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ON THE SPECTRAL DISTRIBUTIONS OF CERTAIN INTEGRO-DIFFERENTIAL OPERATORS WITH RANDOM POTENTIAL

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1. Introduction

Let L be the generator of a d-dimensional symmetric Lévy process. It is well known that L can be represented as

$$(1.1)\qquad \begin{aligned} Lu(x) &= \sum_{i,j=1}^{d} a_{ij}\frac{\partial^2 u(x)}{\partial x^i \partial x^j} \\ &\quad + \int_{R^d-\{0\}} \Big(u(x+y)-u(x)-\frac{\langle y, \nabla u(x)\rangle}{1+|y|^2}\Big) n(dy) \end{aligned}$$

for suitably smooth functions u, where $(a_{ij})_{1\leq i,j\leq d}$ is a symmetric non-negative definite matrix and $n(dy)$, a symmetric measure on $\boldsymbol{R}^d-\{0\}$ satisfying

$$\int_{R^d-\{0\}} |y|^2(1+|y|^2)^{-1} n(dy) < \infty .$$

Let $\{q(x, \omega)\colon x\in \boldsymbol{R}^d,\ \omega\in\Omega\}$ be a stationary random field having continuous sample functions over a probability space $(\Omega, \mathcal{B}, P)$. We will consider the family $\{-L+q(x, \omega)\colon \omega\in\Omega\}$ of operators depending on the random parameter $\omega\in\Omega$. In case of $L=\frac{1}{2}\Delta$ (Δ is the Laplacian), S. Nakao [6] has shown the existence of the spectral distribution function $\rho(\lambda)$ of $\{-\frac{1}{2}\Delta+q(x, \omega)\colon \omega\in\Omega\}$ and investigated the asymptotic behaviour of $\rho(\lambda)$. The purpose of this paper is to extend Nakao's results to the case of the general family $\{-L+q(x, \omega)\colon \omega\in\Omega\}$ satisfying some mild conditions.

The contents of the paper are as follows.

In §§2 and 3 we shall give some preliminary results. Let m denote the Lebesgue measure on $\boldsymbol{R}^d$ and $Q(\xi)$, the exponent of a symmetric Lévy process $\boldsymbol{X}=(X_t, P_x\colon t\geq 0, x\in \boldsymbol{R}^d)$:

$$(1.2)\qquad E_0[\exp\{i\langle\xi, X_t\rangle\}] = \exp\{-tQ(\xi)\} .$$

The exponent $Q(\xi)$ is of the form

$$(1.3)\qquad Q(\xi)=\sum_{i,j=1}^{d}a_{ij}\xi^i\xi^j+\int_{R^d-\{0\}}(1-\cos\langle\xi,y\rangle)n(dy)\,.$$

The process $\boldsymbol{X}$ is said to be a symmetric stable process of order α $(0<\alpha\leqq 2)$ if $Q(\xi)$ has the property that $Q(\lambda\xi)=\lambda^\alpha Q(\xi)$ for $\lambda>0$. We shall assume the existence of the "nice" transition density function $p(t,x,y)$ of the process $\boldsymbol{X}$ relative to $\boldsymbol{m}$. Then we can construct the conditional process of $\boldsymbol{X}$ starting from $x\in\boldsymbol{R}^d$ and terminating in $y\in\boldsymbol{R}^d$ at time $t>0$ for every x, t, and y, which will play an essential role throughout this paper. This conditional process is denoted by $((X_u)_{u\in[0,t]}, P_{0,x}^{t,y})$ and referred to as the $(0,x\colon t,y)$-pinned process of $\boldsymbol{X}$. The fundamental relation between the original process $\boldsymbol{X}$ and the pinned process of $\boldsymbol{X}$ is the following:

$$(1.4)\qquad P_{0,x}^{t,y}(\Lambda)=p(t,x,y)^{-1}E_x[p(t-u,X_u,y);\Lambda]$$

for each $\Lambda\in\sigma(X_s\colon s\in[0,u])$ with $0<u<t$. The principal part of construction of pinned processes will be done in §2 in more general contexts and the case of Lévy processes will be treated in §3. In §3 we shall also collect some known facts on symmetric Lévy processes, which will be systematically used in §4; we mainly follow M. Fukushima [4] for those terminologies and general results on the semigroup, the generator, and the Dirichlet form associated with the symmetric process $\boldsymbol{X}$.

