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A NOTE ON SULLIVAN COMPLETION

Dedicated to Professor Tatsuji Kudo on his 60th birthday

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In this note we give an alternative construction of the Sullivan finite completion for a "good" space by only making use of the standard techniques in homotopy theory.

Let $c: X \to Y$ be a based map of connected based spaces. We say that c is a π_*-finite completion of X if it is the finite completion on π_1 and π_1-finite completion of the higher homotopy.

Theorem 0 (Sullivan [8, Theorem 3.1. ii), Corollary of proof]).
Let X be a connected based space with "good" homotopy groups. A map $c: X \to Y$ is equivalent to the finite completion if and only if c is a π_*-finite completion.

Sullivan [8; Theorem 3.1. i)] also shows that sufficiently many spaces have "good" homotopy groups. Thus, to construct Sullivan finite completion, it is enough to construct a π_*-finite completion.

Since our arguments are quite formal, analogous l-finite construction is also available for a set l of primes.

1. π_*-finite completion and Main Theorem

Let X be a connected based space and let $\{M_i\}_{i\in I}$ be a projective system of finite π_1X-modules and π_1X-(equivariant) homomorphisms. Then we have a projective system $\{H^n(X, *; M_i)\}_{i\in I}$ and compatible homomorphisms $H^n(X, *; \lim M_i) \to H^n(X, *; M_i)$, where $H^n(X, *; M_i)$, $H^n(X, *; \lim M_i)$ are *n-th* cohomology groups with twisted coefficients M_i, $\lim M_i$ respectively.

Theorem 1.1. *We have a natural isomorphism*

$$H^n(X, *; \lim M_i) \cong \lim H^n(X, *; M_i).$$

Following [8], we say that π is a good group (resp. a weakly good group) if

$$H^n(\pi; M) \cong \operatorname{colim} H^n(\pi_\alpha; M) \cong H^n(\hat{\pi}; M)$$
$$(\text{resp. } H^n(\pi; M) \cong H^n(\hat{\pi}; M))$$

for the system of finite quotients $\{\pi_\alpha\}$ of π, each finite π-module M and all n. We say that G is a good π-module (resp. a weakly good π-module) if

$$H^n(G; A) \cong \operatorname{colim} H^n(G_\alpha; A) \cong H^n(\lim G_\alpha; A) < +\infty$$
$$(\text{resp. } H^n(G; A) \cong H^n(\lim G_\alpha; A) < +\infty)$$

for the system of finite π-quotients $\{G_\alpha\}$, all finite coefficient groups A and all n. A connected based space X is said to have good (resp. weakly good) homotopy groups if $\pi_1 X$ is good (resp. weakly good) and $\pi_1 X$-modules $\pi_n X$ are all good (resp. weakly good).

REMARK 1. If G is a good π-module, $\hat{G} \cong \hat{G}_\pi$ $(=\lim G_\alpha)$.

REMARK 2. Sullivan does not use the finiteness condition $H^n(\pi; M) < +\infty$ (see [8]), so we omit it.

REMARK 3. We do not require that $\{\pi_\alpha\}$, $\{G_\alpha\}$ have essentially countable index sets. These conditions are inessential in the proof of Theorem 3.1 of [8].

Let X be a space with weakly good homotopy groups, and let $c: X \to Y$ be a π_*-finite completion. It is easy to see that $c^*: H^n(Y; M) \cong H^n(X; M)$ (and therefore $c^*: H^n(Y, *; M) \cong H^n(X, *; M)$) for every finite $\pi_1 X$-module M by making use of Postnikov decompositions and Serre spectral sequences. Let $\{M_i\}$ be a projective system of finite $\pi_1 X$-modules and $\pi_1 X$-homomorphisms. It is easy to see that the canonical extensions $\widehat{\pi_1 X} \to \operatorname{Aut} M_i$ of $\pi_1 X$-actions $\pi_1 X \to \operatorname{Aut} M_i$ make $\{M_i\}$ a projective system of finite $\widehat{\pi_1 X}$-modules and $\widehat{\pi_1 X}$-homomorphisms. Theorem 1.1 and the commutative diagram

