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TORSION IN BROWN-PETERSON HOMOLOGY AND HUREWICZ HOMOMORPHISMS

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BP is the Brown-Peterson spectrum (for some prime p) and $BP_*X=\pi_*(BP\wedge X)$ is the Brown-Peterson homology of the CW spectrum (or complex) X. BP_*X is a left module over the coefficient ring $BP_*\cong Z_{(p)}[v_1, v_2, \cdots]$ and a left comodule over the coalgebra BP_*BP. A now classical result is that the stable Hurewicz homomorphism $\pi_*^S X\to H_*(X; Z)$ is an isomorphism modulo torsion. In our context, we restate this as: the Hurewicz homomorphism $h_0(X)\colon \pi_*(BP\wedge X)\to H_*(BP\wedge X; Q)$ has as its kernel the p-torsion subgroup of BP_*X. This is a prototype of our results.

Instead of restricting our attention to BP_*X, it is convenient to study abstract BP_*BP-comodules (M, ψ), $\psi\colon M\to BP_*BP\otimes_{BP_*}M$. *A priori*, M is a left BP_*-module. As such, it has a richer potential for torsion than mere p-torsion. For any polynomial generator v_n of BP_* (by convention $v_0=p$), we say that an element $y\in M$ is v_n-torsion if $v_n^s y=0$ for some exponent s. If all elements of M are v_n-torsion ones, we say that M is a v_n-torsion module. If no non-zero element of M is v_n-torsion, we say that M is v_n-torsion free. Being a BP_*BP-comodule severely constrains the BP_*-module structure of M.

Theorem 0.1. *Let M be a BP_*BP-comodule. If $y\in M$ is a v_n-torsion element, then it is a v_{n-1}-torsion element. Consequently, if M is a v_n-torsion module, then it is a v_{n-1}-torsion module. Or: if M is v_n-torsion free, it is v_{n+1}-torsion free* (Lemma 2.3 and Proposition 2.5).

The primitive elements of a BP_*BP-comodule M are those elements a for which $\psi(a)=1\otimes a$ under M's coproduct $\psi\colon M\to BP_*BP\otimes_{BP_*}M$. We find that some qualitative properties of BP_*BP-comodules are determined by these primitives.

Theorem 0.2 *Let M be an associative BP_*BP-comodule. If all the primitives of M are v_n-torsion, then M itself is a v_n-torsion module. Or: if none of the*

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non-zero primitives of M is v_n-torsion, then M is v_n-torsion free (Proposition 2.7).

We may localize a BP_*BP-comodule M with respect to v_n to form $v_n^{-1}M$. Generally, the resulting BP_*-module is not a BP_*BP-comodule; we characterize when it is.

Theorem 0.3. *Let M be an associative BP_*BP-comodule. M is a v_{n-1}-torsion module if and only if $v_n^{-1}M$ is an associative BP_*BP-comodule.* (Proposition 2.9) (The "only if" part is due to Miller and Ravenel [11].)

There is no dearth of homology theories associated to BP, but some of the most interesting are the periodic homology theories $E(n)_*(\ )$. The coefficients of $E(n)_*(\ )$ are $E(n)_*\cong Z_{(p)}[v_1, \cdots, v_{n-1}, v_n, v_n^{-1}]$; the representing spectrum is $E(n)$. $E(0)_*X$ is the familiar rational homology of X. $E(1)_*X$ is a summand of localized (at p) complex K-homology of X. There is a Boardman map $BP\rightarrow E(n)\wedge BP$ which induces a Hurewicz homomorphism $h_n(X)\colon \pi_*(BP\wedge X)\rightarrow E(n)_*(BP\wedge X)$. When $n=1$, this is properly called the Hattori-Stong homomorphism. We prove:

Theorem 0.4. *Let X be a CW spectrum. The kernel of the Hurewicz homomorphism $h_n(X)\colon \pi_*(BP\wedge X)\rightarrow E(n)_*(BP\wedge X)$ is the v_n-torsion subgroup of BP_*X.* (Theorem 4.10)

We can localize BP_*X to form $v_n^{-1}BP_*X$. We prove:

Theorem 0.5. *Let X be a CW spectrum. $v_n^{-1}BP_*X=0$ if and only if $E(n)_*X=0$. Hence $v_n^{-1}BP_*(\ )$ and $E(n)_*(\ )$ have the same acyclic spaces.* (Corollary 4.11)

During a provocative talk at the Northwestern conference of March 1977, Douglas Ravenel shared his insight that Theorem 0.5 should hold. Our attempts to substantiate his intuition led to this paper. We thank Ravenel for making the manuscript [12] of his Northwestern talk available to us, for his stimulating correspondence, and for his kind hospitality.

An obvious generalization presents itself. Let $J=\{q_0, q_1, \cdots, q_{n-1}\}$ be an invariant regular sequence of elements of BP_*. There is a left BP-module spectrum BPJ whose homotopy is $BPJ_*\cong BP_*/(q_0, \cdots, q_{n-1})$. When J is empty, BPJ is just BP. As we do prove our results for BPJ_*BPJ-comodules, we must list properties of such comodules (§1), prove some simple change-of-ring (BPJ_* to BP_*) lemmas in §3, and sketch some proofs of the properties of BPJ (§5). A reader who is interested only in BP_*BP-comodules may neglect the "J" in the BPJ notation and read only the even-numbered sections: §2, "v_n-Torsion Properties," and §4, "Hurewicz Homomorphisms."

1. BPJ_*BPJ-comodules

Let $J=\{q_0, \cdots, q_{n-1}\}$ be an invariant regular sequence of elements of $BP_*=\pi_*BP$. J(and n) will remain fixed throughout this section. There is an associative left BP-module spectrum BPJ which has homotopy $\pi_*BPJ=BPJ_*\cong BP_*/(q_0, \cdots, q_{n-1})$. A map of spectra j: $BP\to BPJ$ induces an epimorphism in homotopy. Let φ_0: $BP\wedge BP\to BP$ give BP its ring spectrum structure and let $\varphi=\varphi_n$: $BP\wedge BPJ\to BPJ$ give BPJ its BP-module structure. Then $\varphi(1\wedge j)=j\circ\varphi_0$ and $\varphi(\varphi_0\wedge 1)=\varphi(1\wedge\varphi)$.

When J is empty, BPJ is just BP. For $J=\{p, v_1, \cdots, v_{n-1}\}$, BPJ is known as $P(n)$ [6; 16; 17]. The following properties have become classical for BP_*BP-comodules [7; 8; 9]. Würgler has established these properties for $P(n)_*P(n)$-comodules (p odd) [16]. We defer some of our exposition and our proof sketches until §5.

Let ι: $S^0\to BP$ be the unit map for the Brown-Peterson spectrum. There are pairings μ: $BPJ\wedge BPJ\to BPJ$ which make BPJ into a quasi-associative ring spectrum with unit $\iota_n=j\circ\iota$: $S^0\to BPJ$. Here $\mu(j\wedge 1)=\varphi$ and $\mu(j\wedge j)=j\circ\varphi_0$. (See Proposition 5.5.) These pairings are not generally unique; the (co) multiplicative structures which follow can depend on the particular (fixed) choice of μ.

Let c: $BPJ_*BPJ\to BPJ_*BPJ$ be the conjugation. BPJ_*BPJ is a free left BPJ_*-module with basis given by symbols $z^{E,A}$ of dimension $\sum_i e_i(2p^i-2)+\sum_j a_j(\dim(q_j)+1)$. Here $E=(e_1, e_2, \cdots)$ is a finite sequence of non-negative integers and $A=(a_0, \cdots, a_{n-1})$ is an n-tuple of zeros and ones. BPJ_*BPJ is an associative left $BP_*BP\cong BP_*[t_1, t_2, \cdots]$-module with structure given by the formula:

$$t^E z^{F,A} = z^{E+F,A} \qquad \text{for} \quad t^E = t_1^{e_1}t_2^{e_2}\cdots\in BP_*BP\,. \tag{1.1}$$

In particular, $(j\wedge j)_*(t^E)=z^{E,0}$. The $c(z^{E,A})$ give a basis for BPJ_*BPJ as a free right BPJ_*-module. Because of this right freeness, there is a natural isomorphism $BPJ_*(BPJ\wedge X)\cong BPJ_*BPJ\otimes_{BPJ_*}BPJ_*X$ for any CW spectrum X. The map $1\wedge\iota_n\wedge 1$: $BPJ\wedge S^0\wedge X\to BPJ\wedge BPJ\wedge X$ induces a coproduct:

$$\psi_X\colon BPJ_*X\to BPJ_*(BPJ\wedge X)\cong BPJ_*BPJ\otimes_{BPJ_*}BPJ_*X\,.$$

We define natural homomorphisms $s_{E,A}$: $BPJ_*X\to BPJ_*X$ by the following recipe

$$\psi_X(x)=\sum_{E,A}c(z^{E,A})\otimes s_{E,A}(x)\,. \tag{1.2}$$

We call these $s_{E,A}$ *elementary BPJ operations*. When J is empty and BPJ is BP, the $s_{E,0}$ coincide with BP operations r_E [2]. The elementary BPJ operations satisfy the following properties.