In §§4 and 5 we shall be concerned with the existence of the spectral distribution function of $\{-L+q(x,\omega)\colon \omega\in\Omega\}$. Given a rectangle V, we consider the eigenvalue problem $(-L+q(x,\omega))u(x)=\lambda u(x)$ with the Dirichlet condition $u=0$ on V^c; the precise formulation of such eigenvalue problem will be given in §4. Let $\lambda_{V,1}^\omega\leqq\lambda_{V,2}^\omega\leqq\cdots$ be the eigenvalues of this problem and define

$$(1.5)\qquad \rho_V^\omega(\lambda)=m(V)^{-1}\sum_{\lambda_{V,i}^\omega\leqq\lambda}1,\ \lambda\in\boldsymbol{R}^1\,.$$

The spectral distribution function $\rho(\lambda)$ will be defined by the limit function of $E[\rho_V^\omega(\lambda)]$ for $V\to\boldsymbol{R}^d$ if it exists, where E denotes the expectation with respect to P. We assume the following two conditions:

(A) $\qquad \exp\{-tQ(\xi)^{1/2}\}\in L^1(\boldsymbol{R}^d)$ for every $t>0$.

(B) There exists a constant $r>2$ such that

$$\exp\{\int_0^t q^-(X_s,\omega)ds\}\in L^r(P\times P_0)\quad\text{for every }\ t>0\,,$$

where $q^-=\max(-q,0)$. In Theorem 5.1, we shall prove the existence of the spectral distribution function $\rho(\lambda)$ of $\{-L+q(x,\omega)\colon\omega\in\Omega\}$ and the relation

$$(1.6)\qquad \int_{-\infty}^{\infty}e^{-t\lambda}d\rho(\lambda)=p(t,0,0)E\times E_{0,0}^{t,0}[\exp\{-\int_0^t q(X_s,\omega)ds\}],\ t>0\,,$$

where $E\times E_{0;0}^{t;0}$ denotes the expectation with respect to the product measure $P\times P_{0;0}^{t;0}$. Further we show in Theorem 5.2 that if $\{q(x, \omega)\}$ is ergodic, then $\rho_V^{\omega}(\lambda)$ tends to $\rho(\lambda)$ almost surely when $V\to \boldsymbol{R}^d$.

Nakao [6] has proved these theorems in the case of the Schrödinger operator under the condition (B); condition (A) is valid for all the symmetric stable processes.

In §6 we shall investigate the asymptotic behaviour of $\rho(\lambda)$ for a special class of non-negative random potentials. Let us consider a potential $\{q(x,\omega)\}$ of the form

$$q(x,\omega)=\int_{\boldsymbol{R}^d}\varphi(x-y)p^{\omega}(dy)\,, \tag{1.7}$$

where $\varphi(x)$ denotes a non-negative continuous function defined on $\boldsymbol{R}^d$ such that $\varphi(x)\not\equiv 0$ and $\{p^{\omega}(dy)\colon \omega\in\Omega\}$, the Poisson random measure with characteristic measure $\nu\cdot m$ (ν a positive constant) over $(\Omega, \mathscr{B}, P)$. It is known that $\{q(x, \omega)\}$ in (1.7) defines an ergodic stationary random field having continuous sample functions if $\varphi(x)=O(|x|^{-(d+\varepsilon)})$ $(|x|\to\infty)$ for some constant $\varepsilon>0$. Therefore the spectral distribution function $\rho(\lambda)$ of $\{-L+q(x,\omega)\colon \omega\in\Omega\}$ exists and satisfies $\rho(0+)=0$. We shall obtain the three different estimates on the exponential decay of $\rho(\lambda)$ for $\lambda\downarrow 0$. Each estimate will be distinguished according to the order of magnitude of $Q(\xi)$ at the origin and that of $\varphi(x)$ at infinity.

