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FACTOR RINGS OF A HEREDITARY AND QF-3 RING

Dedicated to Professor Goro Azumaya on his 60th birthday

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We have been studying many interesting properties of small submodules. W.W. Leonard [8] and M. Rayar [12] defined small modules and gave elementary properties of them. Recently, the author has studied non-small modules and given a class of rings which are concerned with non-small modules and located between QF-rings and QF-3 rings [4] and [5].

In this note we shall consider two conditions (*) and $(*)^*$ in [4] and [5] (see §1) and study a semi-primary ring whose every factor ring satisfies either (*) or $(*)^*$. We shall show such a ring with condition (QS) (see §1) coincides with a generalized uni-serial ring of the first category in the sense of Murase [9].

1. The main theorem

Let R be a ring with identity. We always assume that R is a semi-primary ring, namely the Jacobson radical J of R is nilpotent and R/J is artinian, and every R-module is an unitary right R-module unless otherwise stated. Let M be an R-module. By $E(M)$ and $J(M)$ we denote an injective hull and the Jacobson radical of M, respectively. If M is a small submodule in $E(M)$, we say M is a *small module* [8], [12] and if M is not a small module, we say M is *non-small module* [5]. As the dual concept to the above, we define a *non-cosmall module* N as follows: there exist a projective module P and an epimorphism $f: P \to N$ such that $\ker f$ is not essential in P.

In [4] and [5] we have introduced two conditions:

(*) *Every non-small module contains a non-zero injective module.*

$(*)^*$ *Every non-cosmall module contains a non-zero projective direct summand.*

We have shown that if R satisfies either (*) or $(*)^*$, then R is a right QF-3 ring [13] ($E(R)$ is projective by [7]) and every QF-ring satisfies both (*) and $(*)^*$. Thus, a class of rings satisfying either (*) or $(*)^*$ is located between a class of QF-rings and one of QF-3 rings when R is a left and right artinian ring. If R is left and right artinian and eR, Re have unique composition series for every primitive idempotent e, we call R a *generalized uni-serial ring* [10]. It is easily seen that every generalized uni-serial ring satisfies both (*) and (*)* (Corollary 1 to Lemma 1 below).

Following Murase [9] we say a two-sided indecomposable generalized uni-serial ring is in *the first category*, if there exists a primitive idempotent e such that eR is simple. In order to show that some rings in the new class coincide with the above rings, we introduce the conditions:

(F*) (resp. (F**)) *Every factor ring of R satisfies* (*) (resp. (*)*).

(FQF-3) *Every factor ring of R is right* QF-3. And

(QS) *If a factor ring of R is a* QF-*ring, then it is semi-simple.*

Now, we can state our theorem.

Theorem. *Let R be a semi-primary ring. Then the following statements are equivalent.*

1) *R satisfies* (F*) *and* (QS).
2) *R satisfies* (F**) *and* (QS).
3) *R satisfies* (FQF-3) *and* (QS).
4) *R is isomorphic to a factor ring of* QF-3 *and hereditary ring. And*
5) *R is a direct sum of generalized uni-serial rings of the first category.*

We know from [2], Theorem 2 and [9], Theorems 17 and 18 that the ring R in the theorem is a direct sum of factor rings of rings of tri-angular matrices over division rings when R is basic. Hence, it has a perspective form.

We shall give remarks on the above conditions.

REMARKS 1. If R is a generalized uni-serial ring of the second category [9], R satisfies (F*), (F**) and (FQF-3) but not (QS) (see §2).

2. If R is a left and right artinian, then R is a generalized uni-serial ring if and only if R satisfies (FQF-3) [6].

3. Let $K \subsetneqq L$ be fields with $[L: K] < \infty$ and

$$R = \begin{pmatrix} K & L \\ 0 & K \end{pmatrix}.$$

Then R satisfies (QS) but not any of (F*), (F**) and (FQF-3).

4. If R is a commutative artinian ring and satisfies (QS), then R is a direct sum of fields.

Because, we may assume R is a local ring with maximal ideal M. If $M \neq 0$, we could find a maximal one M' among ideals contained in M. Then R/M' is a QF-ring and so $M/M' = 0$.