(1.3) Under the natural map $j_*: BP_*X \to BPJ_*X$, $s_{E,0}j_*(x) = j_*r_E(x)$.

(1.4) The elementary BPJ operations generate all the BPJ operations in that any BPJ operation θ can be written uniquely as a (possibly infinite) sum

$$\theta = \sum_{E,A} q_{E,A}s_{E,A} \qquad q_{E,A} \in BPJ_* .$$

(See 5.12).

(1.5) The dimension of $s_{E,A}$ is $d = \sum_i e_i(2p^i - 2) + \sum_j a_j(\dim(q_j)+1)$ where $E = (e_1, e_2, \cdots)$ and $A = (a_0, \cdots, a_{n-1})$. That is: if $x \in BP_sX$, then $s_{E,A}(x) \in BP_{s-d}X$. (This follows from (1.2).)

(1.6) For any element $x \in BPJ_*X$, $s_{E,A}(x)$ is zero except for finitely many indices E and A. (The proof is trivial.)

(1.7) There is a Cartan formula. If $y \in BP_*$ and $x \in BPJ_*X$, then

$$s_{E,A}(yx) = \sum_{F+G=E} r_F(y)s_{G,A}(x) .$$

(This follows from (1.1).)

(1.8) There are coefficients $q_A \in BPJ_*$ such that

$$s_{0,0}(x) = x + \sum_{A \neq 0} q_A s_{0,A}(x)$$

for any $x \in BPJ_*X$ and for any X. (See Remark 5.13.)

(1.9) For the elementary BPJ operations $s_{E,A}$ and $s_{F,B}$, there are coefficients $q_{G,C} = q_{G,C}(E, A; F, B) \in BPJ_*$ such that

$$s_{E,A}(s_{F,B}(x)) = \sum_{G,C} q_{G,C}s_{G,C}(x)$$

for any $x \in BPJ_*X$ and for any X. Furthermore, the dimension of $s_{G,C}$ is not less than the sum of the dimensions of $s_{E,A}$ and $s_{F,B}$. (See Remark 5.14.)

Let M be a left BPJ_*-module. M is defined to be a BPJ_*BPJ-comodule if the elementary BPJ operations act on M satisfying (1.5) through (1.8). The BPJ_*BPJ coaction of M is given by $\psi_M: M \to BPJ_*BPJ \otimes_{BPJ_*} M$ with

$$\psi_M(x) = \sum_{E,A} c(z^{E,A}) \otimes s_{E,A}(x) .$$

If (1.9) is also satisfied, we call (M, ψ_M) an *associative* BPJ_*BPJ-comodule. The following remark follows from (1.2).

REMARK 1.10. Let M be a BPJ_*BPJ-comodule and let $x \in M$. The following are equivalent statements.

(i) $\psi_M(x)=1\otimes x$
(ii) $s_{E,A}(x)=0$ if $(E, A)\neq(0,0)$ and $s_{0,0}(x)=x$.
If x satisfies these equivalent statements, we call x *primitive*. Let PM be the subgroup of primitive elements of M.

Define the *primitive degree* $d(x)$ of an element x of a BPJ_*BPJ-comodule M as follows. If there is an elementary operation $s_{E,A}$ of dimension m such that $s_{E,A}(x)\neq 0$, then $d(x)\geqslant m$. Define $d(0)=0$. By (1.6), $d(x)\geqslant 0$ is always finite. We record two observations.

(1.11) If $x\in M$, $d(x)=0$ if and only if x is primitive. (See Remark 1.10.)

(1.12) Let M be an associative BPJ_*BPJ-comodule and let $s_{E,A}$ be an elementary BPJ operation of dimension m. For $x\in M$, $d(s_{E,A}(x))\leqslant$ maximum $\{d(x)-m, 0\}$. (See (1.9).)

Lemma 1.13. *Let M be an associative or a connective BPJ_*BPJ-comodule. Then M coincides with the union of all of its finitely-generated subcomodules.*

Proof. This follows routinely using (1.6) and (1.9) or (1.5).

Lemma 1.14. *Let M be an associative BPJ_*BPJ-comodule. There is an epimorphism of associative BPJ_*BPJ-comodules $f\colon F\to M$ with F BPJ_*-free. F may be chosen to be finitely-generated in the case that M is finitely-generated.*

Proof. Follow the proof of Proposition 2.4 of [9].

Lemma 1.15. *Every associative BPJ_*BPJ-comodule M is a direct limit of finitely-presented associative comodules.*

Proof. See the proof of Lemma 2.11 of [11] or see [17].

Recall that $I_0=(p)$, $I_m=(p, v_1, \cdots, v_{m-1})$, and $I_\infty=(p, v_1, v_2, \cdots)$ are the non-trivial prime ideals of BP_* invariant under the BP_*BP-coaction [7;5]. By Landweber [10], the ideal-theoretic radical of $(q_0, \cdots, q_{n-1})$ is I_n.

Theorem 1.16 (Filtration Theorem). *Let $J=\{q_0, \cdots, q_{n-1}\}$ be an invariant regular sequence in BP_* of length n. Let M be a finitely-presented, associative BPJ_*BPJ-comodule. Then M has a finite filtration*

$$M=M_s\supset M_{s-1}\supset\cdots\supset M_1\supset M_0=\{0\}$$

by finitely-presented, associative BPJ_*BPJ-subcomodules. As a BPJ_*BPJ-comodule, each quotient M_i/M_{i-1}, $1\leqslant i\leqslant s$, is isomorphic to some suspension of some BP_*/I_k, $n\leqslant k$.

Proof. Follow the patterns of the proofs of Theorem 3.3 of [8] and

Theorem 3.4 of [17].

2. v_n-torsion properties

Again, J will be a fixed invariant regular sequence and BPJ will be the resulting spectrum. BPJ_*BPJ-comodules are BP_*-modules through the epimorphism $BP_* \to BPJ_*$. This section studies certain BP_*-module properties of BPJ_*BPJ-comodules which are independent of the particular sequence J. (Here, we use the letter "n" as a variable and not as the length of the fixed sequence J.)

Our study begins with a lemma which descends directly from the "Ballentine Lemma" of Smith (and Stong) [14]. For the exponent sequence $E=(e_1, e_2, \cdots)$, let $|E|=\sum_i e_i(2p^i-2)$. Let $\Delta_k=(0, \cdots, 0, 1, 0, \cdots)$ with the single "1" in the k-th position. Exponent sequences are added (or multiplied by positive integers) term-wise.

Lemma 2.1. *Let E be an exponent sequence with $|E| \geqslant 2kp^s(p^n-p^m)$, $n \geqslant m$, $s \geqslant 0$, and $k \geqslant 1$. Then*

$$r_E(v_n{}^{kps}) = \begin{cases} v_m{}^{kps} \text{ modulo } I_m{}^{s+1} & \text{if } E = kp^{s+m}\Delta_{n-m} \\ 0 \quad \text{modulo } I_m{}^{s+1} & \text{otherwise.} \end{cases}$$

Proof. The $s=0$ case is Corollary 1.8 of [5]. The general case follows by induction on s using the Cartan formula and the fact that $p \in I_m$.

