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ON THE COMMUTATIVITY OF THE RADICAL OF THE GROUP ALGEBRA OF AN INFINITE GROUP

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Throughout K will represent an algebraically closed field of characteristic $p>0$, and G a group. Let G' be the commutator subgroup of G. The Jacobson radical of the group algebra KG will be denoted by $J(KG)$. In case G is a finite group and p is odd, D.A.R. Wallace [6] proved that $J(KG)$ is commutative if and only if G is abelian or $G'P$ is a Frobenius group with complement P and kernel G', where P is a Sylow p-subgroup of G. On the other hand, when we consider the case $p=2$, by the following theorem, we may restrict our attention to the case $|P|\geqq 4$.

Theorem 1 ([5]). *Let G be a group of order $p^a m$. where $(p, m)=1$. Then $J(KG)^2=0$ if and only if $p^a=2$.*

In the previous paper [3], we obtained the following

Theorem 2. *Let $p=2$, and G a non-abelian group of order $2^a m$, where m is odd and $a\geqq 2$. Then the following conditions are equivalent:*

(1) *$J(KG)$ is commutative.*

(2) *G' is of odd order and $|P\cap P^x|\leqq 2$ for each $x\in G'P-P$.*

(3) *G' is of odd order and $C_{G'P}(s)/\langle s\rangle$ is either a 2-group or a Frobenius group with complement $P/\langle s\rangle$ for every involution s of P.*

(4) *G' is of odd order and each block of $KG'P$, except the principal block, is of defect 1 or 0.*

In case G is an infinite group and p is odd, D.A.R. Wallace [8] gave also a necessary and sufficient condition for $J(KG)$ to be commutative. Let G be an infinite non-abelian group. We suppose that *$J(KG)$ is non-trivial.* By [8], Theorem 1.1, if $p=2$ and $J(KG)$ is commutative, then the following three cases can arise:

(α) G' is an infinite group and $J(KG)^2=0$.

(β) G' is a finite group of odd order.

(γ) G' is a finite group of even order and the order of a Sylow 2-group P of G is not greater than 4.

If (α) holds, then $J(KG)$ is trivially commutative. Next, we consider the cases (β) and (γ). If $|P|=2$ then $J(KG'P)^2=0$ by Theorem 1. Since $G/G'P$ is abelian and has no elements of order 2, we have $J(KG)=J(KG'P)KG$ by [4], Theorem 17.7, and so $J(KG)^2=J(KG'P)^2KG=0$. In this paper, we shall therefore investigate the cases (β) and (γ) under the hypothesis that P contains at least four elements, and by making use of Theorem 2 we shall give the conditions for $J(KG)$ to be commutative.

At first, we shall prove the next lemma, which plays an important role in studying the case (β).

Lemma 1. *Let $p=2$. Assume that G' is finite and of odd order. If $J(KG)$ is commutative, then any Sylow 2-subgroup of G is finite.*

Proof. Let Q be a finite subgroup of a Sylow 2-subgroup P of G such that $|Q|\geqq 4$. Suppose $H=G'Q$ is abelian. Since Q is characteristic in H, Q is a normal subgroup of G, and so $J(KQ)KG\subset J(KG)$. Let s, t ($\neq 1$) be distinct elements of Q, and x, y elements of G suth that $xy\neq yx$. Then, since Q is contained in the center of G ([7], Lemma 2.6) and $(1-s)x(1-t)y=(1-t)\cdot y(1-s)x$, we have $(1+s+t+st)xyx^{-1}y^{-1}=1+s+t+st$. But, this is impossible. Hence, H is a non-abelian group. Since H is a finite normal subgroup of G, $J(KH)$ is contained in $J(KG)$, and so $J(KH)$ is commutative. Hence, by Theorem 2, $|Q\cap Q^x|\leqq 2$ for each $x\in H'Q-Q$. If $Q\cap Q^x=1$ for all $x\in H'Q-Q$, then $H'Q$ is a Frobenius group with complement Q, and therefore $|H'|=1+k|Q|$ for some positive integer k, which implies that $|Q|<|H'|\leqq|G'|$. Next, if $Q\cap Q^x=\langle s\rangle$ for some $x\in H'Q-Q$ and some involution s of Q then $sxs^{-1}x^{-1}\in H'\cap Q=1$, and so $C_{H'Q}(s)\neq Q$. Hence, by Theorem 2, $C_{H'Q}(s)/\langle s\rangle$ is a Frobenius group with complement $Q/\langle s\rangle$. Then we have $|N|=1+k'|Q/\langle s\rangle|$ for some positive integer k', where N is the Frobenius kernel of $C_{Q'H}(s)/\langle s\rangle$. This implies that $|Q/\langle s\rangle|<|N|\leqq|H'|\leqq|G'|$. Hence, $|Q|<2|G'|$. Thus, the order of any finite subgroups of the abelian Sylow 2-subgroup P is not greater than $2|G'|$. This is only possible if P itself is finite.

