequalities

$$k(P, \mu_1) \leqq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A$$

almost everywhere with respect to μ_1 and

$$k(P, \nu_1) \leqq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B$$

almost everywhere with respect to ν_1. We have naturally inequalities

$$k(\mu_1, \mu_1) - k(\mu_1, \mu_2) + n(\nu_1 - \nu_2, \mu_1) \leqq a_1 A$$

and

$$k(\nu_1, \nu_1) - k(\nu_1, \nu_2) + n(\nu_1, \mu_1 - \mu_2) \leqq b_1 B .$$

Unless we have the result of the property [P_2], we have the inequality

$$a_1 A + b_1 B \leqq k(\mu_2, \mu_1) + k(\nu_2, \nu_1) - k(\mu_2, \mu_2) + n(\nu_1 - \nu_2, \mu_2) - k(\nu_2, \nu_2) + n(\nu_2, \mu_1 - \mu_2) .$$

Then, we have from those three inequalities

$$I(\mu_1, \mu_2; \nu_1, \nu_2) \leqq 0 ,$$

which is a contradiction. Next, we shall prove that, if a kernel $K(P, Q)$ is of positive definite type and satisfies the property [P_2], it satisfies the energy integral. Let E_1, E_2, F_1 and F_2 be relatively compact sets, E_1 and E_2 being disjoint, F_1 and F_2 being disjoint. Let μ_1, μ_2, ν_1 and ν_2 be positive measures supported by F_1, E_2, F_1 and F_2 respectively and with total mass a_1, a_2, b_1 and $b_2(=1)$ respectively. We are going to prove

$$I(\mu_1, \mu_2; \nu_1, \nu_2) > 0$$

whenever it is finitely determined. If

$$I(\mu_1, \mu_2; \nu_1, \nu_2) = 0 ,$$

the pair $(\mu_1, \mu_2, \nu_1, \nu_2)$ is a minimal pair in Lemma and there hold inequalities

$$k(P, \mu_1) \leqq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A_1$$

almost everywhere with respect to μ_1 and

$$k(P, \nu_1) \leqq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B_1$$

almost everywhere with respect to ν_1. Then, by the property [P_2] we have an inequality

$$k(\mu_2, \mu_1)+k(\nu_2, \nu_1)<k(\mu_2, \mu_2)-n(\nu_1-\nu_2, \mu_2)+k(\nu_2, \nu_2) \\ -n(\nu_2, \mu_1-\mu_2)+a_1A_1+b_1B_1 .$$

Accordingly, we have

$$I(\mu_1, \mu_2; \nu_1, \nu_2)>0 ,$$

which is a contradiction.

Let us consider the kernel of positive definite type in a weaker form.

DEFINITION. A complex-valued function

$$K(P, Q) = k(P, Q)+in(P, Q)$$

is said to be of positive definite type in restricted sense, if the double integral (called energy integral)

$$\iint K(P, Q)d\sigma(Q)d\bar{\sigma}(P)$$

of any complex-valued measure σ supported by a relatively compact Borelian set with total mass 0, whenever it is finitely determined, is non-negative.

DEFINITION. A kernel $K(P, Q)$ of positive definite type in restricted sense is said to satisfy the energy principle, if the double integral (called energy integral)

$$\iint K(P, Q)d\sigma(Q)d\bar{\sigma}(P)$$

of any complex-valued measure σ supported by a relatively compact Borelian set with total mass 0, whenever it is finitely determined, is non-negative and vanishes only when $\sigma\equiv 0$.

For a kernel of positive definite type in restricted sense, Theorems 1 and 2 are expressed in the following styles.

