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ON STABILITY OF FINITELY GENERATED KLEINIAN GROUPS

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1. Introduction. The conformal automorphisms of the extended complex plane $\hat{C}=C\cup\{\infty\}$ form the Möbius group $Möb$ Every element α of $Möb$ is a transformation of the form

$$\alpha(z)=(az+b)/(cz+d),$$

where a,b,c and d are complex numbers with $ad-bc=1$. Hence $Möb$ may be considered as a 3-dimensional complex Lie group, isomorphic to $SL(2,C)$ modulo its center. We denote by e the identity transformation of $Möb$. An element $\alpha\in Möb$, $\alpha(z)=(az+b)/(cz+d)$, different from e, is called parabolic if $\mathrm{tr}^2\alpha=(a+d)^2=4$; α is called elliptic if $\mathrm{tr}^2\alpha=(a+d)^2\in[0\ 4)$; in all other cases α is called loxodromic.

Let G be a finitely generated Kleinian group, $\Omega=\Omega(G)$ the region of discontinuity of G and $\Lambda=\Lambda(G)$ the limit set of G. Let $M(G)$ be the set of Beltrami coefficients $\mu(z)$ for G supported on $\Omega(G)$, that is, the open unit ball in the closed linear subspace of $L_\infty(C)$ determined by the conditions

$$\mu(\gamma z)\overline{\gamma'(z)}/\gamma'(z)=\mu(z),\ (\gamma\in G) \tag{1.1}$$

and

$$\mu_{|\Lambda(G)}=0, \tag{1.2}$$

where $L_\infty(C)$ is the complex Banach space consisting of measurable functions μ on C with finite L_∞ norm $||\mu||$. Let w^μ be the uniquely determined quasiconformal automorphism of $\hat{C}$ with the Beltrami coefficient $\mu=w^\mu_{\bar z}/w^\mu_z$, which keeps the points $0,1,\infty$ fixed. The above condition (1.1) is necessary and sufficient in order that $w^\mu G(w^\mu)^{-1}$ is again a Kleinian group; this is easily checked and is well-known.

Let $\gamma_1,\gamma_2,\cdots,\gamma_k$ be a system of generators for G. A homomorphism χ: $G\rightarrow Möb$ is called parabolic if $\mathrm{tr}^2\chi(\gamma)=4$ for every parabolic element $\gamma\in G$. Let χ: $G\rightarrow Möb$ be a parabolic homomorphism. Then χ is represented by the point $(\chi(\gamma_1), \chi(\gamma_2), \cdots, \chi(\gamma_k)) \in (Möb)^k$ and we may consider the set $X_p(G)$ of all the parabolic homomorphisms as a subset of $(Möb)^k$. The identity isomorphism *id* of G is represented by the point $(\gamma_1, \gamma_2, \cdots, \gamma_k) \in (Möb)^k$.

A homomorphism $\chi: G \to Möb$ will be called, in this paper, Ω-parabolic if $\mathrm{tr}^2\chi(\gamma)=4$ for every parabolic element $\gamma \in G$ determined by a puncture on Ω/G We denote by $X_{\Omega-p}(G)$ the set of all the Ω-parabolic homomorphisms of G into $Möb$.

A quasiconformal deformation of G is a homomorphism χ sending $\gamma \in G$ into $\alpha \circ w^\mu \circ \gamma \circ (\alpha \circ w^\mu)^{-1} \in Möb$, where $\alpha \in Möb$ and $\mu \in M(G)$. The quasiconformal deformations of G form a subset $X_{qc}(G)$ of $X_p(G)$. Now we have the canonical surjection $\Phi_G: Möb \times M(G) \to X_{qc}(G)$ that takes $(\alpha, \mu) \in Möb \times M(G)$ into the homomorphism $\chi \in X_{qc}(G)$ with $\chi(\gamma)=\alpha \circ w^\mu \circ \gamma \circ (\alpha \circ w^\mu)^{-1}$. It is known that Φ_G is holomorphic (see Bers [1], [2]). We note that $X_{qc}(G) \subset X_p(G) \subset X_{\Omega-p}(G)$. Moreover, $X_{\Omega-p}(G)$ is, in a natural way, an affine algebraic variety and a change of generators for G amounts to a biholomorphic transformation of $X_{\Omega-p}(G)$.

Now following Bers [2], we can give the definitions of stability of finitely generated Kleinian groups.

Let G be a finitely generated Kleinian group and $\gamma_1, \gamma_2, \cdots, \gamma_k$ an arbitrarily chosen and fixed system of generators for G. Then G is called quasistable if, for every open neighborhood N of the origin 0 in $M(G)$, there exists an open neighborhood U of $id=(\gamma_1, \gamma_2, \cdots, \gamma_k)$ in $(Möb)^k$ such that $U \cap X_{qc}(G) \subset \Phi_G(Möb \times N)$. If there exists an open neighborhood U of *id* in $(Möb)^k$ such that $U \cap X_p(G)=U \cap X_{qc}(G)$, then G is said to be quasiconformally stable. Analogously, if there exists an open neighborhood U of *id* in $(Möb)^k$ such that $U \cap X_{\Omega-p}(G)=U \cap X_{qc}(G)$, then, in this paper, we will say that G is quasiconformally stable in Ω-parabolic sense. From the fact already stated, it will be easily checked that these definitions are well-defined, that is, are independent of the choice of generators for G.