REMARK 1. In case G' is finite, if a Sylow p-subgroup of G is finite then any two Sylow p-subgroups of G are conjugate. In fact, $G/G'P$ has no elements of order p, and so every Sylow p-subgroup of G is contained in $G'P$.

Given a finite subset S of G, we denote by $\hat{S}$ the element $\sum_{x\in S} x$ of KG.

Lemma 2. *Let G be a non-abelian group with G' finite. Assume that P contains at least three elements. If $J(KG)$ is commutative, then $J(KG'P)$ is commutative and $(G'P)'=O_{p'}(G')$.*

Proof. We put $H=G'P$. Suppose $J(KG)$ is commutative. If G' is a p'-group, then P is a finite group by Lemma 1 and [8], Theorem 1.1. If $|G'|$

is divisible by p, then $p=2$ or 3 and $|P|=4$ or 3 by [8], Theorem 1.1 and our assumption. In either case, H is a finite normal subgroup of G. Thus, $J(KH)$ is commutative as a subset of $J(KG)$. Hence, by [6], Theorem 2, H is a p-nilpotent group with an abelian Sylow p-subgroup, and so H' is a p'-group. Since G' is finite, by [7], Lemma 2.5 (2) we have $\hat{G}'KG \supset J(KG)^2$. It is easy to see that $J(KG) \supset J(KH) \supset J(KH'P) \supset \hat{H}'J(KP)$. Since H' is a normal subgroup of G, the above facts imply that $\hat{G}'KG \supset \hat{H}'^2 J(KP)^2 = \hat{H}'J(KP)^2 \ni \hat{H}'\hat{P}$. Thus, we have $H'[G' \cap P] = G'$, whence it follows $H' = O_{p'}(G')$.

For a finite group H, we denote by $O(H)$ the largest normal subgroup of odd order in H. The next lemma plays an important role in studying the case (γ).

Lemma 3. *Let $p=2$, and G a non-abelian group with G' finite. Assume that $|P|=4$ and $O(G')=1$. Then the following conditions are equivalent*:

(1) *$J(KG)$ is commutative.*

(2) *$G=C_G(P)$ and*

(i) *$|G'|=2$, or*

(ii) *$G'=P$ and P is elementary abelian.*

Proof. (1)$\Rightarrow$(2): Suppose $J(KG)$ is commutative. Since G' is a finite normal subgroup of G, $J(KG')$ is commutative as a subset of $J(KG)$. Hence, by [6], Theorem 2 and $O(G')=1$, G' is included in P, and so P is a normal subgroup of G. Thus, we have $G=C_G(P)$ by [7], Lemma 2.6. Now, we assume that $G'=P$. Since $\hat{G}'KG \supset J(KG)^2 \supset J(KP)^2$ by Lemma 2.5 (2), we have $J(KP)^2 = K\hat{P}$. Hence, P is elementary abelian.

(2)$\Rightarrow$(1): Since G/P is abelian and has no elements of order 2, we have $J(KG)=J(KP)KG$ by [4], Theorem 17.7. We claim here that $J(KP)^2 \subset \hat{G}'KG$. In case P is elementary abelian, the assertion is trivial by $G' \subset P$. In case P is a cyclic group generated by a, $G' = \langle a^2 \rangle$ by our assumption, and hence $J(KP)^2 = (1+a^2)KP \subset \hat{G}'KG$. Now, by making use of this fact we can prove that $J(KG)$ is commutative. In fact, for u, $v \in P-1$ and x, $y \in G$, we have $(1-u)x(1-v)y = (1-v)(1-u)xy = (1-v)(1-u)xyx^{-1}y^{-1}yx = (1-v)(1-u)yx = (1-v)y(1-u)x$, which implies that $J(KG)$ is commutative.