First let α be such as $0<\alpha\leqq 2$ and $Q^{(\alpha)}(\xi)$, the exponent of a symmetric stable process $\boldsymbol{X}^{(\alpha)}$ of order α. We assume that $Q(\xi)$ is close to $Q^{(\alpha)}(\xi)$ near the origin (see the condition (C) of §6) and that $\varphi(x)=o(|x|^{-(d+\alpha)})$ $(|x|\to\infty)$. Then Theorem 6.2 asserts that

$$\lim_{\lambda\downarrow 0}\lambda^{d/\alpha}\log\rho(\lambda)=-\nu(\lambda_{\alpha})^{d/\alpha}\,, \tag{1.8}$$

where λ_{α} is a certain constant determined by $Q^{(\alpha)}(\xi)$; the definition of λ_{α} will be given in §6. This is the case when the contribution of $\varphi(x)$ to the evaluation is negligible.

Next let $0<\beta<\alpha\leqq 2$. We assume that $K=\lim_{|x|\to\infty}|x|^{d+\beta}\varphi(x)>0$ and $Q(\xi)=O(|\xi|^{\alpha})(|\xi|\downarrow 0)$. Then we shall prove in Theorem 6.3 that

$$\lim_{\lambda\downarrow 0}\lambda^{d/\beta}\log\rho(\lambda)=-C_1(\nu,\beta,K); \tag{1.9}$$

the definition of $C_1(\nu,\beta,K)$ will be given in Theorem 6.3. In this case the effect of $Q(\xi)$ to the evaluation is negligible.

Finally we assume that $Q(\xi)=O(|\xi|^{\alpha})$ and $\varphi(x)\asymp|x|^{-(d+\alpha)}$. Then Theorem 6.4 proves that

$$-\infty<\varliminf_{\lambda\downarrow 0}\lambda^{d/\alpha}\log\rho(\lambda)\leqq\varlimsup_{\lambda\downarrow 0}\lambda^{d/\alpha}\log\rho(\lambda)<0\,.$$

Suppose further that $Q(\xi)$ is close to $Q^{(\alpha)}(\xi)$ near the origin and $K=\lim_{|x|\to\infty}|x|^{d+\alpha}\varphi(x)>0$. Then the limit of $\lambda^{d/\alpha}\log\rho(\lambda)$, if exists, would depend on both $Q(\xi)$ and $\varphi(x)$; we have never succeeded in finding any exact formula for the asympototic behaviour of $\rho(\lambda)$.

In the case of $Q(\xi)=Q^{(\alpha)}(\xi)$ the above three results are obviously valid. In the case of the Schrödinger operator ($\alpha=2$ and $Q(\xi)=Q^{(2)}(\xi)=\frac{1}{2}|\xi|^2$) the relations (1.8) and (1.9) were first obtained by Nakao [6] and L.A. Pastur [8] respectively.

The proof of these results will be given in §§8 and 9. We now outline the proof of (1.8) and (1.9). Appealing to the Minlos-Povzner Tauberian theorem and noting the relation (1.6), we can reduce the relations (1.8) and (1.9) to the following relations respectively:

$$\lim_{t\to\infty} t^{-d/(d+\alpha)}\log I(t) = -k(\nu\ L^{(\alpha)})\,, \tag{1.10}$$

$$\lim_{t\to\infty} t^{-d/(d+\beta)}\log I(t) = -\kappa(\nu\ \beta, K)\,, \tag{1.11}$$

where

$$I(t)=p(t,0,0)E\times E^{t;0}_{0;0}[\exp\{-\int_0^t q(X_s,\omega)ds\}],\quad t>0 \tag{1,12}$$

and the definitions of $k(\nu, L^{(\alpha)})$ and $\kappa(\nu,\beta,K)$ are given in Theorems 6.2' and 6.3' respectively. The proof of (1.10) and (1.11) will be split into the lower estimates and the upper estimates.