In this paper we shall be concerned with the above stability of finitely generated Kleinian groups and prove some theorems.

2. Prerequisite lemmas. In this section we prepare some notations and lemmas for the later discussions.

Let G be a Kleinian group. Let D be a non-empty open subset of $\Omega(G)$ invariant under G such that $\hat{C} \setminus D$ contains more than two points. The pair (D, G) is called the configuration (D, G) (see Bers [3]).

A quadratic differential (or an automorphic form of weight -4) for a configuration (D, G) is a holomorphic function $\phi(z)$ in D such that $\phi(\gamma(z))\gamma'(z)^2 = \phi(z)$ for any $\gamma \in G$. It is called integrable if $\|\phi\|_A = \iint_{D/G} |\phi(z)|\, dxdy < \infty$, and is called bounded if $||\phi||_B=||\lambda_D^{-2}\phi||=\sup_{z\in D}\lambda_D^{-2}(z)|\phi(z)|<\infty$, where $\lambda_D(z)|dz|$ is the Poincaré metric on D. Integrable quadratic differentials for (D,G) form a Banach space $A(D,G)$ with the norm $||\ ||_A$. Bounded quadratic differentials for (D,G) form a Banach space $B(D,G)$ with the norm $||\ ||_B$. Let D_G be D with all elliptic fixed points of G removed. Assume that D_G/G is of finite type, that is, D_G/G is a disjoint finite union of Riemann surfaces $S_1+S_2+\cdots+S_m$, where each S_i is obtained from a compact Riemann surface of genus g_i by removing n_i $(<+\infty)$ points. Then, as is well-known, $B(D,G)$ is identical with $A(D,G)$ and the dimension of $B(D,G)$ is equal to $\sum_{i=1}^{m}(3g_i-3+n_i)$ (see Chapter III of Kra [5]).

According to Ahlfors' finiteness theorem (see [5] and the literature quoted there), if G is a non-elementary finitely generated Kleinian group, then Ω_G/G is of finite type and the dimension of $B(\Omega,G)$ is finite. We denote its dimension by $\sigma(G)$. For $\phi\in B(\Omega,G)$ satisfying $||\lambda_\Omega^{-2}\phi||<1$, set $\mu(z)=\lambda_\Omega^{-2}(z)\bar{\phi}(z)$ for $z\in\Omega$, $\mu_{|\Lambda}=0$. Since $\lambda_\Omega^2(\gamma(z))\overline{\gamma'(z)}\gamma'(z)=\lambda_\Omega^2(z)$ for any $\gamma\in G$, we have $\mu\in M(G)$. Such μ are called canonical Beltrami coefficients for G. They form the open unit ball $M_{can}(G)$ in a vector space of dimension $\sigma(G)$.

Lemma 1. *Let G be a finitely generated Kleinian group and let $\gamma_1, \gamma_2, \cdots, \gamma_k$ be a system of generators for G. Put $G_1=\alpha G\alpha^{-1}$ for a fixed $\alpha\in Möb$. Then there are biholomorphic bijections θ: $(Möb)^k\to(Möb)^k$ and τ: $M(G)\to M(G_1)$ satisfying the following properties*:

(1) $\tau(0)=0$, $\tau(M(G))=M(G_1)$,

(2) $\theta((\gamma_1,\gamma_2,\cdots,\gamma_k))=(\alpha\circ\gamma_1\circ\alpha^{-1}, \alpha\circ\gamma_2\circ\alpha^{-1}, \cdots, \alpha\circ\gamma_k\circ\alpha^{-1})$,
$\theta(X_{qc}(G))=X_{qc}(G_1)$, $\theta(X_p(G))=X_p(G_1)$,
$\theta(X_{\Omega-p}(G))=X_{\Omega-p}(G_1)$, and

(3) $\theta(\Phi_G(Möb\times N))=\Phi_{G_1}(Möb\times\tau(N))$ *for any subset N in $M(G)$.*

In particular, if G is quasi-stable or quasiconformally stable or quasiconformally stable in Ω-parabolic sense, then so is G_1, respectively.