2. Proof of Theorem

We always assume that R is a semi-primary ring with identity and every

R-module M is an unitary right R-module. We shall denote the Jacobson radical and the injective hull by $J(M)$ and $E(M)$, respectively. Let R be as above and $1=\sum_{i=1}^{n}\sum_{j=1}^{p(i)} g_{ij}$, where $\{g_{ij}\}$ is a set of mutually orthogonal primitive idempotents such that $g_{ij}R \approx g_{i1}R$ for any j and $g_{ij}R \not\approx g_{i'j'}R$ for $i \neq i'$. We put $g=\sum_{i=1}^{n} g_{i1}$ and $R_0=gRg$ i.e. gRg is the basic ring of R [11] and [2]. It is well known that the category of right R-modules is Morita equivalent to one of right R_0-modules. We have a one to one mapping between the set of two-sided ideals A in R and one of those A_0 in R_0 such that $A_0=gAg$ and $A=RA_0R$.

Lemma 1. *Let A be a two-sided ideal. We put $\bar{R}=R/A$ and $A_0=gAg$. Then $\bar{R}_0=R_0/A_0$ is the basic ring of $\bar{R}$.*

Proof. It is clear that $\bar{1}=\sum_{i=1}^{n}\sum_{j=1}^{p(i)} \bar{g}_{ij}$ and $\bar{g}_{ij}\bar{R} \approx \bar{g}_{i1}\bar{R}$. If $\bar{g}_{ij} \neq \bar{o}$, $\bar{g}_{ij}$ is also a primitive idempotent and $\bar{g}_{ij}\bar{g}_{i'j'}=\delta_{ii'}\delta_{jj'}\bar{g}_{ij}$. We assume $\bar{g}_{i1}\bar{R} \approx \bar{g}_{j1}\bar{R}$ for $i \neq j$. Then there exists x in $g_{j1}Rg_{i1}$ such that $xg_{i1}R+g_{j1}A=g_{j1}R$. Since $g_{i1}R \not\approx g_{j1}R$, $xg_{i1}R \subseteq g_{j1}J(R)$. Hence, $g_{j1}A=g_{j1}R$ by Nakayama's Lemma and so $g_{ik} \in A$ for any k. Thus, $\bar{R}_0$ is the basic ring of $\bar{R}$.

Corollary. *R satisfies one of* (F$*$), (F$*^*$), (FQF-3) *and* (QS) *if and only if so does the basic ring of R.*

Lemma 2. *Let R be a generalized uni-serial ring. Then every idecomposable non-small* (*resp. non-cosmall*) *module is injective* (*resp. projective*).

Proof. Every indecomposable module is uni-serial by [10]. Hence, the lemma is trivial from the defintions.

Corollary 1. *Every generalized uni-serial ring satisfies* (F$*$), (F$*^*$) *and* (FQF-3).

Corollary 2. *Let R be left and right artinian. Then the following statements are equivalent.*

1) *R satisfies* (FQF-3).
2) *R satisfies* (F$*$).
3) *R satisfies* (F$*^*$). *And*
4) *R is a generalized uni-serial ring.*

Proof. 1)$\leftrightarrow$4) is proved in [6]. Corollary 1 gives 4)$\rightarrow$2) and 3). We know 2)$\rightarrow$1) and 3)$\rightarrow$1) from [5], Propositions 2.5 and 3.4.

In order to prove the theorem, we may always assume from Lemma 1 that R is basic and $g_{i1}Rg_{i1}/g_{i1}Jg_{i1}=\Delta_i$ is a division ring. Let M_{ij} be a $\Delta_i-\Delta_j$ bimodule ($i<j$). We defined the ring of generalized upper tri-angular ma-

trices $T_n(\Delta_i; M_{ij})$ [3]. When $\Delta_i=\Delta$ for all i and $M_{ij}=\Delta$, we shall denote the usual upper tri-angular matrix ring by $T_n(\Delta)$ and the set of matrix units by $\{e_{ij}\}_{i<j}$.