Now, concerning the cases (β) and (γ) we shall give the conditions for $J(KG)$ to be commutative. At first, concerning the case (β), we have the following:

Theorem 3. *Let $p=2$, and G a non-abelian group. Assume that P contains at least four elements. If G' is a finite group of odd order, then the following conditions are equivalent*:

(1) *$J(KG)$ is commutative.*

(2) *P is a finite group with $(G'P)'=G'$, and for every involution s of P, $C_{G'P}(s)/\langle s\rangle$ is either a 2-group or a Frobenius group with complement $P/\langle s\rangle$.*

Next, concerning the case (γ), we have the following:

Theorem 4. *Let $p=2$, and G a non-abelian group. Assume that $|P|=4$. If G' is a finite group of order $2m$ with odd m, then the following conditions are equivalent:*

(1) *$J(KG)$ is commutative.*

(2) (i) *$|G'|=2$ and $G=C_G(P)$, or*

(ii) *$1\neq(G'P)'=O(G')\supset[G, P]$, and for every involution s of P, $C_{G'P}(s)/\langle s\rangle$ is either a 2-group or a Frobenius group with complement $P/\langle s\rangle$.*

Theorem 5. *Let $p=2$, and G a non-abelian group. Assume that $|P|=4$. If G' is a finite group of order $4m$ with odd m, then the following conditions are equivalent:*

(1) *$J(KG)$ is commutative.*

(2) *P is elementary abelian and*

(i) *$G'=P$ and $G=C_G(P)$,* or

(ii) *$1\neq G''=O(G')\supset[G,P]$, and for every involution s of P, $C_{G'}(s)/\langle s\rangle$ is either a 2-group or a Frobenius group with complement $P/\langle s\rangle$.*

In order to prove these theorems, we require a result of K. Morita [2]: *If G is a finite p-nilpotent group and B is a block of KG with defect group D, then B is isomorphic to the matrix ring $(KD)_f$ for some f.* Especially, this implies the following:

Theorem 6. *Let $p=2$, and G a finite 2-nilpotent group. If B is a block of KG of defect 1, then $J(B)^2=0$.*

Now, we shall prove Theorems 3, 4 and 5 together.

Proof of Theorems 3-5. We put $N=O(G')$, and $e=|N|^{-1}\hat{N}$.

Suppose $J(KG)$ is commutative. In case G' is of odd order, P is finite by Lemma 1. Since $J(KG'P)$ is commutative and $(G'P)'=G'$ (Lemma 2), we obtain (2) of Theorem 3 by Theorem 2. Next, we assume that G' is of even order. If $G'P$ is abelian, then $1=(G'P)'=N$ by Lemma 2. Hence, by Lemma 3 G satisfies the condition (2)(i) of Theorem 4 or that of Theorem 5. In case $G'P$ is non-abelian, since e is a central idempotent of KG, KGe ($\cong KG/N$) is a direct summand of KG, and so $J(KG/N)$ is commutative. Furtheremore, since $J(KG'P)$ is commutative and $(G'P)'=N$ (Lemma 2), the rest of the verification of (2) in Theorems 4 and 5 is easy by Lemma 3 and Theorem 2.

Now, we shall prove the converse implication. We put $H=G'P$. Then we have $J(KG)=eJ(KG)\oplus(1-e)J(KH)KG$ by $J(KG)=J(KH)KG$ ([4], Theorem 17.7). Firstly, $eJ(KG)$ is commutative. In fact, $eJ(KG)\cong J(KG/N)$ by $eKG\cong KG/N$. If G' is of odd order then G/N is abelian; if G' is of even order then the assertion is immediate by Lemma 3. Secondly, since $J(KH)$ is commutative and $H'=N$, $(1-e)KH$ is a direct sum of blocks of defect 1 or 0 (Theorem 2), and so $(1-e)J(KH)^2=0$ by Theorem 6. Then $[(1-e)J(KH)KG]^2=(1-e)J(KH)^2KG=0$, and hence $(1-e)J(KH)KG$ is commutative.

By Theorem 3 and [3], Corollary, we readily obtain the following:

Corollary 1. *Let $p=2$, and G a non-abelian group with G' finite. If $J(KG)$ is commutative, then P is a finite cyclic group or a finite abelian group of type $(2, 2^{a-1})$.*

Corollary 2. *Let G be a non-abelian group with G' finite. Assume that P contains at least three elements. If $J(KG)$ is commutative, then G is a semi-direct product of $O_{p'}(G')$ by $N_G(P)$.*

Proof. If $J(KG)$ is commutative then G' is a p-nilpotent group. Hence, one can easily see that $G=G'N_G(P)=O_{p'}(G')N_G(P)$. Since $J(KG'P)$ is commutative and $(G'P)'=O_{p'}(G')$ (Lemma 2), by [3], Remark we have $O_{p'}(G')\cap N_G(P)=(G'P)'\cap N_{G'P}(P)=1$.