Theorem 1′. *Suppose that a kernel $K(P, Q)$ is symmetric. A necessary and sufficient condition that the kernel $K(P, Q)$ is of positive definite type in restricted sense is that the following property is satisfied*:

[P_1'] *Let E_1, E_2, F_1 and F_2 be compact sets, E_1 and E_2 being disjoint, F_1 and F_2 being disjoint. Let μ_1 and μ_2 be positive measures supported by E_1 and E_2 with total mass a (>0) respectively and ν_1 and ν_2 positive measures supported by F_1 and F_2 with total mass 1 respectively. For certain constants A and B, if there hold inequalities*

$$k(P, \mu_1)\leqq k(P, \mu_2)-n(\nu_1-\nu_2, P)+A$$

on the support of μ_1 *and*

$$k(P, \nu_1) \leqq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B$$

on the support of ν_1, *then there holds the inequality*

$$k(\mu_2, \mu_1) + k(\nu_2, \nu_1) \leqq k(\mu_2, \mu_2) - n(\nu_1 - \nu_1, \mu_2) + k(\nu_2, \nu_2) - n(\nu_2, \mu_1 - \mu_2) + aA + B .$$

Theorem 2′. *Suppose that a kernal* $K(P, Q)$ *is of positive definite type in restricted sense. A necessary and sufficient condition that the kernel* $K(P, Q)$ *satisfies the energy principle is that the following property is satisfied*:

[P_2'] *Let* E_1, E_2, F_1 *and* F_2 *be relatively compact Borelian sets*, E_1 *and* E_2 *being disjoint*, F_1 *and* F_2 *being disjoint. Let* μ_1 *and* μ_2 *be positive measures supported by* E_1 *and* E_2 *with total mass* a (>0) *respectively and* ν_1 *and* ν_2 *positive measures supported by* F_1 *and* F_2 *with total mass* 1 *respectively. For certain constants* A *and* B, *if there hold inequalities*

$$k(P, \mu_1) \leqq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A$$

almost everywhere with respect to μ_1 *and*

$$k(P, \nu_1) \leqq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B$$

almost everywhere with respect to ν_1, *then there holds the inequality*

$$k(\mu_2, \mu_1) + k(\nu_2, \nu_1) < k(\mu_2, \mu_2) - n(\nu_1 - \nu_2, \mu_2) + k(\nu_2, \nu_2) - n(\nu_2, \mu_1 - \mu_2) + aA + B .$$

Corollary. *A kernel* $K(P, Q)$, *which is symmetric, is of positive definite type in restricted sense if the following property is satisfied*:

[P*] *Let* E_1, E_2, F_1 *and* F_2 *be relatively compact Borelian sets*, E_1 *and* E_2 *being disjoint*, F_1 *and* F_2 *being disjoint. Let* μ_1 *and* μ_2 *be positive measures supported by* E_1 *and* E_2 *with total mass* a (>0) *respectively and* ν_1 *and* ν_2 *positive measures supported by* F_1 *and* F_2 *with total mass* 1 *respectively. For certain constants* A *and* B, *if there hold inequalities*

$$k(P, \mu_1) \leqq k(P, \mu_2) - n(\nu_1 - \nu_2, P) + A$$

on the support of μ_1 *and*

$$k(P, \nu_1) \leqq k(P, \nu_2) - n(P, \mu_1 - \mu_2) + B$$

on the support of ν_1, *then these two inequalities hold at the same time in the whole space.*

Using this corollary, we should like to terminate the paper presenting a simple example of a kernel satisfying the above property [P^*], therefore of positive definite type in restricted sense.

EXAMPLE. On the plane R^2, let

$$P = (x_1, x_2)\,, \qquad Q = (y_1, y_2)\,,$$
$$k(P, Q) = \log \frac{1}{PQ} = \log \frac{1}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}}$$

and

$$n(P, Q) = c_1(x_1-y_1)+c_2(x_2-y_2)\,,$$

c_1 and c_2 being any real constants. A complex-valued function, which is symmetric,

$$K(P, Q) = k(P, Q)+in(P, Q)$$

is of positive definite type in restricted sense.