$$\begin{array}{ccc} H^n(Y, *; \lim M_i) & \xrightarrow{c^*} & H^n(X, *; \lim M_i) \\ \downarrow & & \downarrow \\ \lim H^n(Y, *; M_i) & \xrightarrow{c^*} & \lim H^n(X, *; M_i) \end{array}$$

lead to

Corollary 1.2 (Technical Lemma). *We have an isomorphism*

$$H^n(Y; \lim M_i) \cong H^n(X; \lim M_i).$$

Corollary 1.3 (Main Theorem). *A space with weakly good (resp. good) homotopy groups has a π_*-finite completion (resp. the Sullivan finite completion).*

Proof. Let X be a connected based space with weakly good homotopy groups, and let

$$\cdots \to X(n) \xrightarrow{p(n)} X(n-1) \to \cdots \to X(1) \to *$$

be its Postnikov decomposition, so that $p(n)$ is a fibration with k-invariant $k(n)\in H^{n+1}(X(n-1);\ \pi_n X)\cong[X(n-1),\ L_{\Phi_n}(\pi_n X,\ n+1)]_{\bar{W}\pi_1 X}$ (see [5]). Assume that we have a π_*-finite completion $c(n-1)\colon X(n-1)\to Y(n-1)$. By Technical Lemma we have a unique extension $\widehat{k(n)}$ in the diagram

$$\begin{array}{ccc} X(n-1) & \xrightarrow{c(n-1)} & Y(n-1) \\ \downarrow{\scriptstyle k(n)} & & \downarrow{\scriptstyle \widehat{k(n)}} \\ L_{\Phi_n}(\pi_n X,\ n+1) & \longrightarrow & L_{\hat{\Phi}_n}(\widehat{\pi_n X}_{\widehat{\pi_1 X}},\ n+1)\,. \end{array}$$

Thus we have $Y(n)$ and $c(n)\colon X(n)\to Y(n)$ with desired properties. Inductively we obtain a π_*-finite completion, which is the finite completion of X by Theorem 0 if X has good homotopy groups. This completes the proof.

2. Proof of Theorem 1.1

In this section we construct a spectral sequence

$$\text{(2.1)}\qquad \lim{}^s H^{n-t}(X,\ *;\ M_i)\Rightarrow H^{n-(t-s)}(X,\ *;\ \lim M_i)\,.$$

Thus, to prove Theorem 1.1, it is enough to prove that $\lim^s H^{n-t}(X,\ *;\ M_i)=0$, $s>0$. So we must construct a second spectral sequence which is analogous one due to Araki-Yosimura [1].

$$\text{(2.2)}\qquad \lim{}^s H^{n-t}(X_\alpha,\ *;\ M)\Rightarrow H^{n-(t-s)}(X,\ *;\ M)$$

where $\{X_\alpha\}$ is the diagram of finite connected pointed subspaces of X, M is a finite π_1-X-module. Since $H^m(X_\alpha,\ *;\ M)$ are all finite groups by definition (see [5]), we have $\lim^s H^m(X_\alpha,\ *;\ M)=0$, $s>0$, and therefore $H^m(X,\ *;\ M)\cong \lim H^m(X_\alpha,\ *;\ M)$, so that we can give $H^m(X,\ *;\ M)$ a compact Hausdorff topology. For the projective system $\{H^m(X,\ *;\ M_i)\}$ of compact Hausdorff abelian groups and continuous homomorphisms, $\lim^s H^m(X,\ *;\ M_i)=0$, $s>0$, therefore Theorem 1.1 is proved.

To construct spectral sequences (2.1), (2.2) we use simplicial notations in [3; Part II], [4] and [6].