In Theorem 7.1 (a generalization of the lemma of Pastur [8]) we will give the bounds for $I(t)$. It should be noted that the lower bound of $I(t)$ involves the Dirichlet form of $\boldsymbol{X}$. By making use of these bounds we can prove, in §8, the lower estimates for (1.10) and (1.11) and, in §9, the upper estimate for (1.11). The upper bound of $I(t)$ in Theorem 7.1, however, is not sufficient to prove the upper estimate for (1.10). In the case of the Schrödinger operator Nakao [6] has shown that, using the relation (1.4), the upper estimate for (1.10) can be reduced to the asymptotic evaluation for the Wiener sausage by M.D. Donsker and S.R.S. Varadhan [2]. Since Nakao's method is quite general and since Donsker and Varadhan [2] have also given the asymptotic evaluation for the sausages of symmetric stable processes, we can immediately establish the case of $Q(\xi)=Q^{(\alpha)}(\xi)$. To prove the upper estimate for (1.10) in the case of Theorem 6.2 we have only to extend the results of Donsker and Varadhan [2] to the case of processes which are close to the stable process $\boldsymbol{X}^{(\alpha)}$ in the sense that the condition (C) in §6 is satisfied. The proof of this extension will be given elsewhere.

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2. Construction of pinned processes

Let S be a locally compact separable Hausdorff space and let $\mathscr{B}(S)$ be the topological Borel field of S. Let $\boldsymbol{X}=(W, X_t, P_x)$ be a conservative Hunt process with state space S. Assume that there exist a positive Radon measure m on $(S, \mathscr{B}(S))$ and a transition density function $p(t,x,y)$, $t>0$, $x\in S$, $y\in S$ of $\boldsymbol{X}$ relative to m satisfying the following three conditions:

(p.1) For each $t>0, p(t,\cdot,\cdot)$ is a $\mathscr{B}(S)\times\mathscr{B}(S)$-measurable function defined throughout $S\times S$.

(p.2) $0<p(t,x,y)<\infty$ for all $t>0$, $x\in S$, and $y\in S$.

(p.3) $p(s+t,x,y)=\int_S p(s,x,z)p(t,z,y)m(dz)$ for all $s>0$, $t>0$, $x\in S$, and $y\in S$.

Under these assumptions we can define, for each $t>0$ and each $y\in S$, a time-inhomogeneous Markov transition function $P^{t,y}(s, x, u, E)$, $0\leqq s<u<t$, $x\in S$, $E\in\mathscr{B}(S)$ by

$$P^{t,y}(s,x,u,E) = p(t-s,x,y)^{-1}\int_E p(u-s,x,z)p(t-u,z,y)m(dz)\,. \tag{2.1}$$

Later we will further assume the following condition:

(p.4) $p(t,x,y)=p(t,y,x)$ for all $t>0$, $x\in S$, and $y\in S$.

Before stating the theorem we introduce some notations and prepare a lemma.

For each $0<t\leqq\infty$, let $W[0,t)$ be the set of all S-valued right continuous functions on $[0,t)$ having left hand limits on $(0,t)$ and Y_s^t, the coordinate function $w\to w(s)$ on $W[0,t)$. Let $\mathscr{F}_I^t=\sigma(Y_s^t\colon s\in I)$ for each interval $I\subset[0,t)$. Since the process $\boldsymbol{X}$ is conservative, we can assume that the basic space W is identical with $W[0,\infty)$ and $X_t=Y_t^\infty$ for all $t\in[0,\infty)$. We write $\mathscr{F}_I$ for $\mathscr{F}_I^\infty$. Define, for each $s\geqq 0$, the shift operator θ_s on W by $\theta_s w=w(\cdot+s)$ and define, for each $t>0$, the restriction mapping π_t of W into $W[0,t)$ by $\pi_t w=w|_{[0,t)}$.