Proof. We define θ: $(Möb)^k\to(Möb)^k$ by $\theta((\tilde{\gamma}_1,\tilde{\gamma}_2,\cdots,\tilde{\gamma}_k))=(\alpha\circ\tilde{\gamma}_1\circ\alpha^{-1}, \alpha\circ\tilde{\gamma}_2\circ\alpha^{-1}, \cdots, \alpha\circ\tilde{\gamma}_k\circ\alpha^{-1})$ for $(\tilde{\gamma}_1,\tilde{\gamma}_2,\cdots,\tilde{\gamma}_k)\in(Möb)^k$. We define τ by $\tau(\mu)(z)=\mu(\alpha^{-1}(z))\overline{\alpha^{-1\prime}(z)}/\alpha^{-1\prime}(z)$ for $\mu\in M(G)$ and $z\in\hat{C}$. If we set $W=w^\mu\circ\alpha^{-1}$ for $\mu\in M(G)$, then the complex dilatation μ_W of W satisfies $\mu_W=\tau(\mu)$. By this fact, we readily see the properties (1), (2), (3) to hold and the latter part of the lemma follows from these properties and the definitions of stability.

REMARK 1. In the case where G is non-elementary, if we set $\psi(z)=\phi(\alpha^{-1}(z))\alpha^{-1\prime}(z)^2$ for $\phi\in B(\Omega(G),G)$, then we can easily check that ψ belongs to $B(\Omega(G_1),G_1)$ and that $\tau(M_{can}(G))=M_{can}(G_1)$.

Lemma 2. *Let G be a non-elementary finitely generated Kleinian group* *and* $\gamma_1, \gamma_2, \cdots, \gamma_k$ *a system of generators for* G. *Then the three following conditions are equivalent to each other*:

(1) G *is quasi-stable,*

(2) *for every open neighborhood* $\tilde{N}$ *of the origin* 0 *in* $M_{can}(G)$, *there exists an open neighborhood* U *of* $id=(\gamma_1, \gamma_2, \cdots, \gamma_k)$ *in* $(M\ddot{o}b)^k$ *such that* $U\cap X_{qc}(G)\subset \Phi_G(M\ddot{o}b\times\tilde{N})$,

and

(3) *if* V *is a sufficiently small open neighborhood of* $(e,0)$ *in* $M\ddot{o}b\times M_{can}(G)$, *then the restriction* $\Phi_{G|V}$ *of* Φ_G *to* V *maps* V *biholomorphically onto some open neighborhood of* $id=(\gamma_1, \gamma_2, \cdots, \gamma_k)$ *in* $X_{qc}(G)$. *In particular, there exists an open neighborhood* U *of id in* $(M\ddot{o}b)^k$ *such that* $U\cap X_{qc}(G)$ *is a complex analytic submanifold of dimension* $\sigma(G)+3$ *of* U.

Proof. According to Lemma 1 and REMARK 1, we may assume that G is normalized such that the points $0, 1, \infty$ are the attractive fixed points of some loxodromic elements, say, $\gamma_1^*, \gamma_2^*, \gamma_3^*$ of G, respectively. In this case, if we fix a sufficiently small open neighborhood $\hat{N}$ of 0 in $M(G)$, it has been known that there exists an open continuous mapping $\Pi_\Omega\colon \hat{N}\to M_{can}(G)$ such that $\Pi_\Omega(0)=0$ and $\Phi_G(\alpha,\mu)=\Phi_G(\alpha,\Pi_\Omega(\mu))$ for $\alpha\in M\ddot{o}b$ and $\mu\in\hat{N}$ (see Theorem 1 in Bers [3]).

Assume that G is quasi-stable. Let $\tilde{N}$ be a given open neighborhood of 0 in $M_{can}(G)$. Then there exists an open neighborhood U of $id=(\gamma_1, \gamma_2, \cdots, \gamma_k)$ in $(M\ddot{o}b)^k$ such that $U\cap X_{qc}(G)\subset\Phi_G(M\ddot{o}b\times\Pi_\Omega^{-1}(\tilde{N}))$. Since $\Phi_G(M\ddot{o}b\times \Pi_\Omega^{-1}(\tilde{N}))\subset\Phi_G(M\ddot{o}b\times\tilde{N})$, (1) implies (2).

By the definition of quasi-stability, (3) clearly implies (1). Thus it remains to prove that (2) implies (3). Put $F=\alpha\circ w^\mu$ for $\alpha\in M\ddot{o}b$ and $\mu\in M(G)$. Then α is uniquely determined by the homomorphism $\chi\colon \gamma\mapsto F\circ\gamma\circ F^{-1}$. Indeed, α takes $0, 1, \infty$ into the attractive fixed points of $\chi(\gamma_1^*)$, $\chi(\gamma_2^*)$ and $\chi(\gamma_3^*)$, respectively. Furthermore, we can easily verify that α depends continuously on $\chi\in X_{qc}(G)$. Thus, for any open neighborhood W of e in $M\ddot{o}b$, there exists some open neighborhood U_1 of id in $(M\ddot{o}b)^k$ such that

$$U_1\cap X_{qc}(G)\subset\Phi_G(W\times M(G))\,. \tag{2.1}$$

Let $\tilde{N}$ be any open neighborhood of 0 in $M_{can}(G)$. Assume the property (2). Then there exists an open neighborhood U of id in $(M\ddot{o}b)^k$ such that