Let X be an one vertexed fibrant simplicial set and let $\{M_i\}$ be a projective system of finite $\pi_1 X$-modules. Put $K(i)=K(M_i,\ n)$, $K(\omega)=K(\lim M_i,\ n)$, $L(i)=K(i)\underset{\pi_1 X}{\times} W\pi_1 X$ and $L(\omega)=K(\omega)\underset{\pi_1 X}{\times} W\pi_1 X$. We have canonical fibrations $\theta(i)\colon L(i)\to \bar{W}\pi_1 X$, $\theta(\omega)\colon L(\omega)\to \bar{W}\pi_1 X$, and thus we obtain the fibrations $\theta(i)_*\colon \hom_*(X,\ L(i))\to \hom_*(X,\ \bar{W}\pi_1 X)$, $\theta(\omega)_*\colon \hom_*(X,\ L(\omega))\to \hom_*(X,\ \bar{W}\pi_1 X)$, where $\hom_*(X,\ Y)$ is the pointed simplicial function space (see [3; Ch. V III, 4]). Since X is an one vertexed fibrant simplicial set, we have the canonical map $\theta\colon X\to \bar{W}\pi_1 X$ defined by the twisting function $\tau(x)=$

$[\partial_2\partial_3\cdots\partial_n x]$, $x\in X_n$ (see [5]). Put $F(i)=\theta(i)_*^{-1}\theta$, $F(\omega)=\theta(\omega)_*^{-1}\theta$, where θ is regarded as a vertex of $\hom_*(X,\bar{W}\pi_1X)$. We can define abelian group structures on $F(i)_k$ and $F(\omega)_k$ for any k by fibrewise additions (see [5]), and these group structures commute with the simplicial structures. So that we obtain a projective system of simplicial abelian groups $\{F(i)\}_{i\in I}$ and compatible simplicial abelian group homomorphisms $p(i)\colon F(\omega)\to F(i)$. By Bousfied-Kan [3; Ch. XI, 7.1] we define the spectral sequence

$$\lim{}^s \pi_t F(i) \Rightarrow \pi_{t-s}\operatorname{holim} F(i)\,. \tag{2.3}$$

Examining the functors Π^*, Tot_n and Tot (see [3; Ch. X, XI]), we find that the tower of fibrations

$$\operatorname{holim} F(i) \cdots\to \mathrm{Tot}_n\,\Pi^*F(i) \to \mathrm{Tot}_{n-1}\Pi^*F(i) \to\cdots\to \mathrm{Tot}_{-1}\,\Pi^*F(i)$$

is a tower of simplicial abelian groups. So that we have

Proposition 2.4. *The spectral sequence* (2.3) *is that of abelian groups.*

Let $\{F(\omega)\}_{i\in I}$ be the constant projective system. Since I is a directed set, the opposite category is a "left filtering" and the trivial map $I\to *$ is left cofinal (see [3; Ch. XI, 9.3]), which induces the homomorphism of simplicial abelian groups $F(\omega)\to\operatorname{holim} F(\omega)$ and is a homotopy equivalence (see [3; Ch. XI, 9.1,9.2 and 9.4]). Canonical homomorphisms $p(i)$'s induce the homomorphism of simplicial abelian groups holim $F(\omega)\to\operatorname{holim} F(i)$. By Fibration lemma (see [3; Ch. XI, 5.5]) and right adjointness of the functor holim we have the maps of fibrations

$$\begin{array}{ccccc}
\operatorname{holim} F(\omega) & \to & \operatorname{holim}\hom_*(X, L(\omega)) & \to & \operatorname{holim}\hom_*(X, \bar{W}\pi_1X) \\
\downarrow & & \downarrow & & \| \\
\operatorname{holim} F(i) & \to & \operatorname{holim}\hom_*(X, L(i)) & \to & \operatorname{holim}\hom_*(X, \bar{W}\pi_1X)\,.
\end{array}$$