In fact, let E_1, E_2, F_1 and F_2 be compact sets, E_1 and E_2 being disjoint, F_1 and F_2 being disjoint. Let μ_1 and μ_2 be positive measures with total mass $a(>0)$ whose supports are E_1 and E_2 respectively, and ν_1 and ν_2 be positive measures with total mass 1 whose supports are F_1 and F_2 respectively. For certain constants A and B, suppose that there hold inequalities

$$k(P, \mu_1) \leqq k(P, \mu_2)-n(\nu_1-\nu_2, P)+A$$

on E_1 and

$$k(P, \nu_1) \leqq k(P, \nu_2)-n(P, \mu_1-\mu_2)+B$$

on E_2. We are going to prove that the former inequality holds everywhere. Then, we shall see that the latter one holds everywhere, too. E_1 being a compact set of logarithmic capacity positive, let λ be the equilibrium measure on E_1. That is, λ is a positive measure supported by E_1 with total mass 1 such that

$$k(P, \lambda) = \int \log \frac{1}{PQ}\, d\lambda(Q)$$

is equal to a constant V on E_1 except for a set of logarithmic capacity zero, and $\leqq V$ everywhere. The logarithmic potential of a positive measures with compact support is superharmonic in the whole plane and harmonic in each component outside of the support, and the n-potential is harmonic in the whole plane. Then, ε being any positive number, consider the function

$$g(P) = k(P, \mu_1)+\varepsilon k(P, \lambda)-k(P, \mu_2)+n(\nu_1-\nu_2, P)-(A+\varepsilon V)\,.$$

This is subharmonic in each component outside of E_1, and we have at each boundary point M of E_1

$$\overline{\lim_{P\to M}}\, g(P) \leqq \overline{\lim_{P'\to M}}\, g(P') \leqq 0^{3)}\,,$$

P being outside of E_1 and P' being in E_1. We should like to prove

$$\lim_{P\to\infty} g(P) = -\infty\,.$$

First, we have

$$\lim_{P\to\infty} k(P, \lambda) = -\infty\,.$$

Next, we have, 0 denoting the origin,

$$\lim_{P\to\infty} \{k(P, \mu_1)-k(P, \mu_2)\} = \lim_{P\to\infty} \left\{\int \log\frac{PO}{PQ}\, d\mu_1(Q) - \int \log\frac{PO}{PQ}\, d\mu_2(Q)\right\} = 0$$

Finally, let us notice that $n(\nu_1-\nu_2, P)$ is bounded. Since, taking the Dirac measure $\mathcal{E}$ at the origin 0, we have

$$|n(\nu_1-\nu_2, P)| = |n(P, \nu_1)-n(P, \nu_2)| \leqq |n(P, \nu_1)-n(P, \mathcal{E})| + |n(P, \nu_2)-n(P, \mathcal{E})|$$
$$\leqq \int \{|c_1 y_1| + |c_2 y_2|\}\, \{d\nu_1(Q)+d\nu_2(Q)\}\,.$$

So, we have

$$g(P) \leqq 0$$

in each component outside of E_1. Making $\mathcal{E}\to 0$, we have

$$k(P, \mu_1) \leqq k(P, \mu_2) - n(\nu_1-\nu_2, P) + A$$

everywhere.

QUESTION. A kernel $K(P, Q)$ which is of positive definite type in restricted sense but is not symmetric, does it exist ?

References

[1] O. Frostman: *Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions*, Medd. Lunds Univ. Mat. Sem. **3** (1935), 1–118.

[2] N. Ninomiya: *On the potential taken with respect to complex-valued and symmetric kernels*, Osaka J. Math. **9** (1972), 1–9.

Department of Mathematics, Osaka City University
Sugimoto-cho, Sumiyoshi-ku, Osaka 558, Japan

---

3) For a kernel $k(P, Q)$ logarithmic on R^2, Newtonian in R^3 or generally satisfying the maximum principle of Frostman, the potential $k(P, \mu)$ of a positive measure μ with compact support satisfies at each boundary point M of the support of μ an inequality

$$\overline{\lim_{P\to M}}\, k(P, \mu) \leqq \overline{\lim_{P'\to M}}\, k(P', \mu)\,,$$

P being outside of the support of μ and P' being on the support of μ (cf.: [1], p. 69).