$$U\cap X_{qc}(G)\subset\Phi_G(M\ddot{o}b\times\tilde{N})\,. \tag{2.2}$$

We may assume that $U\subset U_1$. Owing to (2.1) and (2.2), we have $U\cap X_{qc}(G)\subset\Phi_G(W\times\tilde{N})$. This means that, for any open neighborhood $V=W\times\tilde{N}$ of $(e,0)$ in $M\ddot{o}b\times M_{can}(G)$, there exists an open neighborhood U of id in $(M\ddot{o}b)^k$ such that

$$U\cap X_{qc}(G)\subset\Phi_G(V)\,. \tag{2.3}$$

Since we know by Lemma 1 in Bers [2] that the restriction $\Phi_{G|M\ddot{o}b\times M_{can}(G)}$ to $M\ddot{o}b\times M_{can}(G)$ of Φ_G has maximal rank $\sigma(G)+3$ at the point $(e,0)$, the above (2.3) implies (3). Thus we see that (2) implies (3).

3. Criteria for stability. A Kleinian group G acts on the right on the vector space Π of quadratic polynomials via

$$p\gamma(z)=p(\gamma z)/\gamma'(z),\ (p\in\Pi,\ \gamma\in G,\ z\in\boldsymbol{C}).$$

One can thus define the (first) Eichler cohomology group $H^1(G,\Pi)$, that is, $H^1(G,\Pi)$ is the space of cocycles $Z^1(G,\Pi)$ factored by the space of coboundaries $B^1(G,\Pi)$. If $p\in Z^1(G,\Pi)$ satisfies

$$p\,|\,G_0\in B^1(G_0,\Pi) \tag{3.1}$$

for every parabolic cyclic subgroup G_0 of G, we say that p belongs to $PZ^1(G,\Pi)$, the space of parabolic cocycles. We denote by $PH^1(G,\Pi)$ the space of parabolic cohomology, that is, the space of parabolic cocycles factored by the space of coboundaries. Analogously, the space $PZ^1_\Omega(G,\Pi)$ of Ω-parabolic cocycles is defined as the space of those $p\in Z^1(G,\Pi)$ for which (3.1) holds for every parabolic cyclic subgroup G_0 of G which corresponds to a puncture on Ω/G, and we denote by $PH^1_\Omega(G,\Pi)$ the space of Ω-parabolic cohomology. From these definitions we have the equalities

$$\dim PH^1(G,\Pi)=\dim PZ^1(G,\Pi)-\dim B^1(G,\Pi) \tag{3.2}$$

and

$$\dim PH^1_\Omega(G,\Pi)=\dim PZ^1_\Omega(G,\Pi)-\dim B^1(G,\Pi)\,. \tag{3.3}$$

For a non-elementary Kleinian group G, we have the so-called Bers' map $\beta^*\colon B(\Omega,G)\to H^1(G,\Pi)$ which is anti-linear and injective, and we know that $\beta^*(B(\Omega,G))\subset PH^1(G,\Pi)$ (see Kra [5]).

Gardiner and Kra have discussed in [4] the intimate relation between $X_{\Omega-p}(G)$ and $PH^1_\Omega(G,\Pi)$ and the analogous relation between $X_p(G)$ and $PH^1(G,\Pi)$. Here we state slightly stronger versions of some results in [4] as the following lemmas (see Theorem 8.4 in [4]). The proofs of these stronger versions are already accomplished in [4].

Lemma 3. *Let G be a finitely generated Kleinian group and $\gamma_1,\gamma_2,\cdots,\gamma_k$ a system of generators for G. Then there exist an open neighborhood U of $id=(\gamma_1,\gamma_2,\cdots,\gamma_k)$ in $(M\ddot{o}b)^k$ and a complex submanifold V of dimension* $\dim PZ^1(G,\Pi)$ *of U such that $U\cap X_p(G)\subset V$. If, further, $U\cap X_p(G)$ itself is a submanifold of U, then the holomorphic tangent space of $U\cap X_p(G)$ at id is isomorphic to $PZ^1(G,\Pi)$ and hence the dimension of $U\cap X_p(G)$ equals* $\dim PZ^1(G,\Pi)$. *Analogously there exist an open neighborhood $\tilde{U}$ of id in $(M\ddot{o}b)^k$ and a submanifold $\tilde{V}$ of* *dimension* $\dim PZ^1_\Omega(G,\Pi)$ *of* $\tilde{U}$ *such that* $\tilde{U}\cap X_{\Omega-p}(G)\subset\tilde{V}$ *and such that, if* $\tilde{U}\cap X_{\Omega-p}(G)$ *itself is a submanifold, then the dimension of* $\tilde{U}\cap X_{\Omega-p}(G)$ *is equal to* $\dim PZ^1_\Omega(G,\Pi)$.