Lemma 3. *Let Δ_i be division rings and $R=T_n(\Delta_i; M_{ij})$. 1) We assume $e_{ii}R$ is injective and $M_{ik}\neq 0$ and $M_{it}=0$ for all $t>k$. Then $\mathrm{Hom}_{\Delta_k}(Re_{kk}/(M_{i-1k}\oplus M_{i-2k}\oplus\cdots\oplus M_{1k}), \Delta_k)$ is isomorphic to $e_{ii}R$ by multiplications of elements in $e_{ii}R$ from the left side. Hence, $M_{ip}\neq 0$ if and only if $M_{pk}\neq 0$.*
2) *If R is a right* QF-3, *$e_{11}R$ is injective.*

Proof. 1) Since M_{ik} is the socle of $e_{ii}R$, $[M_{ik}: \Delta_k]=1$. We have the natural homomorphism φ: $e_{ii}R\to\mathrm{Hom}_{\Delta_k}(Re_{kk}/(M_{i-1k}\oplus\cdots\oplus M_{1k}), \Delta_k)$. Since $\varphi(M_{ik})\neq 0$, φ is monomorphic. Let f be in $\mathrm{Hom}_{\Delta_k}(Re_{kk}/(M_{i-1k}\oplus\cdots\oplus M_{1k}), \Delta_k)$ $=\sum_{p=i}^{k}\oplus\mathrm{Hom}_{\Delta_k}(M_{pk}, M_{ik})$ and $f=\sum f_p$; $f_p\in\mathrm{Hom}_{\Delta_k}(M_{pk}, M_{ik})$. Put $F_p\,|\,M_{pk}=f_p$ $F_p(M_{pt})=0$ for $t>k$. Then $F_p\in\mathrm{Hom}_R(M_{pk}R, e_{ii}R)$, since $M_{it}=0$ for $t>k$. Hence, there exists an element x_p in $e_{ii}R$ such that $F_p(m_{pk})=x_pm_{pk}(x_p\in M_{ip})$ for every m_{pk}, since $e_{ii}R$ is injective. Therefore, $f=\varphi(\sum x_p)$. Hence, φ is isomorphic.
2) If R is right QF-3, $E(R)\approx\sum_{i\in K}\oplus(e_{ii}R)^{n_i}$ since R is semi-primary. Being $e_{ii}Re_{11}=0$ for $i>1$, the index set K must contain 1. Hence, $e_{11}R$ is injective.

Let $R=T_n(\Delta)$ and A a two-sided ideal. It is clear [9]

$$R/A=\begin{pmatrix} \Delta & & \lfloor\!\_\ 0 & & & \\ & \Delta & \Delta\rfloor\!\_ & & & \\ & & \ddots & \lfloor\!\_ & & \\ & 0 & & \ddots & & \\ & & & & \Delta & \\ & & & & & \Delta \end{pmatrix} \qquad (2.1)\,.$$

We call such a form the *standard form* of R/A. It is easily seen that R/A is a generalized uni-serial ring of the first category. Hence, from Lemmas 1 and 2 and [2], Theorem 2 (consider $e_{nn}R$) we have

Lemma 4. *Let R be a factor ring of a* QF-3 *and hereditary ring. Then R satisfies* (F*), (F**), (FQF-3) *and* (QS).

Now, we shall consider the converse case.

Lemma 5. *If R satisfies one of* (F*), (F**), (FQF-3) *and* (QS), *then so does every factor ring of R.*
It is clear.

Lemma 6. *Let $R=T_2(\Delta_1, R_2; M_{12})$. If R is two-sided indecomposable and $e_{11}R$ is injective, then R_2 is indecomposable, where Δ_1 is a division ring and R_2 is a* *semi-primary ring.*

Proof. From the assumptions $e_{11}R$ contains a unique minimal submodule. Hence, R_2 is indecomposable if so is R.

Lemma 7. *Let R be a semi-primary, two-sided indecomposable and basic ring. We assume $J^2=0$. If R satisfies* (FQF-3) *and* (QS), *then R is isomorphic to $T_n(\Delta)/J(T_n(\Delta))^2$, where Δ is a division ring.*