Lemma 2.2. *Let M be a BPJ_*BPJ-comodule and let $s_{E,A}$ be any elementary BPJ operation. If an element $x \in M$ is v_m-torsion for all m satisfying $0 \leqslant m \leqslant n$, then $s_{E,A}(x)$ is also v_m-torsion for such m, $0 \leqslant m \leqslant n$.*

Proof. Assume inductively that $s_{F,B}(x)$ is v_k-torsion for every elementary BPJ operation $s_{F,B}$ and for all k satisfying $0 \leqslant k < m$. (The initial $m=0$ case is the same as the inductive step.) Recalling (1.6), there is a non-negative integer $s=s(x, m)$ such that $v_m{}^{p^s}x=0$ and $I_m{}^{s+1}s_{F,B}(x)=0$ for all elementary BPJ operations $s_{F,B}$. By (1.7) and Lemma 2.1,

$$0 = s_{E,A}(v_m{}^{p^s}x) = v_m{}^{p^s}s_{E,A}(x)$$

and so $s_{E,A}(x)$ is v_m-torsion.

Lemma 2.3. *Let M be a BPJ_*BPJ-comodule. If an element $x \in M$ is v_n-torsion, then it is v_m-torsion for each m satisfying $0 \leqslant m \leqslant n$.*

Proof. Our proof is by double induction. The first induction (on m) assumes that if x is v_n-torsion, then x is v_k-torsion for $k<m$. For such an x and for any elementary BPJ operation $s_{E,A}$, $s_{E,A}(x)$ is v_k-torsion for all $k<m$ by Lemma

2.2. We may choose an $s\geqslant 0$ such that $v_n^{p^s}x=0$ and $I_m^{s+1}s_{E,A}(x)=0$ for all $s_{E,A}$. Suppose $s_{H,A}$ is an elementary BPJ-operation of dimension $d(x)$. (See (1.11) and (1.12).) Let $G=p^{m+s}\Delta_{n-m}$. By (1.7) and Lemma 2.1,

$$0=s_{G+H,A}(v_n^{p^s}x)=\sum_{E+F=G+H}r_F(v_n^{p^s})s_{E,A}(x)=r_G(v_n^{p^s})s_{H,A}(x)$$
$$=v_m^{p^s}s_{H,A}(x)=s_{H,A}(v_m^{p^s}x)\,.$$

If $d(x)=0$, this computation shows that x is v_m-torsion. If $d(x)>0$, it shows that $d(v_m^{p^s}x)<d(x)$. By a second induction on the primitive degree $d(\ )$, $v_m^{p^s}x$ is assumed to be v_m-torsion. Hence x is v_m-torsion as desired.

Corollary 2.4. *Let M be a BPJ_*BPJ-comodule. If $x\in M$ is v_n-torsion, then $s_{E,A}(x)$ is v_m-torsion for all m satisfying $0\leqslant m\leqslant n$ and for all elementary BPJ operations $s_{E,A}$.*

Proof. Lemmas 2.2 and 2.3.

Recall that a BP_*-module M (e.g. a BPJ_*BPJ-comodule) is v_n-torsion if every element $x\in M$ is v_n-torsion. M is v_n-torsion free if no non-zero element is v_n-torsion. The following proposition follows immediately from Lemma 2.3.

Proposition 2.5. *Let M be a BPJ_*BPJ-comodule. If M is v_n-torsion, then it is v_{n-1}-torsion. At the other extreme: if M is v_n-torsion free, then it is v_{n+1}-torsion free.*

Let Y be an associative BP-module spectrum. We can form a new spectrum $v_n^{-1}Y$ which is defined to be the mapping telescope $\lim S^{-2t(p^n-1)}Y$ of the map

$$S^{2p^n-2}Y\xrightarrow{v_n\wedge 1}BP\wedge Y\to Y\,.$$

Note that *$v_n^{-1}Y$ is a BP-module spectrum which is possibly non-associative. We have a canonical isomorphism $v_n^{-1}(Y_*(X))\to(v_n^{-1}Y)_*X$.*

Corollary 2.6. *Let X be a CW spectrum. If $(v_n^{-1}BPJ)_*X=0$, then $(v_{n-1}^{-1}BPJ)_*X=0$.*

Proposition 2.7. *Let M be a BPJ_*BPJ-comodule which is either associative or connective.*
(i) *If all the primitive elements of M are v_n-torsion, then M is a v_n-torsion module.*
(ii) *If none of the non-zero primitive elements of M is v_n-torsion, then M is a v_n-torsion free module.*

Proof. To prove (i), assume M is an associative comodule with v_n-torsion primitives. Assume inductively that M is a v_k-torsion module for $k<m\leqslant n$. If $y\in M$ with $d(y)=0$ (see (1.11)), y is v_k-torsion for all $k\leqslant n$ by our hypothesis and by Lemma 2.3. Let $x\in M$ with $d(x)>0$. Let $s_{E,A}$ be any positive dimensional elementary BPJ operation. Since M is associative, $d(s_{E,A}(x))<d(x)$ by (1.12). By a subsidiary induction on $d(y)$, we may assume that such $s_{E,A}(x)$ are v_m-torsion. Hence there is an $s\geqslant 0$ such that $I_m^{s+1}x=0$ and $I_{m+1}^{s+1}s_{E,A}(x)=0$. Note that (1.8) implies that $s_{0,0}(x)=x+z$ with $I_{m+1}^{s+1}z=0$. For any positive dimensional $s_{E,A}$,

$$s_{E,A}(v_m^{p^s}x)=\sum_{F+G=E} r_F(v_m^{p^s})s_{G,A}(x)=0\,.$$

So $v_m^{p^s}x$ is primitive and hence v_m-torsion. Thus x itself is v_m-torsion. This completes both the auxiliary and the original inductions.

We turn to (ii). Let M be an associative comodule with no non-zero v_n-torsion primitives. We assume inductively that all non-zero elements $y\in M$ with $d(y)<l$ are not v_n-torsion. If non-primitive $x\in M$ has $d(x)=l$, there is an elementary BPJ operation $s_{E,A}$ with $s_{E,A}(x)\neq 0$ and $d(s_{E,A}(x))<d(x)$ (1.12). So $s_{E,A}(x)$ is not v_n-torsion. By Corollary 2.4, x fails to be v_n-torsion also. Thus M is v_n-torsion free.

Finally, assume M is connective. With a few minor modifications, the above proofs of (i) and (ii) work if we replace the primitive degree $d(x)$ of the element x by x's dimension $|x|$.

A BP_*-module (e.g. a BPJ_*BPJ-comodule) M is said to be *v_n-divisible* if multiplication by v_n on M is epic.

Proposition 2.8. *If an associative BPJ_*BPJ-comodule M is v_n-divisible, then it is v_{n-1}-torsion.* (Cf. [11, Proposition 3.5].)

Proof. Assume inductively that M is v_k-torsion for $k<m<n$. Let $0\neq x\in M$ be a primitive element. By Proposition 2.7, it will suffice to show that x is v_m-torsion. There is an integer $t\geqslant 0$ such that $I_m^{t+1}x=0$. Note that this implies that $I_m^{t+1}s_{0,A}(x)=0$ (1.7). By the divisibility of M, there is an element $y\in M$ with $v_n^{p^t}y=x$. In preparation for a second induction, we do a curious computation. For any integer $u\geqslant 0$, our (primary) inductive hypothesis gives us an integer $s\geqslant t$ such $I_m^{s+1}s_{E,A}(v_m^{up^t}y)=0$ for all elementary BPJ operations $s_{E,A}$. Suppose $d(v_m^{up^t}y)=l$ and let $s_{H,A}$ be any elementary BPJ operation of that maximal dimension l. Let $G=p^{m+s}\Delta_{n-m}$. Using (1.7) and Lemma 2.1 repeatedly, we compute:

$$\begin{aligned}0&=r_{G+H}(v_n^{p^s-p^t})v_m^{up^t}s_{0,A}(x)=r_{G+H}(v_n^{p^s-p^t})s_{0,A}(v_m^{up^t}x)=s_{G+H,A}(v_n^{p^s-p^t}v_m^{up^t}x)\\&=s_{G+H,A}(v_n^{p^s-p^t}v_m^{up^t}v_n^{p^t}y)=s_{G+H,A}(v_n^{p^s}v_m^{up^t}y)=r_G(v_n^{p^s})s_{H,A}(v_m^{up^t}y)\\&=v_m^{p^s}s_{H,A}(v_m^{up^t}y)=s_{H,A}(v_m^{p^t(p^{s-t}+u)}y)\,.\end{aligned}$$

If $d(v_m^{up^t}y)=0$, this shows that $v_m^{up^t}y$ –and hence y and x– are v_m-torsion. If $d(v_m^{up^t}y)>0$, the computation shows that $d(v_m^{p^t(p^{s-t}+u)}y)<d(v_m^{up^t}y)$. This indicates a proof that x is v_m-torsion by induction on the primitive degrees of

the $v_m^{up^t}y$.