Since holim $\hom_*(\ ,\ )\cong\hom_*(\ ,\ \operatorname{holim}\ )$ (see [3; Ch. XI. 7.6]) the above diagram is obtained by applying $\hom_*(X,\ )$ to the following diagram (of fibrations)

$$\begin{array}{ccc}
\operatorname{holim} L(\omega) & \to & \operatorname{holim}\ \bar{W}\pi_1X \\
\downarrow & & \| \\
\operatorname{holim} L(i) & \to & \operatorname{holim}\ \bar{W}\pi_1X
\end{array}$$

To prove that holim $F(\omega)\to\operatorname{holim} F(i)$ is a weak homotopy equivalence it is enough to prove that the corresponding map of the fibres of the above diagram, holim $K(\omega)\to\operatorname{holim} K(i)$, is a (weak) homotopy equivalence. It is proved by routine spectral sequence arguments (see [3; Ch. XI, 7.1 and 7.2]). Thus we have

Proposition 2.5. *The canonical homomorphism* $F(\omega)\to\operatorname{holim} F(i)$ *is a weak homotopy equivalence.*

It is easy to see that $\pi_0 F(i) \cong H^n(X, *: M_i)$, $\pi_0 F(\omega) \cong H^n(X, *; \lim M_i)$ (see [5]). Put $\alpha = i$ or ω. It remains to consider what $\pi_k F(\alpha)$ is, or in other words, what $\Omega^k F(\alpha)$ is.

We have the canonical isomorphism ϕ: $\hom(S^k, \hom(X, L(\alpha)) \to \hom(X, \hom(S^k, L(\alpha))$ defined by

$$\phi(f)(x, a)(y, b) = f(y, b^* a)(b^* x, (0, 1, \cdots, p)) \qquad \text{for}$$

$f \in \hom(S^k, \hom(X, L(\alpha)))_n$, $x \in X_q$, $a \in \Delta[n]_q$, $y \in S^k_p$ and $b \in \Delta[q]_p$, where $\hom(\ ,\ ) = \hom_*(\ \amalg_*,\ )$. The inverse ψ is also defined by

$$\psi(g)(y, a)(x, b) = g(x, b^* a)(b^* y, (0, 1, \cdots, p)) \qquad \text{for}$$

$g \in \hom(X, \hom(S^k, L(\alpha)))_n$, $y \in S^k_q$, $a \in \Delta[n]_q$, $x \in X_p$ and $b \in \Delta[q]_p$. Let $N(\alpha)$ ($\subset \hom(S^k, L(\alpha))$) be the simplicial set defined by

$$N(\alpha)_q = \left\{ u \,\middle|\, \begin{array}{ccc} (S^k \times \Delta[q], * \times \Delta[q]) & \xrightarrow{u} & (L(\alpha), \bar{W}\pi_1 X) \\ \downarrow p_r & & \downarrow \theta(\alpha) \\ \Delta[q] & \longrightarrow & \bar{W}\pi_1 X \end{array} \right\}$$

with the usual simplicial structure. We have the map $q(\alpha)$: $N(\alpha) \to \bar{W}\pi_1 X$, $q(\alpha)(u) = \theta(\alpha)(u(*, (0, 1, \cdots, q)))$ which induces the map $q(\alpha)_*$: $\hom_*(X, N(\alpha)) \to \hom_*(X, \bar{W}\pi_1 X)$. Then we have the following

Lemma 2.6. $\phi(\Omega^k F(\alpha)) = q(\alpha)_*^{-1} \theta$.

Lemma 2.7. *We have an isomorphism* $\Omega^k K(\alpha) \underset{\pi_1 X}{\times} W\pi_1 X \to N(\alpha)$ *as a map of spaces over* $\bar{W}\pi_1 X$.

By these lemmas we have

Proposition 2.8. $\pi_k F(\alpha) \cong H^{n-k}(X, *; M_\alpha)$.