REMARK 2. In Theorems 4 and 5, the condition $O(G')\supset[G, P]$ may be replaced by the condition $N_G(P)=C_G(P)$. In fact, if $O(G')\supset[G, P]$ then $[N_G(P), P]\subset O(G')\cap P=1$, and so $N_G(P)=C_G(P)$. Conversely, suppose $N_G(P)=C_G(P)$. Since G' is a 2-nilpotent group by $(G'P)'=O(G')$ (Theorem 4) or by $G''=O(G')$ (Theorem 5), we have $[G, P]=[G'N_G(P), P]=[O(G')C_G(P), P]\subset O(G')$.

In what follows, we shall give examples which satisfy the conditions of Theorems 3, 4 and 5, respectively (cf. also Corollary 1).

EXAMPLE 1 (cf. [1], Example). Let $G=Z\times H$, where Z is an infinite cyclic group and $H=\langle a, b \mid a^4=b^3=1,\ aba^{-1}=b^{-1}\rangle$, Then $G'=\langle b\rangle$ and G has a cyclic Sylow 2-subgroup $P=\langle a\rangle$. Hence $G'P=H$. It is easy to see that G satisfies the condition (2) of Theorem 3.

Next, we consider $G=Z\times D$, where $D=\langle a, b \mid a^6=b^2=1,\ bab^{-1}=a^{-1}\rangle$ is a dihedral group of order 12. Then $G'=D'$ and a Sylow 2-subgroup P of G is an elementary abelian group $\langle a^3, b\rangle$ of order 4. Hence $G'P=D$. Again we can easily see that G satisfies the condition (2) of Theorem 3.

EXAMPLE 2 (cf. [7], Example 6.3). Let C be the complex field. Let U

be the subgroup of $GL(2, C)$ generated by $x=\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ and $y=\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$. We put $z=xyx^{-1}y^{-1}=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Let H be the group defined in Example 1. Identifying z with a^2, we construct the central product G of U and H with respect to $\langle z\rangle$. As is easily seen, H includes a Sylow 2-subgroup P of G. Hence P is a cyclic group of order 4. Since $G'=\langle z, b\rangle$, we have $G'P=H$, whence it follows $(G'P)'=H'=\langle b\rangle=O(G')$. Since $[G, P]=[H, P]\subset H'$, G satisfies the condition (2) (ii) of Theorem 4. Furtheremore, $G/O(G')$ satisfies the condition (2) (i) of Theorem 4, and this is isomorphic to the subgroup of $GL(2, C)$ generated by x, y and $\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}$.

Next, let D be the dihedral group of order 12 in Example 1. Identifying z with a^3, we construct the central product G of U and D with respect to $\langle z\rangle$. We can see that D includes a Sylow 2-subgroup P of G. Hence P is an elementary abelian group of order 4. Since $G'=\langle z, a^2\rangle$, we have $G'P=D$, whence it follows $(G'P)'=D'=\langle a^2\rangle=O(G')$. Since $[G, P]=[D, P]\subset D'$, again G satisfies the condition (2) (ii) of Theorem 4. Furtheremore, $G/O(G')$ satisfies the condition (2) (i) of Theorem 4, and this is isomorphic to the direct product of U and a group of order 2.

EXAMPLE 3. Let U be the infinite group defined in Example 2, and Q an elementary abelian group of order 9 generated by b_1 and b_2. We define a homomorphism θ: $U\to GL(2, 3)$ ($\cong Aut\ Q$) by

$$\theta(x)=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \theta(y)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now, let V be the semi-direct product of Q by U with respect to θ. Then the following relations hold:

$$xb_1x^{-1}=b_1, \quad xb_2x^{-1}=b_2^{-1}, \quad yb_1y^{-1}=b_2, \quad yb_2y^{-1}=b_1.$$

Now let $U_0=\langle x_0, y_0\rangle$ be a group which is isomorphic to U, where $x_0\leftrightarrow x$, $y_0\leftrightarrow y$. We put $G=U_0\times V$, and $z_0=x_0y_0x_0^{-1}y_0^{-1}$. Then the elementary abelian group $\langle z_0\rangle\times\langle z\rangle$ is a Sylow 2-subgroup P of G. Since $zb_1z^{-1}=b_1^{-1}$, $zb_2z^{-1}=b_2^{-1}$ and $G'=\langle z_0\rangle\times\langle z, b_1, b_2\rangle$, we have $G''=\langle b_1, b_2\rangle=O(G')=[G, P]$. As is easily seen, $C_{G'}(z)=C_{G'}(z_0z)=P$ and $C_{G'}(z_0)/\langle z_0\rangle$ is a Frobenius group with complement $P/\langle z_0\rangle$. Hence, G satisfies the condition (2) (ii) of Theorem 5. Furtheremore, $G/O(G')$ ($\cong U\times U$) satisfies the condition (2) (i) of Theorem 5.

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