The "only if" part of the following proposition is due to Miller and Ravenel [11, Lemma 3.2].

Proposition 2.9. *Let M be an associative BPJ_*BPJ-comodule. Then M is v_{n-1}-torsion if and only if the localization $v_n^{-1}M$ is an associative BPJ_*BPJ-comodule.*

Proof. By Lemma 1.13, we may assume M is finitely-generated. Assuming M is v_{n-1}-torsion, there is an $s\geqslant 0$ such that $I_n^{s+1}M=0$ (Proposition 2.5). By Lemma 2.1, multiplication by $v_n^{p^s}$ on M is a comodule map. Hence the localization $v_n^{-1}M$, considered as the direct limit of the system

$$M \xrightarrow{v_n^{p^s}} M \xrightarrow{v_n^{p^s}} M\cdots,$$

is an associative BPJ_*BPJ-comodule. Furthermore, $M\to v_n^{-1}M$ is a comodule map.

Now assume that $v_n^{-1}M$ is an associative comodule. As a v_n-divisible associative comodule, $v_n^{-1}M$ is v_{n-1}-torsion by Proposition 2.8. Thus $v_k^{-1}v_n^{-1}M=0$ for each k satisfying $0\leqslant k<n$ by Proposition 2.5. Assume inductively that M is v_{k-1}-torsion. By the "only if" part of this proposition, $v_k^{-1}M$ is an associative comodule. Since $v_n^{-1}v_k^{-1}M=v_k^{-1}v_n^{-1}M=0$, the associative comodule $v_k^{-1}M$ is v_n-torsion, By Proposition 2.5, $v_k^{-1}M$ is v_k-torsion and thus is zero. So M is v_k-torsion.

3. More BP_*-module properties of BPJ_*BPJ-comodules

This section develops some algebraic preliminaries to Section 4. All of the results here are well-known or trivial when $BPJ=BP$. Our point of departure is the BPJ_*BPJ version of Landweber's Filtration Theorem (1.16). A unifying technique is the following.

Lemma 3.1. *Let $j\colon\Lambda\to\Gamma$ be a homomorphism of commutative rings with unit. Let A be a right Λ-module and let B and C be two-sided Γ-modules such that there is an isomorphism $B\otimes_\Gamma C\cong C\otimes_\Gamma B$ of left Γ-modules. Further assume that B is Γ-flat. If $\mathrm{Tor}_1^\Lambda(A, C)=0$, then $\mathrm{Tor}_1^\Gamma(A\otimes_\Lambda B, C)=0$.*

Proof. If either B or C is Γ-flat, we have a Künneth exact sequence

$$\mathrm{Tor}_2^\Gamma(A\otimes_\Lambda B, C)\to\mathrm{Tor}_1^\Lambda(A, B)\otimes_\Gamma C\to\mathrm{Tor}_1^\Lambda(A, B\otimes_\Gamma C)\to\mathrm{Tor}_1^\Gamma(A\otimes_\Lambda B, C)\to 0\,.$$

When B is Γ-flat (and the roles of B and C are interchanged), this gives an isomorphism $\mathrm{Tor}_1^\Lambda(A, C)\otimes_\Gamma B\xrightarrow{\cong}\mathrm{Tor}_1^\Lambda(A, C\otimes_\Gamma B)$. The lemma now follows immediately from the isomorphism $B\otimes_\Gamma C\cong C\otimes_\Gamma B$.

Throughout this section, let $J=\{q_0, \cdots, q_{n-1}\}$ be an invariant regular sequence of length n. Let $\Lambda=BP_*$ and $\Gamma=BPJ_*$.

For any commutative ring R and any R-module M, we have two dimensions of concern. The projective dimension, $\text{h.}\dim_R M$, is the greatest integer k such that $\text{Ext}_R^k(M, N)\neq 0$ for some R module N. The weak dimension, $\text{w.}\dim_R M$, is the greatest integer k such that $\text{Tor}_R^k(M, N)\neq 0$ for some R module N. Of course, $\text{w.}\dim_R M\leqslant \text{h.}\dim_R M$.

Lemma 3.2. *The projective dimension of* Γ *as a* Λ*-module is* n.

Proof. Let $J_m=\{q_0, \cdots, q_{m-1}\}\subseteq J$, $m\leqslant n$. For $m<n$, there are short exact sequences of Λ-modules

$$0 \to BPJ_{m*} \xrightarrow{q_m} BPJ_{m*} \to BPJ_{m+1*} \to 0$$

showing inductively that $\text{h.}\dim_\Lambda BPJ_{m*}\leqslant m$. The ideal $(q_0, \cdots, q_{n-1})$ has radical $(v_0, \cdots, v_{n-1})$ [10, Proposition 2.5]; so $\Gamma=BPJ_*$ is a v_{n-1}-torsion module. By the "ideal annihilator estimate" [6, Proposition 4.6], $\text{h.}\dim_\Lambda \Gamma\geqslant n$.

Corollary 3.3. $\text{Tor}_1^\Lambda(Z_{(p)}[v_1, \cdots, v_m], \Gamma)=0$ *for all* $m>n$.

Proof. Apply Landweber's Theorem 4.2 of [9] to the connective, associative BP_*BP-comodule $BPJ_*=\Gamma$.

Lemma 3.4. *For any* m *satisfying* $n\leqslant m\leqslant n+k+1$,

$$\text{Tor}_1^\Gamma (Z_{(p)}[v_1, \cdots, v_{n+k}]\otimes_\Lambda BPJ_*BPJ, BP_*/I_m) = 0\,.$$

Proof. Recall that BPJ_*BPJ is Γ-free. $\text{Tor}_1^\Lambda(Z_{(p)}[v_1, \cdots, v_{n+k}], BP_*/I_m)=0$ for $m\leqslant n+k+1$. For $n\leqslant m$, BP_*/I_m is a Γ-module. Apply Lemma 3.1.

Recall that $E(m)_*=Z_{(p)}[v_1, \cdots, v_{m-1}, v_m, v_m^{-1}]$.

Lemma 3.5. *Let* M *be an associative* BPJ_*BPJ*-comodule. Let* B *be* (i) Γ, (ii) BPJ_*BPJ, *or* (iii) $BPJ_*(v_m^{-1}BPJ)$. *Then* $\text{Tor}_1^\Gamma(E(m)_*\otimes_\Lambda B, M)=0$.

Proof. Both Γ and BPJ_*BPJ are Γ-free. As a direct limit of copies of BPJ_*BPJ, $BPJ_*(v_m^{-1}BPJ)$ is Γ-flat. By Landweber's Exact Functor Theorem [9], $\text{Tor}_1^\Lambda(E(m)_*, BP_*/I_k)=0$, $k\geqslant -1$. If $k\geqslant n$, BP_*/I_k is a Γ-module and so Lemma 3.1 implies that $\text{Tor}_1^\Gamma(E(m)_*\otimes_\Lambda B, BP_*/I_k)=0$, $k\geqslant n$. If M is finitely presented, M has a finite filtration whose subquotients are isomorphic to suspended copies of BP_*/I_k, $k\geqslant n$ (1.14). By an induction over M's filtration, $\text{Tor}_1^\Gamma(E(m)_*\otimes_\Lambda B, M)=0$ when M is finitely presented. By (1.13), this suffices to prove the lemma.