Thus by Proposition 2.4, 2.5, 2.8 and the spectral sequence (2.3) we can construct the first spectral sequence (2.1).

Proof of Lemma 2.6. Straightforward, routine calculations complete the proof.

Proof of Lemma 2.7. Define λ: $\Omega^k K(\alpha) \underset{\tau(\pi_1 X)}{\times} \bar{W}\pi_1 X \to N(\alpha)$ and μ: $N(\alpha) \to \Omega^k K(\alpha) \underset{\tau(\pi_1 X)}{\times} \bar{W}\pi_1 X$ by

$$\lambda(h, w)(y, a) = \tau(\pi_1 X)(\bar{w}(0, a_0)) h(y, a) \text{ for}$$

$(h, w) \in (\Omega^k K(\alpha) \underset{\tau(\pi_1 X)}{\times} \bar{W}\pi_1 X)_q$ and $(y, a) = (y, (a_0, a_1, \cdots, a_p)) \in (S^k \times \Delta[q])_p$,

$$\mu(u) = (h', \theta(\alpha)(u(*, (0, 1, \cdots, q))) \text{ and}$$
$$h'(y, a) = \tau(\pi_1 X)(\theta(\alpha)(u(*, (0, a_0)))) l(y, a) \text{ for}$$

$u=(l, \theta(\alpha)(u(*, \ )))\in N(\alpha)_q$ and $(y, a)\in(S^k\times\Delta[q])_p$, where all the bundles are written by *T.C.P.*'s (i.e. $L(\alpha)\cong K(\alpha)\underset{\tau(\pi_1X)}{\times}\bar{W}\pi_1X$, $\Omega^kK(\alpha)\underset{\pi_1X}{\times}W\pi_1X\cong\Omega^kK(\alpha)\underset{\tau(\pi_1X)}{\times}\bar{W}\pi_1X$). These maps are all well defined, simplicial and inverse to each other. This completes the proof.

We next construct the second spectral sequence (2.2). By making use of [3; Ch. XII, Corollary 3.6] we have the map hocolim $X_\alpha\to$ colim $X_\alpha=X$ which is a weak equivalence. Let $F'(\alpha)$ (resp. $F'(\omega)$) be the fibre of the fibration $\hom_*(X_\alpha, L)\to\hom_*(X_\alpha, \bar{W}\pi_1X)$ with base point $\theta_{|X_\alpha}\colon X_\alpha\to X\to\bar{W}\pi_1X$ (resp. $\hom_*(X, L)\to\hom_*(X, \bar{W}\pi_1X)$), where $L=L_{\phi_M}(M, n)$. Since $\hom_*(\text{hocolim},\ )\cong\text{holim}\hom_*(\ ,\ )$ (see [3; Ch. XII, Proposition 4.1]) we have the following diagram of fibrations

$$\begin{array}{ccccc}
\text{holim}\, F'(\alpha) & \cong & F' & \leftarrow & F'(\omega) \\
\downarrow & & \downarrow & & \downarrow \\
\text{holim}\hom_*(X_\alpha, L) & \cong & \hom_*(\text{hocolim}\, X_\alpha, L) & \leftarrow & \hom_*(X, L) \\
\downarrow & & \downarrow & & \downarrow \\
\text{holim}\hom_*(X_\alpha, \bar{W}\pi_1X) & \cong & \hom_*(\text{hocolim}\, X_\alpha, \bar{W}\pi_1X) & \leftarrow & \hom_*(X, \bar{W}\pi_1X)\,,
\end{array}$$

which leads to

Proposition 2.9. *The canonical homomorphism (of simplicial abelian groups) $F'(\omega)\to$ holim $F'(\alpha)$ is a weak homotopy equivalence.*

By Proposition 2.8, 2.9 and the spectral sequence (2.3) we can construct the spectral sequence (2.2).

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