Lemma 2.1. *Let $t>0$, $I\subset[0,t)$, and $0\leqq s<u$. Then*

(i) $\mathscr{F}_I=\pi_t^{-1}\mathscr{F}_I^t$;

(ii) $\mathscr{F}_{[s,u]}=\theta_s^{-1}\mathscr{F}_{[0,u-s]}$;

(iii) *if $B\in\mathscr{F}_{[s,u]}$, then $\theta_s(B)\in\mathscr{F}_{[0,u-s]}$ and $B=\theta_s^{-1}\theta_s(B)$;*

(iv) *the mapping $\mathscr{F}_{[s,u]}\ni B\to\theta_s(B)\in\mathscr{F}_{[0,u-s]}$ preserves set operations.*

Proof. The first two assertions are obvious. The third assertion (iii) follows from (ii) and the fact that the mapping θ_s: $W\to W$ is onto. To see (iv) it suffices to show that $\theta_s(B^c)=\theta_s(B)^c$ for $B\in\mathscr{F}_{[s,u]}$. But, by (iii), we get $B^c=(\theta_s^{-1}\theta_s(B))^c=\theta_s^{-1}(\theta_s(B)^c)$. So we have $\theta_s(B^c)=\theta_s(B)^c$ since θ_s: $W\to W$ is onto.

Theorem 2.1. *Let $X=(W,X_t,P_x)$ be a Hunt process on S with a transition density function $\{p(t,x,y)\}$ relative to a positive Radon measure m satisfying* (p.1)~(p.3). *Then, for each $t>0$ and each $y\in S$, there exists a time-inhomogeneous Markov process $Y^{t,y}=(W[0,t),(Y_u^t)_{u\in[s,t)}, Q_{s,x}^{t,y}: s\in[0,t), x\in S)$ with the transition function $\{P^{t,y}(s,x,u,E)\}$ defined by* (2.1).

DEFINITION 2.1. We call the Markov process $Y^{t,y}$ of the above theorem the *(t,y)-conditional process* of X(corresponding to $\{p(t,x,y)\}$ and m).

Proof of Theorem 2.1. Let $0\leqq s<t$, $x\in S$, and $y\in S$ be fixed. We have only to construct a probability measure $Q=Q_{s,x}^{t,y}$ on $(W[0,t),\mathcal{F}_{[s,t)}^t)$ such that

$$(2.2)\qquad \begin{aligned} &Q(Y_{u_i}^t\in E_i,\ i=1,\cdots,k) \\ &=\int_{E_1}\int_{E_2}\cdots\int_{E_k} P^{t,y}(s,x,u_1,dz_1)P^{t,y}(u_1,z_1,u_2,dz_2) \\ &\quad\cdots P^{t,y}(u_{k-1},z_{k-1},u_k,dz_k) \end{aligned}$$

holds for each $s=u_0<u_1<\cdots<u_k<u_{k+1}=t$ and $E_i\in\mathcal{B}(S)$, $i=1,\cdots,k$. But, by (2.1), the right hand side of (2.2) is equal to

$$p(t-s,x,y)^{-1}\int_{E_1\times\cdots\times E_k}\!\!\cdots\int \Pi_{i=1}^{k+1}p(u_i-u_{i-1},z_{i-1},z_i)m(dz_1)\cdots m(dz_k)\,,$$

where $z_0=x$ and $z_{k+1}=y$. Further, by the Markov property of $X=(W,X_u,P_x)$, this is equal to

$$p(t-s,x,y)^{-1}E_x[p(t-u_k,X_{u_k-s},y);\ X_{u_i-s}\in E_i,\ i=1,\cdots,k]\,,$$

where E_x denotes the expectation with respect to P_x. Since

$$\{X_{u_i-s}\in E_i,\ i=1,\cdots,k\}=\theta_s(\pi_t^{-1}\{Y_{u_i}^t\in E_i,\ i=1,\cdots,k\})\,,$$

it suffices to construct a probability measure Q on $(W[0,t),\mathcal{F}_{[s,t)}^t)$ such that if $s<u<t$, then, for all $B\in\mathcal{F}_{[s,u]}^t$,

$$(2.3)\qquad Q(B)=p(t-s,x,y)^{-1}E_x[p(t-u,X_{u-s},y);\ \theta_s(\pi_t^{-1}B)]\,.$$