Lemma 3.6. *Let* M *be an associative* BPJ_*BPJ*-comodule. If* $\text{w.}\dim_\Gamma$

$M \leqslant m-n+1$, then:
(i) $\mathrm{Tor}_1^{\Gamma}(Z_{(p)}[v_1, \cdots, v_m]\otimes_{\Lambda}\Gamma, M)=0$;
(ii) *the sequence*

$$0 \to Z_{(p)}[v_1, \cdots, v_{m+1}]\otimes_{\Lambda}M \xrightarrow{v_{m+1}\otimes M} Z_{(p)}[v_1, \cdots, v_{m+1}]\otimes_{\Lambda}M \to Z_{(p)}[v_1, \cdots, v_m]\otimes_{\Lambda}M \to 0$$

is exact.

Proof. By Corollary 3.3, the endomorphism $v_{m+1}\otimes\Gamma$ of $Z_{(p)}[v_1, \cdots, v_{m+1}]\otimes_{\Lambda}\Gamma$ is injective; part (ii) follows from the resulting short exact sequence and from (i). Let $A=Z_{(p)}[v_1, \cdots, v_m]$ and note that $v_m^{-1}A=E(m)_*$. So $v_m^{-1}\mathrm{Tor}_1^{\Gamma}(A\otimes_{\Lambda}\Gamma, M)\cong\mathrm{Tor}_1^{\Gamma}(E(m)_*\otimes_{\Lambda}\Gamma, M)=0$ by Lemma 3.5 (i). So $\mathrm{Tor}_1^{\Gamma}(A\otimes_{\Lambda}\Gamma, M)$ is a v_m-torsion module. Part (i) is obvious when $\mathrm{w.\,dim}_{\Gamma}M=0$; we may assume $\mathrm{w.\,dim}_{\Gamma}=k>0$. Using (1.13) we construct an exact sequence of BPJ_*BPJ-comodules

$$0 \to K \to F \to M \to 0$$

where F is Γ-free. Consequently, $\mathrm{w.\,dim}_{\Gamma}K \leqslant k-1$ and we assume inductively (ii) that $A\otimes_{\Lambda}K$ is v_m-torsion free. (*N.B.* $k-1\leqslant m-1-n+1$.) So the v_m-torsion module $\mathrm{Tor}_1^{\Gamma}(A\otimes_{\Lambda}\Gamma, M)$ injects into the v_m-torsion free module $A\otimes_{\Lambda}\Gamma\otimes_{\Gamma}K\cong A\otimes_{\Lambda}K$ thus establishing (i).

Proposition 3.7. *Let J be a finite invariant regular sequence of length n. Let M be a connective associative BPJ_*BPJ-comodule. If $\mathrm{w.\,dim}_{BPJ_*}M\leqslant k-n$, then M is v_k-torsion free.*

Proof. It suffices to prove by an induction on $l\geqslant k$ that v_k acts injectively on $Z_{(p)}[v_1, \cdots, v_l]\otimes_{\Lambda}M$. When $l=k$, this follows from the $m+1=k$ case of Lemma 3.6 (ii). The general proof follows from a five lemma argument involving a diagram whose horizontal rows are two copies of the short exact sequences of 3.6(ii) ($l=m$) and whose vertical arrows represent multiplication by v_k. The left vertical arrow would be injective by a subsidiary induction on dimension; the right one by the original induction on l.

4. Hurewicz homomorphisms

Let $J=\{q_0, \cdots, q_{n-1}\}$ continue to be an invariant regular sequence of length n. Let $\Gamma=BPJ_*\cong BP_*/(q_0, \cdots, q_{n-1})$ and let $\Lambda=BP_*$. The ring epimorphism $\Lambda\to\Gamma$ results in the identification $A\otimes_{\Lambda}B\cong A\otimes_{\Gamma}B$ for arbitrary Γ-modules A_{Γ} and ${}_{\Gamma}B$. Hence, throughout this section, we can adopt the convention that $A\otimes B$ means $A\otimes_{\Lambda}B$.

By Lemma 3.5(i), $X\mapsto E(m)_*\otimes BPJ_*X$ defines a homology theory; let $E(m, J)$ be its representing CW spectrum. $E(m, J)_*X\cong E(m)_*\otimes BPJ_*X$ for any CW spectrum X. Recall from §2 that we can form the spectrum $v_m^{-1}BPJ$

with $\pi_*(v_m^{-1}BPJ)\cong v_m^{-1}BPJ_*$. For a spectrum Y, we have the Boardman map $Y\to E(m,J)\wedge Y$ which induces a Hurewicz homomorphism $\pi_*(Y\wedge X)\to E(m,J)_*(Y\wedge X)$. The key topological results of this section compute the kernels of these homomorphisms when $Y=BPJ$ or $v_m^{-1}BPJ$. These computations depend on a theorem of Ravenel concerning the right unit of the BP spectrum.

Theorem 4.1 (Ravenel [13]). *Let η_m be the composition*

$$\eta_m\colon BP_* \xrightarrow{\eta_R} BP_*BP \to Z/p[v_m]\otimes BP_*BP .$$

Then $\eta_m(v_m)=v_m$ and for $k\geqslant 1$,

$$\eta_m(v_{m+k}) \equiv v_m t_k^{p^m} - v_m^{p^k} t_k \qquad \text{modulo} \quad (\eta_m(v_{m+1}), \cdots, \eta_m(v_{m+k-1}))\,.$$

Let $G(m,J)_*=Z_{(p)}[v_1,\cdots,v_m]\otimes BPJ_*BPJ$ and let $\lambda\colon G(m,J)_*\to v_m^{-1}G(m,J)_*\cong E(m,J)_*BPJ$ be the localization homomorphism. We have two Hurewicz homomorphisms induced by BPJ's right unit:

$$h_m'(\Gamma)\colon\ \Gamma = BPJ_* \xrightarrow{\eta_R} BPJ_*BPJ \to G(m,J)_*$$

$$h_m(\Gamma) = \lambda\circ h_m'(\Gamma)\colon\ \Gamma \xrightarrow{\eta_R} BPJ_*BPJ \to E(m,J)_*BPJ\,.$$

For a left Γ module M, we define:

$$h_m'(M) = h_m'(\Gamma)\otimes 1\colon\ M\cong\Gamma\otimes M \to G(m,J)_*\otimes M$$
$$h_m(M) = h_m(\Gamma)\otimes 1\colon\ M\cong\Gamma\otimes M \to E(m,J)_*BPJ\otimes M\,.$$

Lemma 4.2. *Let $m\geqslant n$ Let $h'=h_m'(BP_*/I_m)$. For any non-zero element y in BP_*/I_m, left multiplication by $h'(y)$ in $G(m,J)_*\otimes BP_*/I_m$ is injective.*

Proof. The $m=n=0$ case is well-known; so we assume that $m>0$. Since $\eta_R(v_s)\equiv v_s$ modulo $I_s\cdot BP_*BP$ [2,11.16.1], we have the isomorphisms ζ and ρ in commutative diagram 4.3.

$$(4.3)\qquad \begin{array}{ccccc}
BP_* & \xrightarrow{\eta_m} & Z/p[v_m]\otimes BP_*BP & \xrightarrow{1\otimes(j\wedge j)_*} & Z/p[v_m]\otimes BPJ_*BPJ \\
\big\downarrow & & & & \zeta\big\downarrow\cong \\
 & & & & Z/p[v_m]\otimes BPJ_*BPJ\otimes BP_*/I_m \\
 & & & & \rho\big\uparrow\cong \\
BP_*/I_m & \xrightarrow{h'} & G(m,J)_*\otimes BP_*/I_m & = & Z_{(p)}[v_1,\cdots,v_m]\otimes BPJ_*BPJ\otimes BP_*/I_m
\end{array}$$

A Z/p-basis element of BP_*/I_m is (represented by) a monomial of form $v_m^{i_0}v_{m+1}^{i_1}v_{m+2}^{i_2}\cdots=v^I$. Let $i=i_0+i_1+i_2\cdots$ and let

$$t^{p^m I} = t_0^{p^m i_0}t_1^{p^m i_1}t_2^{p^m i_2}\cdots = t_1^{p^m i_1}t_2^{p^m i_2}\cdots \qquad (t_0 = 1)\,.$$

If $E=(e_1, e_2, \cdots)$ and $t^E=t_1^{e_1}t_2^{e_2}\cdots$, observe that $(j\wedge j)_* t^E = z^{E,0}$ which is a left BPJ_*-basis element of BPJ_*BPJ (Lemma 5.10). Filter each gradation of the image of $1\otimes(j\wedge j)_*$ by defining $v_m{}^a\otimes z^{E,0}$ to be of lower filtration than $v_m{}^b\otimes z^{F,0}$ provided that $\cdots$, $e_{s+2}=f_{s+2}$, $e_{s+1}=f_{s+1}$, but $e_s<f_s$. We now interpret Theorem 4.1 as saying that $\zeta^{-1}\rho h'(v^I)=(1\otimes(j\wedge j)_*)\eta_m(v^I)\equiv v_m{}^i\otimes z^{p^m I,0}$ modulo terms of lower filtration. The result is now evident.