REMARK 2. For a non-elementary Kleinian group G, we know that dim $B^1(G,\Pi)=3$ (see [5]). Thus, for a non-elementary finitely generated Kleinian group G, in view of (3.2) and (3.3), we have

$$\dim PZ^1(G,\Pi) = \dim PH^1(G,\Pi)+3 \tag{3.4}$$

and

$$\dim PZ^1_\Omega(G,\Pi) = \dim PH^1_\Omega(G,\Pi)+3\,. \tag{3.5}$$

Lemma 4. *Let G be a non-elementary finitely generated Kleinian group. If* $\beta^*(B(\Omega,G))=PH^1(G,\Pi)$, *then G is both quasiconformally stable and quasi-stable. If* $\beta^*(B(\Omega,G))=PH^1_\Omega(G,\Pi)$, *then G is both quasiconformally stable in* Ω*-parabolic sense and quasi-stable.*

Now we can prove the converse of Lemma 4 by using Lemmas 2 and 3.

Theorem 1. *Let G be a non-elementary finitely generated Kleinian group. Then G is both quasiconformally stable and quasi-stable if and only if* $\beta^*(B(\Omega,G))=PH^1(G,\Pi)$. *Analogously, G is both quasiconformally stable in* Ω*-parabolic sense and quasi-stable if and only if* $\beta^*(B(\Omega,G))=PH^1_\Omega(G,\Pi)$.

Proof. According to Lemma 4, it suffices to prove the necessity. Assume that G is both quasiconformally stable and quasi-stable. Then, by Lemma 2, there exists an open neighborhood U of $(\gamma_1,\gamma_2,\cdots,\gamma_k)$ in $(M\ddot{o}b)^k$ such that $U\cap X_{qc}(G)$ is a submanifold of dimension $\sigma(G)+3$ of U, where $(\gamma_1,\gamma_2,\cdots,\gamma_k)$ is a system of generators for G. Since G is quasiconformally stable, we may assume that

$$U\cap X_p(G) = U\cap X_{qc}(G)\,. \tag{3.6}$$

Then, according to Lemma 3 and REMARK 2, $U\cap X_p(G)$ is a submanifold of dimension $\dim PH^1(G,\Pi)+3$ of U. Thus, by (3.6), we see that $\sigma(G)+3=\dim PH^1(G,\Pi)+3$, and because of the injectivity of β^*, we have $\beta^*(B(\Omega,G))=PH^1(G,\Pi)$.

In a similar manner, if G is both quasiconformally stable in Ω-parabolic sense and quasi-stable, we see that $\beta^*(B(\Omega,G))=PH^1_\Omega(G,\Pi)$.

REMARK 3. Bers conjectured in [2] that all finitely generated Kleinian groups are quasi-stable. Kruškal' [7] responded to this conjecture in the affirmative for all non-elementary finitely generated Kleinian groups, but it seems hard to follow.

Let G be a non-elementary finitely generated Kleinian group and $\gamma_1, \gamma_2, \cdots, \gamma_k$ a system of generators for G. Let $\chi: G \rightarrow Möb$ be an isomorphism such that both χ and χ^{-1} preserve parabolic elements and such that $\chi(G)$ is a Kleinian group. All such χ form a subset S of $X_p(G)$. Using the fact that a Kleinian group is elementary if and only if it has a commutative subgroup of finite index, we see that if $\chi \in S$, then $\chi(G)$ is non-elementary. Let S_1 be the subset of S which consists of all $\chi \in S$ such that $\chi(G)$ is quasiconformally stable and let S_2 be the subset of S which consists of all $\chi \in S$ with $\beta^*(B(\Omega(\chi(G)), \chi(G))) = PH^1(\chi(G), \Pi)$.

Then we can prove the following theorem.

Theorem 2. *Let G be a non-elementary finitely generated Kleinian group and $\gamma_1, \gamma_2, \cdots, \gamma_k$ a system of generators for G. Then S_1 and S_2 are empty or open in $X_p(G)$ and S_2 is a subset of S_1.*

Proof. By Lemma 4, S_2 is clearly a subset of S_1. Let χ be any element of S. As both χ and χ^{-1} are isomorphisms which preserve parabolic elements, we can see $X_p(\chi(G)) = X_p(G)$ as subsets of $(Möb)^k$. Hence it suffices to prove that, if $id = (\gamma_1, \gamma_2, \cdots, \gamma_k) \in S_i$, then S_i includes an open neighborhood of id in $X_p(G)$ for $i = 1, 2$.

First we treat the case $id \in S_1$. Then G is quasiconformally stable and thus there exists an open neighborhood U of id in $(Möb)^k$ such that $U \cap X_p(G) = U \cap X_{qc}(G)$. Let χ be any element of $U \cap X_p(G)$. As χ is a quasiconformal deformation of G, we can see $X_p(\chi(G)) = X_p(G)$ and $X_{qc}(\chi(G)) = X_{qc}(G)$ as subsets of $(Möb)^k$. Hence we have

$$U \cap X_p(\chi(G)) = U \cap X_p(G) = U \cap X_{qc}(G) = U \cap X_{qc}(\chi(G)) .$$

Since U is also an open neighborhood of χ in $(Möb)^k$, this means that $\chi(G)$ is quasiconformally stable and $\chi \in S_1$. That is, we have $U \cap X_p(G) \subset S_1$. This proves the theorem for S_1.