It follows from Lemma 2.1 that, for each $u\in(s,t)$, the mapping $\mathcal{F}_{[s,u]}^t\ni B\to\theta_s(\pi_t^{-1}B)\in\mathcal{F}_{[0,u-s]}$ preserves set operations. Let $M_u=p(t-u,X_{u-s},y)$, $u\in[s,t)$. Then $\{M_u,\mathcal{F}_{[0,u-s]};\ u\in[s,t)\}$ is a martingale over $(W,\mathcal{F}_{[0,t-s)},P_x)$ such that $E_x[M_u]=p(t-s,x,y)$, $u\in[s,t)$. Thus the right hand side of (2.3) is a probability measure in $B\in\mathcal{F}_{[s,u]}^t$. Let $\{t_n\}\subset(s,t)$ be a sequence such that $t_n\uparrow t$ and let $\mathcal{F}_n=\mathcal{F}_{[s,t_n]}^t$. Define probability measures Q_n, $n=1,2,\cdots$ by

$$Q_n(B)=p(t-s,x,y)^{-1}E_x[M_{t_n};\ \theta_s(\pi_t^{-1}B)],\quad B\in\mathcal{F}_n\,.$$

Then, from the martingale property of M_u, $u\in[s,t)$, $Q_1,Q_2,\cdots$ is a consistent sequence of probability measures on $\mathcal{F}_1, \mathcal{F}_2, \cdots$. To complete the proof it suffices to show that $Q_1, Q_2, \cdots$ is extendable to a probability measure on $\mathcal{F} \equiv \mathcal{F}^t_{[s,t]}$.

To this end we note the following: (i) $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ and $\bigcup_n \mathcal{F}_n$ generates $\mathcal{F}$, and (ii) $(W[0,t), \mathcal{F}_n)$ is a standard Borel space (see K.R. Parthasarathy[7]) for each $n=1,2 \cdots$; the first assertion is obvious and the second follows from the following observation: Let $W[s,t_n]$ be the set of all S-valued right continuous functions on $[s,t_n]$ having left hand limits on $(s,t_n]$ and let $\mathcal{F}'_n$ be the σ-algebra of $W[s,t_n]$ generated by all cylinder sets of $W[s,t_n]$. Then one can see that the measurable space $(W[s,t_n], \mathcal{F}'_n)$ is a Lusin space (see C. Dellacherie [1: §1]). Noting that any Lusin space is a standard Borel space and that the σ-algebras $(W[0,t), \mathcal{F}_n)$ and $(W[s,t_n], \mathcal{F}'_n)$ are σ-isomorphic, we have the desired assertion (ii). Let $A_1, A_2, \cdots$ be any sequence of subsets of $W[0,t)$ such that A_n is an atom of $\mathcal{F}_n$ (*i.e.*, $A_n \in \mathcal{F}_n$ and the relations $A \subset A_n$, $A \in \mathcal{F}_n$ imply that $A=A_n$ or $A=\phi$) for each n and that $A_1 \supset A_2 \supset \cdots$. If we check that $\bigcap_n A_n \neq \phi$, then it follows from Theorem 4.1 of [7; V] that every consistent sequence of probability measures is extendable to a probability measure on $\mathcal{F}$. But it is easy to see that, for each atom A_n, there exists an S-valued right continuous function w_n on $[s,t_n]$ having left hand limits on $(s,t_n]$ such that

$$A_n = \{w \in W[0,t) \colon w(u) = w_n(u) \quad \text{for all} \quad u \in [s,t_n]\} .$$

Thus the condition $A_1 \supset A_2 \supset \cdots$ implies that an S-valued right continuous function w_0 on $[s,t)$ is well defined by $w_0(u)=w_n(u)$, $u \in [s,t_n]$, $n=1,2,\cdots$, and w_0 has left hand limits on (s,t). Therefore

$$\bigcap_n A_n = \{w \in W[0,t) \colon w(u) = w_0(u) \quad \text{for all} \quad u \in [s,t)\} \neq \phi .$$

This completes the proof.