Corollary 4.4. *Let $m\geqslant n$. Then $h_m(BP_*/I_m)$: $BP_*/I_m\to E(m,J)_*BPJ\otimes BP_*/I_m$ is a monomorphism.*

Proof. By Lemma 4.2, left multiplication by $v_m\otimes 1=h'(v_m)$ on $G(m,J)_*\otimes BP_*/I_m$ is monic; thus the localization map $\lambda\otimes 1$: $G(m,J)_*\otimes BP_*/I_m\to E(m,J)_*BPJ\otimes BP_*/I_m$ is monic. A second application of Lemma 4.2 shows that $h_m'(BP_*/I_m)$ is injective. But $h_m(BP_*/I_m)=(\lambda\otimes 1)h_m'(BP_*/I_m)$.

Let us adopt the notation

$$\tilde{h}_m(\Gamma)\colon\ v_m{}^{-1}\Gamma = v_m{}^{-1}BPJ_* = \pi_*(v_m{}^{-1}BPJ)\to E(m,J)_*(v_m{}^{-1}BPJ)$$

for the Hurewicz homomorphism induced by the Boardman map $v_m{}^{-1}BPJ\to E(m,J)\wedge v_m{}^{-1}BPJ$. For any Γ-module M, we define $\tilde{h}_m(M)$ by

$$\tilde{h}_m(M)=\tilde{h}_m(\Gamma)\otimes 1\colon\ v_m{}^{-1}M = v_m{}^{-1}\Gamma\otimes M\to E(m,J)_*(v_m{}^{-1}BPJ)\otimes M\,.$$

Using the notation $\lambda(M)$: $M\to v_m{}^{-1}M$ for the algebraic localization of M and λ: $BPJ\to v_m{}^{-1}BPJ$ for the topological localization of the spectrum BPJ, we have the commutative diagram 4.5.

$$\text{(4.5)}\qquad \begin{array}{ccc} M & \xrightarrow{h_m(M)} & E(m,J)_*BPJ\otimes M \\ {\scriptstyle\lambda(M)}\downarrow & & \downarrow{\scriptstyle E(m,J)_*(\lambda)\otimes 1} \\ v_m{}^{-1}M & \xrightarrow{\tilde{h}_m(M)} & E(m,J)_*(v_m{}^{-1}BPJ)\otimes M \end{array}$$

Lemma 4.6. *Let $m\geqslant n$. For any associative BPJ_*BPJ-comodule M, $\tilde{h}_m(M)$ is monic.*

Proof. For a future analogy and some present simplicity, let $A=BPJ_*=\Gamma$, $B=E(m,J)_*BPJ$, and $f=h_m(\Gamma)$: $A\to B$. We record four essential facts.
(i) By Corollary 4.4, $f\otimes BP_*/I_m$: $A\otimes BP_*/I_m\to B\otimes BP_*/I_m$ is monic.
(ii) By Lemma 3.5(ii), $\mathrm{Tor}_1^\Gamma(B, BP_*/I_{j+1})=0$, $n\leqslant j+1$ $(\leqslant m+1)$.
(iii) By Lemma 3.5(iii), $\mathrm{Tor}_1^\Gamma(B\otimes v_m{}^{-1}\Gamma, BP_*/I_j)=0$, $n\leqslant j$.
(iv) A is connective.
The lemma is well known when $m=n=0$; so we assume $m>0$. Multiplication by v_j on BP_*/I_j induces commutative diagram 4.7 which has exact rows. We assume $n\leqslant j\leqslant m$ so that the bottom torsion term is zero as indicated.

$$(4.7)\quad \begin{array}{ccccccc} \mathrm{Tor}_1^\Gamma(A, BP_*/I_{j+1}) & \to & A\otimes BP_*/I_j & \xrightarrow{1\otimes v_j} & A\otimes BP_*/I_j & \to & A\otimes BP_*/I_{j+1} \to 0 \\ \downarrow & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\ 0=\mathrm{Tor}_1^\Gamma(B, BP_*/I_{j+1}) & \to & B\otimes BP_*/I_j & \xrightarrow{1\otimes v_j} & B\otimes BP_*/I_j & \to & B\otimes BP_*/I_{j+1} \to 0 \end{array}$$

The vertical f_i's are $f\otimes BP_*/I_k$'s as appropriate. By a downward induction beginning with $j=m$ (i), we assume f_3 is monic. By an upward induction on the dimension of elements in connective $A\otimes BP_*/I_j$ (iv), we assume f_1 is monic in the dimension of interest. Note that $1\otimes v_j$ raises dimensions. Thus f_2 is monic by the five lemma. By this double induction $f\otimes BP_*/I_j$ is monic for j satisfying $n\leqslant j\leqslant m$. Upon v_m-localization f induces $\tilde{f}$: $v_m^{-1}A\to E(m, J)_*(v_m^{-1}BPJ)$. Since $(v_m^{-1}A)\otimes BP_*/I_j=0$ for $j>m$, we have that $\tilde{f}\otimes BP_*/I_j$: $v_m^{-1}BP_*/I_j\to E(m, J)_*(v_m^{-1}BPJ)\otimes BP_*/I_j$ is monic for all j satisfying $n\leqslant j$. To prove $\tilde{f}\otimes M$ is monic for all associative BPJ_*BPJ-comodules, it suffices to prove $\tilde{f}\otimes M$ is monic where M is finitely presented (Lemma 1.13). Such a finitely presented comodule M has a finite filtration by sub-comodules whose subquotients are suspended copies of BP_*/I_j, $j\geqslant n$. The remainder of the proof is a five-lemma-aided induction over the filtration of M using fact (iii) to have the "bottom-left torsion term" zero.

Lemma 4.8. *Let $m\geqslant n$ and let M be an associative BPJ_*BPJ-comodule. Left multiplication by $h_m(M)(v_m)$ acts injectively on $E(m, J)_*BPJ\otimes M$.*

Proof. Let $A=B=G(m, J)_*=Z_{(p)}[v_1, \cdots, v_m]\otimes BPJ_*BPJ$. Let f: $A\to B$ be left multiplication by $h_m'(\Gamma)(v_m)$. We record four essential facts.

(i) By Lemma 4.2, $f\otimes BP_*/I_m$ is monic.

(ii) By Lemma 3.4, $\mathrm{Tor}_1^\Gamma(B, BP_*/I_{j+1})=0$, $n\leqslant j+1\leqslant m+1$.

(iii) By Lemma 3.5 (ii), $\mathrm{Tor}_1^\Gamma(v_m^{-1}B, BP_*/I_j)=0$, $n\leqslant j$.

(iv) A is connective.

Follow the pattern of the proof of Lemma 4.6.

Theorem 4.9. *Let $J=\{q_0, \cdots, q_{n-1}\}$ be an invariant regular sequence of length n and let $m\geqslant n$. Let M be an associative BPJ_*BPJ-comodule. The kernel of*

$$h_m(M)\colon\ M\to E(m, J)_*BPJ\otimes_{BPJ_*}M$$

is the v_m-torsion subgroup of M.

Proof. In diagram 4.5, $\tilde{h}_m(M)$ is monic by Lemma 4.6. By Lemma 4.8, left multiplication by $h_m(M)(v_m)$—i.e. right multiplication by $v_m\otimes 1$—is monic on $E(m, J)_*BPJ\otimes M$. Thus the localization map $E(m, J)_*(\lambda)\otimes 1$ is monic in 4.5. Thus the kernel of $h_m(M)$ coincides with that of $\lambda(M)$ which is the v_m-torsion subgroup of M.