Next we consider the case $id \in S_2$. This means $\beta^*(B(\Omega(G), G)) = PH^1(G, \Pi)$. By Theorem 1, G is both quasiconformally stable and quasi-stable. Thus, by Lemma 2, there exists an open neighborhood U of id in $(Möb)^k$ such that $U \cap X_p(G)$ is a submanifold of dimension $\sigma(G)+3$ of U and such that $U \cap X_p(G) = U \cap X_{qc}(G)$. Let χ be any element of $U \cap X_p(G)$. As χ is a quasiconformal deformation of G, the open neighborhood $U \cap X_p(\chi(G))$ of χ in $X_p(\chi(G))$ $(= X_p(G))$ is identical with $U \cap X_p(G)$. Hence $U \cap X_p(\chi(G))$ is a submanifold of dimension $\sigma(G)+3 = \sigma(\chi(G))+3$ of U. On the other hand, in this case, the dimension of $U \cap X_p(\chi(G))$ must be equal to dim $PZ^1(\chi(G), \Pi) = \dim PH^1(\chi(G), \Pi)+3$ by Lemma 3 and (3.4). Hence we see $\sigma(\chi(G))+3 = \dim PH^1(\chi(G), \Pi)+3$. So we have $\beta^*(B(\Omega(\chi(G)), \chi(G))) = PH^1(\chi(G), \Pi)$ and $\chi \in S_2$, which show $U \cap X_p(G) \subset S_2$. This completes the proof of the theorem.

REMARK 4. Under the hypothesis of Theorem 2, let $\chi: G \rightarrow M\ddot{o}b$ be an isomorphism such that $\chi(G)$ is a Kleinian group and $X_{\Omega-p}(\chi(G))=X_{\Omega-p}(G)$ as subsets of $(M\ddot{o}b)^k$. All such χ form a subset $\hat{S}$ of $X_{\Omega-p}(G)$. Let $\hat{S}_1$ be the subset of $\hat{S}$ which consists of all $\chi \in \hat{S}$ such that $\chi(G)$ is quasiconformally stable in Ω-parabolic sense. Let $\hat{S}_2$ be the subset of $\hat{S}$ which consists of all $\chi \in \hat{S}$ with $\beta^*(B(\Omega(\chi(G)), \chi(G)))=PH^1_\Omega(\chi(G), \Pi)$. Then, using the above lemmas, Theorem 1 and (3.5), we can similarly prove that $\hat{S}_1$ and $\hat{S}_2$ are empty or open in $X_{\Omega-p}(G)$ and that $\hat{S}_2$ is a subset of $\hat{S}_1$.

4. Some results on stable groups. In this section, using some more lemmas, we shall prove some results on stable groups.

Let $K \geqq 1$ be a finite real number and w_n a K-quasiconformal automorphism of $\hat{C}$ with the complex dilatation μ_n $(n=1,2,\cdots)$. We say that the sequence $w_n, n=1,2,\cdots$, is a good approximation of a quasiconformal automorphism w of $\hat{C}$ with the complex dilatation ν if the two following conditions are satisfied:

(1) $\lim_{n\to\infty} w_n(z)=w(z)$ uniformly on $\hat{C}$ with the spherical metric

and

(2) $\lim_{n\to\infty} \mu_n(z)=\nu(z)$ for almost every point z in $\hat{C}$.

Now we draw the two following facts from Lehto-Virtanen [8] (see Chapter II and IV in [8]).

Lemma 5. *Let $K \geqq 1$ be a finite real number and W a family of K-quasiconformal automorphisms of $\hat{C}$. If there is a finite positive number $d>0$ such that, for every mapping $w \in W$ and for three distinct fixed points z_1, z_2, $z_3 \in \hat{C}$, the spherical distances between $w(z_i)$ and $w(z_j)$ $(i,j=1,2,3, i\neq j)$ are greater than d, then W forms a normal family with respect to the spherical metric. That is, every infinite sequence of elements of W contains a subsequence which converges uniformly on $\hat{C}$ with respect to the spherical metric. Furthermore, all its limit functions are also K-quasiconformal automorphisms of $\hat{C}$.*

Lemma 6. *Let $K \geqq 1$ be a finite real number and let $w_n, n=1,2,\cdots$, be a sequence of K-quasiconformal automorphisms of $\hat{C}$ which converges to a quasiconformal automorphism w of $\hat{C}$ with the complex dilatation ν uniformly on $\hat{C}$ with respect to the spherical metric. If the complex dilatations $\mu_n(z)$ of w_n tend to a limit $\mu(z)$ almost everywhere on $\hat{C}$, then the sequence $w_n, n=1,2,\cdots$, is a good approximation of w, that is, $\mu(z)=\nu(z)$ for almost every point z in $\hat{C}$.*

Now we prove the following.