Proof. Let $R=\sum_{i=1}^{n}\oplus e_iR\oplus\sum_{j=1}^{m}\oplus f_jR$ be a decomposition of R with indecomposable modules e_iR and f_jR, where the e_iR is injective and the f_jR is small (see [5], Theorem 1.3). We quote here the argument in [6], Lemma in pp. 404–405. We know $\sum\oplus e_iR$ is faithful. Let $x\neq 0$ be in f_jR. Then $(\sum\oplus e_iR)x \neq 0$ and so there exists e_ir such that $0\neq e_irx=e_irf_jx\in Jx$. Hence, $x\notin f_jJ$ since $J^2=0$. Therefore, f_jR is simple if $f_jR\neq 0$. Since e_iR is injective and $J^2=0$, e_iR is uni-serial. Accordingly, R is right artinian. First, we assume $m=0$. Then R is self-injective and so a QF-ring (see [1], Theorem 1). Therefore, R is a division ring by (QS). Thus, we may assume $m\neq 0$. We know from the above that f_1R is simple. Hence, $f_1Rg=0$ for any primitive idempotent g ($\not\approx f_1$) and $f_1Rf_1=\Delta$ is a division ring. Thus, we have

$$R=\begin{pmatrix} R_1 & FRf_1 \\ 0 & \Delta \end{pmatrix} \qquad (2.2),$$

where $F=1-f_1$ and $R_1=FRF$ satisfies (QS) and (FQF-3). We first assume $s=n+m=2$. Then $n=m=1$. Hence, R_1 is a division ring from the case $m=0$. Therefore, $R\approx T_2(\Delta)$ by [2], Theorem 2 and [3], Theorem 1. Now, we shall prove the lemma by induction on $s=s(R)$ (we assume $m\neq 0$). We have done it when $s\leqslant 2$. Since $s(R)>s(R_1)$, $R_1\approx\sum\oplus T_{n_i}(\Delta_i)/J(T_{n_i}(\Delta_i))^2$ by the induction, where the Δ_i is a division ring. Hence, we obtain $R=T_s(\Delta_1, \Delta_2\cdots, \Delta_{s-1}, \Delta; M_{ij})$. Lemma 3,2) shows that $e_{11}R$ is injective. It is clear $e_{kk}Re_{11}=0$ for $k\neq 1$. We put $F'=1-e_{11}$ and $R_1'=F'RF'$. Then we have

$$R=\begin{pmatrix} \Delta_1 & e_{11}RF' \\ 0 & R_1' \end{pmatrix} \qquad (2.3).$$

Here R_1' is two-sided idecomposable by Lemma 6. Hence, $R_1'\approx T_{s-1}(\Delta')/J(T_{s-1}(\Delta'))^2$ by the hypothesis of induction. Now R is of the form

$$\begin{pmatrix} \Delta_1 & A_2 & \cdots\cdots & & A_s \\ & \Delta' & \Delta' & & \\ & & \ddots & \ddots & 0 \\ & & & \ddots & \ddots \\ & 0 & & & \ddots\ \Delta' \\ & & & & \Delta' \end{pmatrix} \qquad (2.4)$$

Since $e_{11}R$ contains a unique (minimal) submodule, only one A_i is not zero. If $i\neq 2$, $A_2=0$ implies $M_{i-1i}=\Delta'=0$ by Lemma 3. Hence, $A_i=0$ for $i>2$. Since $s\geqslant 3$, we have $\Delta'\approx\Delta_1$ and $A_2=\Delta_1$ by the induction (cf. [3], Lemma 13).

Lemma 8. *If R satisfies* (FQF-3) *and* (QS), *then R is isomorphic to a factor ring of a semi-primary hereditary ring R' such that $R/J(R)\approx R'/J(R')$.*

Proof. We know $R/J^2\approx\sum\oplus T_{n_i}(\Delta_i)/J(T_{n_i}(\Delta_i))^2$ by Lemmas 5 and 7. Hence, gl. dim $R/J^2<\infty$ by [3], Theorem 3. Therefore, we obtain the lemma by [3], Theorem 5 and its proof.

Since $R/J(R)\approx R'/J(R')$, R' is basic and $R'\approx T_n(\Delta_i; M_{ij})$ by [3], Theorem 4′. Let $\{f_{ij}\}$ be the usual matrix units in R'. Then $gR'f_{11}=0$ for any primitive idempotent g with $gR'\not\approx f_{11}R'$. Let φ: $R'\to R$ be the ring epimorphism. Then $J(R')=\varphi^{-1}(J(R))$ and $\{e_{ii}=\varphi(f_{ii})\}$ is a complete set of mutually orthogonal primitive idempotents in R. If $0\neq e_{jj}Re_{11}=\varphi(f_{jj}R'f_{11})$ implies $j=1$. Furthermore, $e_{11}J(R)e_{11}=\varphi(f_{11}J(R')f_{11})=0$. From now on, we shall denote e_{ii} by e_i. Then $\Delta_1=e_1Re_1$ is a division ring from the above.