Theorem 4.10. *Let $J=\{q_0, \cdots, q_{n-1}\}$ be an invariant regular sequence of length n and let $m \geqslant n$. Let X be any CW spectrum.*
(i) *The Boardman map $v_m^{-1}BPJ \to E(m, J) \wedge v_m^{-1}BPJ$ induces a Hurewicz monomorphism*

$$\tilde{h}_m(X): \pi_*(v_m^{-1}BPJ \wedge X) \to E(m, J)_*(v_m^{-1}BPJ \wedge X).$$

(ii) *The Boardman map $BPJ \to E(m, J) \wedge BPJ$ induces a Hurewicz homomorphism*

$$h_m(X): \pi_*(BPJ \wedge X) \to E(m, J)_*(BPJ \wedge X)$$

whose kernel is precisely the v_m-torsion subgroup of $BPJ_*X=\pi_*(BPJ \wedge X)$.

Proof. The latter part follows immediately from Theorem 4.9 and the isomorphism $E(m, J)_*(BPJ \wedge X) \cong E(m, J)_*BPJ \otimes BPJ_*X$ (Lemma 3.5(ii)). Similarly, the first part follows from Lemma 4.6.

Corollary 4.11. *Let J continue to be an invariant regular sequence of length n and let $m \geqslant n$. For any CW spectrum X, $(v_m^{-1}BPJ)_*X=0$ if and only if $E(m, J)_*X=0$.*

Proof. The "only if" statement follows from a Conner-Floyd type isomorphism:

$$E(m, J)_*X \cong E(m)_* \otimes_{BP_*} BPJ_*X \cong E(m)_* \otimes_{v_m^{-1}BP_*} BPJ_*X.$$

Its converse follows from Theorem 4.10(i) and the isomorphisms

$$\begin{aligned} E(m, J)_*X \otimes_{BPJ_*} BPJ_*(v_m^{-1}BPJ) &\cong E(m, J)_*(X \wedge v_m^{-1}BPJ) \\ &\cong E(m, J)_*(v_m^{-1}BPJ \wedge X). \end{aligned}$$

Corollary 4.12. *Let J be a finite invariant regular sequence of length n. Let $m \geqslant n$. Let X be a connective CW spectrum. If w. $\dim_{BPJ_*}BPJ_*X \leqslant m-n$, then the Hurewicz homomorphism*

$$h_m(X): \pi_*(BPJ \wedge X) \to E(m, J)_*(BPJ \wedge X)$$

is injective. (Cf. [6, Theorem 6.1].)

Proof. Proposition 3.7 and Theorem 4.10 (ii).

5. BPJ_*BPJ and BPJ^*BPJ

Let A be an algebra over the ground ring R, N be an R-module, and M be an associative A-module. Then there is an isomorphism

$$\theta: \mathrm{Hom}_A(A \otimes_R N, M) \to \mathrm{Hom}_R(N, M)$$

defined by $\theta(f)=f(\eta \otimes 1)$ where $\eta: R \to A$ is the unit map. $\theta^{-1}(g)=\varphi(1 \otimes g)$

where $\varphi: A\otimes_R M\to M$ gives M's A-module structure. (Adams [1, p. 320].)

Lemma 5.1. *Let $h: M\to A\otimes_R N$ be an A-module homomorphism. If h is split epic as an R-module homomorphism, then it is also split epic as an A-module map.*

Proof. Let the R-module map $f: A\otimes_R N\to M$ be a right inverse for h. Then $\theta^{-1}(f(\eta\otimes 1))$ is the desired A-module splitting of h.

Lemma 5.2. *Let C be a coalgebra over R, N be an R-module, and M be an associative C-comodule. Let $h: C\otimes_R N\to M$ be a C-comodule map. If h is split monic as an R-module homomorphism, then it is also split monic as a C-comodule map.*

Proof. This is the formal dual of Lemma 5.1.

Let $J=\{q_0, \cdots, q_{n-1}\}$ be a finite invariant regular sequence in BP_*. By Baas [4], there exists an associative left BP-module spectrum BPJ with pairing $\varphi: BP\wedge BPJ\to BPJ$ such that $\pi_*BPJ=BPJ_*\cong BP_*/(q_0, \cdots, q_{n-1})$. Let $J_{m+1}=\{q_0, \cdots, q_m\}$, $m<n$. BPJ_m and BPJ_{m+1} are related by a cofibration of BP-module spectra

$$S^d BPJ_m \xrightarrow{\varphi_m(q_m\wedge 1)} BPJ_m \xrightarrow{j_m} BPJ_{m+1} \xrightarrow{k_m} S^{d+1}BPJ_m\,.$$

Here d is the dimension of q_m in BP_*. $\varphi_m: BP\wedge BPJ_m\to BPJ_m$ defines BPJ_m's BP-module structure; $\varphi_m(1\wedge j_{m-1})=j_{m-1}\circ\varphi_{m-1}$ and $\varphi_n=\varphi$. $BPJ_0=BP$ and $\varphi_0=m: BP\wedge BP\to BP$. Let

$$j_{m+s,m}=j_{m+s-1}\circ\cdots\circ j_m: BPJ_m\to BPJ_{m+s}\,.$$

Let $\iota_m=j_{m,0}\circ\iota: S^0\to BPJ_m$ where $\iota: S^0\to BP$ is the unit for the Brown-Peterson spectrum.

The homomorphism

$$\psi: BPJ^*(BPJ_l\wedge BPJ_m)\xrightarrow{(\varphi_l\wedge\varphi_m)^*(1\wedge T\wedge 1)^*} BPJ^*(BP\wedge BP\wedge BPJ_l\wedge BPJ_m)$$
$$\xleftarrow{\cong} BPJ^*(BP\wedge BP)\hat{\otimes}_{BPJ^*} BPJ^*(BPJ_l\wedge BPJ_m)$$

makes $BPJ^*(BPJ_l\wedge BPJ_m)$ into an associative $BPJ^*(BP\wedge BP)$-comodule. (See Würgler [16].) $\varphi(1\wedge j_{n,0})=j_{n,0}\circ\varphi_0$ gives a distinguished element of $BPJ^*(BP\wedge BP)$. A map $f: BPJ_l\wedge BPJ_m\to BPJ$ is said to be primitive if $\psi[f]=[j_{n,0}\circ\varphi_0]\otimes[f]$. In other words, $f(\varphi_l\wedge\varphi_m)(1\wedge T\wedge 1)=\varphi(\varphi_0\wedge 1)(1\wedge 1\wedge f)$. We follow Würgler in denoting the set of primitives of $BPJ^*(BPJ_l\wedge BPJ_m)$ by $Pr\, BPJ^*(BPJ_l\wedge BPJ_m)$.

REMARK 5.3. When $BPJ_l=BPJ_m=BPJ$, a multiplication $\mu: BPJ\wedge BPJ\to$

BPJ in $BPJ^*(BPJ\wedge BPJ)$ is primitive if and only if the following three conditions hold.

(i) $\mu(\varphi\wedge 1)=\varphi(1\wedge\mu)$: $BP\wedge BPJ\wedge BPJ\to BPJ$

(ii) $\mu(\varphi\wedge 1)(T\wedge 1)=\mu(1\wedge\varphi)$: $BPJ\wedge BP\wedge BPJ\to BPJ$

(iii) $\mu(1\wedge\varphi)(1\wedge T)=\varphi\circ T(\mu\wedge 1)$: $BPJ\wedge BPJ\wedge BP\to BPJ$

(The first two conditions imply the third.) Conditions (i), (ii), and (iii) give Araki-Toda's characterization of a *quasi-associative* multiplication [3].

Lemma 5.4. *Let a multiplication* μ: $BPJ\wedge BPJ\to BPJ$ *be quasi-associative and let* ι_n: $S^0\to BPJ_n=BPJ$ *be a unit for* μ. *Then the following diagram commutes.*

$$\begin{array}{ccccc} BP\wedge BPJ & \xrightarrow{j_{n,0}\wedge 1} & BPJ\wedge BPJ & \xleftarrow{1\wedge j_{n,0}} & BPJ\wedge BP \\ & \searrow^{\varphi} & \downarrow{\mu} & & \downarrow{T} \\ & & BPJ & \xleftarrow{\varphi} & BP\wedge BPJ \end{array}$$

Proof. Routine.