Theorem 3. *Let G be a non-elementary finitely generated Kleinian group and $\gamma_1, \gamma_2, \cdots, \gamma_k$ a system of generators for G. Assume that G is quasi-stable. Then there exists an open neighborhood U of $id=(\gamma_1, \gamma_2, \cdots, \gamma_k)$ in $(M\ddot{o}b)^k$ with the following properties;*

(1) *if $\chi_n \in U \cap X_{qc}(G)$, then there exists a quasiconformal automorphism w_n of $\hat{C}$ with the complex dilatation $\mu_n \in M_{can}(G)$ such that $w_n \circ \gamma(z) = \chi_n(\gamma) \circ w_n(z)$ for $\gamma \in G$ and $z \in \hat{C}$, and*

(2) *if a sequence $\chi_n \in U \cap X_{qc}(G), n=1,2,\cdots$, converges to the identity isomorphism $id=(\gamma_1, \gamma_2, \cdots, \gamma_k)$, then the sequence $w_n, n=1,2,\cdots$, gives a good approximation of the identity transformation e of $\hat{C}$.*

Proof. By our assumption, G is quasi-stable. Thus, by Lemma 2, there exist an open neighborhood V of $(e, 0)$ in $M\ddot{o}b \times M_{can}(G)$ and an open neighborhood U of id in $(M\ddot{o}b)^k$ such that $\Phi_{G|M\ddot{o}b \times M_{can}(G)}$ maps V biholomorphically onto $U \cap X_{qc}(G)$. Hence every $\chi_n \in U \cap X_{qc}(G)$ is induced by the quasiconformal automorphism $w_n = \alpha_n \circ w^{\mu_n}$ of $\hat{C}$ satisfying $w_n \circ \gamma(z) = \chi_n(\gamma) \circ w_n(z)$ for $\gamma \in G$ and $z \in \hat{C}$, where $(\alpha_n, \mu_n) \in V$ is uniquely determined by χ_n.

It remains to prove the latter part of the theorem. Because of Lemma 1, we may assume that the points $0, 1, \infty$ are the attractive fixed points of some loxodromic elements of G. Now assume that a sequence $\chi_n \in U \cap X_{qc}(G), n=1,2,\cdots$, converges to id. Then, as remarked in the proof of Lemma 2, the sequence $\alpha_n \in M\ddot{o}b$, $n=1,2,\cdots$, converges to the identity transformation e and thus we have $\lim_{n\to\infty} \alpha_n(z) = z$ uniformly on $\hat{C}$ with the spherical metric. In this case, by Lemma 5, we see that the sequence $w_n = \alpha_n \circ w^{\mu_n}, n=1,2,\cdots$, forms a normal family with respect to the spherical metric and that, if $w_{n_k} = \alpha_{n_k} \circ w^{\mu_{n_k}}$, $k=1,2,\cdots$, is any convergent subsequence of the sequence, then the limit function w is a quasiconformal automorphism of $\hat{C}$ which keeps the points $0, 1, \infty$ fixed. On the other hand, since $\lim_{n\to\infty} \mu_n(z) = 0$ almost everywhere in $\hat{C}$, w is conformal by Lemma 6. Hence w is the conformal automorphism of $\hat{C}$ which keeps $0, 1, \infty$ fixed, that is, w is the identity transformation e of $\hat{C}$. Since we have shown that the sequence $w_n, n=1,2,\cdots$, forms a normal family with respect to the spherical metric and that any convergent subsequence $w_{n_k}, k=1,2,\cdots$, is a good approximation of the identity transformation e of $\hat{C}$, the sequence w_n, $n=1,2,\cdots$, itself is a good approximation of e.

Corollary. *Under the hypothesis of Theorem 3, if a sequence $\chi_n \in X_{qc}(G)$, $n=1,2,\cdots$, converges to the identity isomorphism $id=(\gamma_1, \gamma_2, \cdots, \gamma_k)$, then the spherical distance between the attractive fixed point of $\chi_n(\gamma)$ and that of γ converges to zero uniformly for all loxodromic elements $\gamma \in G$.*

Now we shall state one more lemma. The essential part of our proof of the lemma is due to Maskit [9] and, for the sake of completeness, we shall give the proof.

Lemma 7. *Let G be a Kleinian group and let F and w be quasiconformal automorphisms of $\hat{C}$ with $F\circ\gamma\circ F^{-1}=w\circ\gamma\circ w^{-1}\in Möb$ for $\gamma\in G$. If the complex dilatation μ_w of w satisfies $\mu_{w|\Lambda(G)}=0$, then so does that of F.*

Proof. If we set $\hat{G}=FGF^{-1}$, then the quasiconformal automorphism $F\circ w^{-1}$ of $\hat{C}$ satisfies $F\circ w^{-1}\circ\hat{\gamma}(z)=\hat{\gamma}\circ F\circ w^{-1}(z)$ for $z\in\hat{C}$ and $\hat{\gamma}\in\hat{G}$. It is well-known that $F\circ w^{-1}(z)=z$ holds for all $z\in\Lambda(\hat{G})$ (see Lemma 1 in Kra [6]).