Lemma 9. *If R satisfies* (FQF-3) *and* (QS), *then R is isomorphic to $\sum\oplus T_{n_i}(\Delta_i)/C_i$, where C_i is a two-sided ideal in $T_{n_i}(\Delta_i)$.*

Proof. We may assume R is a two-sided indecomposable. We shall use the notations above. Put $F=1-e_1$. Then

$$R\approx\begin{pmatrix}\Delta_1 & A\\ 0 & R_1\end{pmatrix}\qquad(2.5).$$

We shall prove the lemma by induction on n, where $1=\sum_{i=1}^{n}e_i$. If $n\leqslant 2$, the lemma is true by Lemma 7. We assume $n\geqslant 3$. Then since $e_{11}R$ is injective by Lemma 3, 2), $R_1\approx T_{n-1}(\Delta)/C$ by Lemma 6 and the induction. Thus, we obtain

$$R=\begin{pmatrix}\Delta_1 & A_2\cdots\cdots A_n & & \\ & \Delta & & 0\\ & & \ddots\ \Delta & \\ 0 & & & \ddots \\ & & & \Delta\end{pmatrix}\qquad(2.6).$$

If we take a two-sided ideal Re_n and use the induction hypothesis, we know $\Delta_1=\Delta$ and $A_i(i<n)$ is equal to either zero or Δ (cf. [3], Lemma 13). We assume $A_n\neq 0$. Since e_1R is injective and has a simple socle, $[A_n:\Delta]=1$ as a right Δ-module. Put $A_n=u\Delta$. We know by Lemma 3 that every Δ-endomorphism of $u\Delta$ is given by a unique element of $\Delta=e_1Re_1$. Let x be in e_1Re_1,

then $xu=u\delta(x)$, where δ is a ring homomorphism of Δ. Therefore, $\delta(\Delta)=\Delta$ from the above and so $A_n=\Delta$ as a two-sided Δ-module, if $A_n\neq 0$. Now we may assume $A_k=\Delta$ and $A_{k+1}=\cdots=A_n=0$. We shall show $A_2\neq 0$. Assume $A_2=A_3=\cdots=A_{s-1}=0$ and $A_s=\Delta$ for some $s\leqslant k$. We put $D=\sum_{p>s+1}^{n}\oplus Re_p$. Then $\bar{R}=R/D$

$$\approx\begin{pmatrix} \Delta & 0 & 0 & \cdots\cdots & \Delta \\ & \Delta & & \lfloor\_ \; 0 & E_2 \\ & & \ddots \; \Delta & \lfloor\_ & E_3 \\ & 0 & & \ddots & \vdots \\ & & & \ddots & E_{s-1} \\ & & & & \Delta \end{pmatrix} \qquad (2.7).$$

Since $\overline{e_1R}$ is $\bar{R}$-injective, $E_2=\cdots=E_{s-1}=0$ by Lemma 3. However, R_1 is indecomposable and is of the standard form. Hence, $E_{s-1}\neq 0$, which is a contradiction. Accordingly, $A_2\neq 0$ and $e_2Re_k\neq 0$ by Lemma 3. Again, since R_1 is of standard form, $e_jRe_k\neq 0$ for $j\leqslant k$. Therefore, $R\approx T_n(\Delta)/C$.

Lemma 10 ([9], Theorems 17 and 18). *Let R be a two-sided indecomposable basic and generalized uni-serial ring. If there exists a primitive idempotent e such that eR is simple, then R is isomorphic to* $\mathrm{T}_n(\Delta)/C$.

Proof. R satisfies (F*) by Corollary 1 to Lemma 2. First we assume $J^2=0$. We use the same notations in the proof of Lemma 7. We assume $m=0$ and e_nR is simple. Since R is a QF-ring, e_nR is a two-sided ideal. Hence, R is a division ring. If $m\neq 0$, we obtain the form (2.2) and so (2.3). Hence, we can use the same argument. In general case, noting that e_1R is not simple in (2.3), we can use the induction. Therefore, Lemma 8 is true for the ring in the lemma. Again we can use the same argument in the proof of Lemma 9.

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