Proposition 5.5 (Würgler [16, Theorem 5.1]). *Let* $J=\{q_0,\cdots,q_{n-1}\}$ *be an invariant regular sequence of* BP_*. *For* $0\leqslant m\leqslant n$, *there is a quasi-associative multiplication* μ_m: $BPJ_m\wedge BPJ_m\to BPJ_m$ *with unit* ι_m: $S^0\to BPJ_m$ *such that* $j_{m-1}\circ\mu_{m-1}=\mu_m(j_{m-1}\wedge j_{m-1})$ *as maps* $BPJ_{m-1}\wedge BPJ_{m-1}\to BPJ_m$.

Proof. For $0\leqslant m\leqslant n$, we construct primitive maps μ_m': $BPJ_{m-1}\wedge BPJ_m\to BPJ_m$ and μ_m: $BPJ_m\wedge BPJ_m\to BPJ_m$ such that all of the obvious compositions commute:

(i) $\mu_m'(1\wedge j_{m-1})=j_{m-1}\circ\mu_{m-1}$;

(ii) $\mu_m'(j_{m-1,0}\wedge 1)=\phi_m$;

(iii) $\mu_m(j_{m-1}\wedge 1)=\mu_m'$;

(iv) $\mu_m(1\wedge j_{m,0})=\phi_m\circ T$

where T: $BPJ_m\wedge BP\to BP\wedge BPJ_m$ is the switching map. (Compare Lemma 5.4.) The proof is by induction on m. We sketch the inductive step.

Since $\eta_R(q_l)\in(q_0,\cdots,q_l)\cdot BP_*BP$, the cofibration

$$BPJ_l\xrightarrow{j_l}BPJ_{l+1}\xrightarrow{k_l}S^{d+1}BPJ_{l+1} \tag{5.6}$$

induces a split short exact sequence of BPJ_m^*-modules, $0\leqslant k$, $l+1\leqslant m\leqslant n$.

$$0\to BPJ_m^*(BPJ_k\wedge BPJ_l)\to BPJ_m^*(BPJ_k\wedge BPJ_{l+1})\xrightarrow{(1\wedge j_l)^*}BPJ_m^*(BPJ_k\wedge BPJ_l)\to 0\,. \tag{5.7}$$

We assume inductively that $BPJ_m^*(BPJ_k\wedge BPJ_l)\cong BPJ_m^*(BP\wedge BP)\otimes_{BPJ_{m*}}N$

for some BPJ_{m*}-module N. By Lemma 5.1, (5.7) splits as $BPJ_m^*(BP\wedge BP)$-modules and the middle term of (5.7) has the desired inductive structure. By Lemma 5.2, (5.7) splits as $BPJ_m^*(BP\wedge BP)$-comodules. Hence the functor $Pr(—)$ preserves the exactness of (5.7). We first pick $\mu_m'\in PrBPJ_m{}^*(BPJ_{m-1}\wedge BPJ_m)$ satisfying (i) and (ii), and next $\mu_m\in PrBPJ_m{}^*(BPJ_m\wedge BPJ_m)$ satisfying (iii) and (iv).

Quasi-associative multiplications μ: $BPJ\wedge BPJ\to BPJ$ with unit ι_n: $S^0\to BPJ$ exist by Proposition 5.5. We assume that a choice of such a μ is fixed throughout this paper.

The cofibration (5.6) induces two split short exact sequences of $BPJ_*\cong BPJ^*$-modules.

$$0\to BPJ_*BPJ_l\xrightarrow{j_l^*}BPJ_*BPJ_{l+1}\xrightarrow{k_l^*}BPJ_*BPJ_l\to 0 \tag{5.8}$$

$$0\to BPJ^*BPJ_l\xrightarrow{k_l^*}BPJ^*BPJ_{l+1}\xrightarrow{j_l^*}BPJ^*BPJ_l\to 0 \tag{5.9}$$

Recall that $BP_*BP\cong BP_*[t_1, t_2, \cdots]$ where the indeterminate t_i is of dimension $2p^i-2$. (Let $t_0=1$.) Thus $BPJ_*BP\cong BPJ_*[t_1, t_2, \cdots]$. An argument using Lemma 5.1, similar to that of the proof of Proposition 5.5, shows that BPJ_*BPJ_l is a free left BPJ_*BP-module. Let $A=(a_0, \cdots, a_{l-1})$ be an l-tuple of 0's and 1's. A free left BPJ_*BP-basis of BPJ_*BPJ_l is given by the symbols

$$\partial^A=\partial_0{}^{a_0}\cdots\partial_{l-1}{}^{a_{l-1}}$$

of dimension $\sum_j a_j$ (dimension $(q_j)+1$). In (5.8), j_{l*} sends ∂^A to a symbol of the same name. We choose elements $\partial^A\partial_l\in BPJ_*BPJ_{l+1}$ so that $k_{l*}(\partial^A\partial_l)=\partial^A$. Let $z^{E,A}\in BPJ_*BPJ$ be the element corresponding to $t_1{}^{e_1}\cdots t_m{}^{e_m}\partial^A$ where $E=(e_1, \cdots, e_m, 0, \cdots)$. Let c: $BPJ_*BPJ\to BPJ_*BPJ$ be the conjugation induced by interchange of the BPJ factors of $BPJ\wedge BPJ$.

Lemma 5.10. *Let $J=\{q_0, \cdots, q_{n-1}\}$ be an invariant regular sequence of BP_*. A free BPJ_* basis for BPJ_*BPJ is given by the elements $z^{E,A}$ where $A=(a_0, \cdots, a_{n-1})$ is a sequence of* 0's *and* 1's. *The left action of BP_*BP on BPJ_*BPJ is given by*

$$t^F z^{E,A}=z^{E+F,A}\,. \tag{5.11}$$

BPJ_*BPJ is free as a right BPJ_*-module on the basis $c(z^{E,A})$.

As explained in §1, $1\wedge\iota_n$: $BPJ\wedge S^0\to BPJ\wedge BPJ$ induces a coaction

$$\psi_X:\ BPJ_*X\to BPJ_*(BPJ\wedge X)\cong BPJ_*BPJ\otimes_{BPJ_*}BPJ_*X$$

and we define elementary BPJ operations $s_{E,A}$ by the formula

$$\psi_X(x) = \sum_{E,A} c(z^{E,A}) \otimes s_{E,A}(x) . \tag{5.12}$$

Since BPJ^*BPJ is Hausdorff, each elementary BPJ operation

$$s_{E,A}\colon BPJ^*(\ ) \to BPJ^{*+d}(\ ) ,$$

$d = \sum e_i(2p^i - 2) + \sum a_j(\dim(q_j) + 1)$, is induced by a unique map of spectra $S_{E,A}$: $BPJ \to S^d BPJ$. By an induction (over $l \leqslant n$) using (5.9), one can prove:

Lemma 5.12. *BPJ^*BPJ is a direct product of copies of BPJ^* indexed by the maps $S_{E,A}$. Each element $\theta \in BPJ^tBPJ$ has a unique representation as a convergent infinite sum*

$$\theta = \sum_{E,A} q_{E,A} S_{E,A} , \qquad q_{E,A} \in BPJ^{t-d}$$

where d is the dimension of $S_{E,A}$.

REMARK 5.13. Another induction using the exactness of (5.9) shows that

$$S_{0,0} \circ j_{n,l} - j_{n,l} = \sum_{A \neq 0} q_A S_{0,A} \circ j_{n,l}$$

for $0 \leqslant l \leqslant n$ and $q_A \in BPJ^*$. This establishes (1.8).

REMARK 5.14. The composition $S_{E,A} \circ S_{F,B}$ has a representation $\sum q_{G,C} S_{G,C}$ by Lemma 5.12. Here the dimension of $S_{G,C}$ must be greater than or equal to the sum of the dimensions of $S_{E,A}$ and $S_{F,B}$. Since $s_{E,A} \circ s_{F,B} = S_{E,A*} \circ S_{E,B*} = (S_{E,A} \circ S_{F,B})_*$, (1.9) is established. In general, the relations given by $S_{E,A} \circ S_{F,B}$ will be even more frightful than the ones given by $r_E \circ r_F$ in BP theory. In particular: $S_{0,0} \circ S_{0,0}$ will not be $S_{0,0}$ unless the q_A's in Remark 5.13 happen to be all zero.

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