Since $F\circ w^{-1}$ is quasiconformal, $F\circ w^{-1}$ is absolutely continuous on lines and has finite partial derivatives at almost every point (see [8]). For any fixed line S, almost every point of $\Lambda(\hat{G})\cap S$ is a point of density of $\Lambda(\hat{G})\cap S$. Hence, for almost every point $z_0\in\Lambda(\hat{G})$, we can find a sequence of points $z_n\in\Lambda(\hat{G})$ with $z_n\to z_0$ and with $Re(z_n-z_0)=0$ (or $Im(z_n-z_0)=0$). Since $F\circ w^{-1}$ is the identity on $\Lambda(\hat{G})$, we see $\partial(F\circ w^{-1})/\partial z=1$ and $\partial(F\circ w^{-1})/\partial\bar{z}=0$ almost everywhere on $\Lambda(\hat{G})$. In particular, we have

$$\mu_{F\circ w^{-1}|\Lambda(\hat{G})}=0\,. \tag{4.1}$$

On the other hand, if we set $z=w(\zeta)$, then

$$\mu_{F\circ w^{-1}}(z)=\frac{\mu_F(\zeta)-\mu_w(\zeta)}{1-\mu_F(\zeta)\overline{\mu_w(\zeta)}}\,e^{2i\arg w_\zeta(\zeta)}\,. \tag{4.2}$$

By our assumption, we have

$$\mu_{w|\Lambda(G)}=0\,. \tag{4.3}$$

Since $w(\Lambda(G))=\Lambda(\hat{G})$, we see $\mu_{F|\Lambda(G)}=0$ from (4.1), (4.2) and (4.3).

Gardiner and Kra have remarked in [4] that, for a non-elementary finitely generated Kleinian group G, if $\beta^*(B(\Omega,G))=PH^1_\Omega(G,\Pi)$, then there exist no other Beltrami coefficients for G supported on $\Lambda(G)$ than 0. On the other hand, by Lemma 4, if $\beta^*(B(\Omega,G))=PH^1_\Omega(G,\Pi)$, then G is quasiconformally stable in Ω-parabolic sense and hence G is, of course, quasiconformally stable. Noting these facts, we can prove the following slightly modified form of Gardiner and Kra's result.

Theorem 4. *Let G be a non-elementary finitely generated Kleinian group. If G is quasiconformally stable, then there exist no other Beltrami coefficients for G supported on the limit set $\Lambda(G)$ than 0.*

Proof. Assume the contrary. Then there exists a Beltrami coefficient

μ for G supported on $\Lambda(G)$ such that $\mu_{|\Lambda(G)}\neq 0$. We set $\mu_n(z)=\frac{1}{n}\mu(z)$. Since $\lim_{n\to\infty}\mu_n(z)=0$ for $z\in\hat{C}$, we see by Lemma 5 and Lemma 6 that the sequence w^{μ_n}, $n=1,2,\cdots$, is a good approximation of the identity transformation e of $\hat{C}$. Let χ_n: $G\to M\ddot{o}b$ be the homomorphism defined by $\chi_n(\gamma)(z)=w^{\mu_n}\circ\gamma\circ(w^{\mu_n})^{-1}(z)$ for $\gamma\in G$ and $z\in\hat{C}$. Then the sequence $\chi_n, n=1,2,\cdots$, converges to the identity isomorphism $id=(\gamma_1,\gamma_2,\cdots,\gamma_k)$, where $\gamma_1,\gamma_2,\cdots,\gamma_k$ is some arbitrarily chosen and fixed system of generators for G.

Since G is quasiconformally stable, there exists an open neighborhood U of id in $(M\ddot{o}b)^k$ such that

$$U\cap X_p(G)\subset X_{qc}(G)\,. \tag{4.4}$$

If we choose a sufficiently large n_0, then we see $\chi_{n_0}\in U$. Also clearly $\chi_{n_0}\in X_p(G)$ and

$$\chi_{n_0}\in U\cap X_p(G)\,. \tag{4.5}$$

Owing to (4.4) and (4.5), we can see that there exists a quasiconformal automorphism F of $\hat{C}$ with $\mu_{F|\Lambda(G)}=0$ and that $w^{\mu_{n_0}}\circ\gamma\circ(w^{\mu_{n_0}})^{-1}=F\circ\gamma\circ F^{-1}$ for $\gamma\in G$. In this case, for the complex dilatation $\mu_{n_0}=\frac{1}{n_0}\mu$ of $w^{\mu_{n_0}}$, we have $\mu_{n_0|\Lambda(G)}=0$ by Lemma 7. This contradiction proves the theorem.

Recently the author learned from Prof. Ahlfors and Prof. Kra that the following result holds as a translation of Dennis Sullivan's theorem: Let G be any finitely generated Kleinian group. Then there exist no other Beltrami coefficients for G supported on $\Lambda(G)$ than 0.

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