Corollary. *Let $Y^{t,y}$ be as in Theorem* 2.1. *Then we have the following*:
(i) *If $0 \leqq s < u < t$, then, for all $B \in \mathcal{F}^t_{[s,u]}$,*

$$\text{(2.4)} \qquad Q^{t;y}_{s;x}(B) = p(t-s,x,y)^{-1} E_x[p(t-u, X_{u-s}, y);\ \theta_s(\pi_t^{-1} B)] .$$

(ii) *For each $s=u_0<u_1<\cdots<u_n<u_{n+1}=t$ and $E_i \in \mathcal{B}(S)$, $i=1,\cdots,n$*

$$\begin{aligned}\text{(2.5)} \qquad & Q^{t;y}_{s;x}(Y^t_{u_i} \in E_i,\ i = 1,\cdots,n) \\ & = p(t-s,x,y)^{-1} \int\cdots\int_{E_1\times\cdots\times E_n} \prod_{i=1}^{n+1} p(u_i-u_{i-1}, z_{i-1}, z_i) m(dz_1)\cdots m(dz_n) ,\end{aligned}$$

where $z_0=x$, $z_{n+1}=y$.

We next consider the family $\{Y^{t,y} \colon t>0,\ y \in S\}$.

Proposition 2.1. *Under the assumption of Theorem* 2.1, $\{Y^{t,y} \colon t>0, y \in S\}$ *satisfies the following*:

(i) $(x,y) \to Q^{t;y}_{s,x}(B)$ *is* $\mathcal{B}(S)\times\mathcal{B}(S)$*-measurable for each* $B\in\mathcal{F}^t_{[s,t)}$.
(ii) *For each* $B\in\mathcal{F}^t_{[s,t)}$ *and each* $E\in\mathcal{B}(S)$

$$(2.6)\qquad P_x(\theta_s(\pi_t^{-1}B)\cap\{X_t\in E\}) = \int_E Q^{t;y}_{s,x}(B)p(t-s,x,y)m(dy)\,.$$

(iii) *For each* $0\leqq s<t$, $x\in S$ *and* $y\in S$

$$\{(Y^t_u)_{u\in[s,t)},\ Q^{t;y}_{s,x}\}\approx\{(Y^t_{u-s})_{u\in[s,t)},\ Q^{t-s;y}_{0,x}\}\,.$$

REMARK. Throughout this paper "$\boldsymbol{Y}\approx\boldsymbol{Z}$" will always mean that two stochastic processes $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are identical in law.

Proof of Proposition 2.1. The first and the third assertions immediately follow from (2.5). To see (ii) we may assume that $B\in\mathcal{F}^t_{[s,u]}$ for some $u\in(s,t)$. Then by (2.4), we have

$$\begin{aligned}&\int_E Q^{t;y}_{s,x}(B)p(t-s,x,y)m(dy)\\ &\quad = E_x[\int_E p(t-u,X_{u-s},y)m(dy);\ \theta_s(\pi_t^{-1}B)]\\ &\quad = E_x[P_{X_{u-s}}(X_{t-u}\in E);\ \theta_s(\pi_t^{-1}B)]\,.\end{aligned}$$

Making use of the Markov property of $\boldsymbol{X}=(X_u,\ P_x)$ we have (ii).

From now on we further assume the following:
(p.4) $p(t,x,y)=p(t,y,x)$ for all $t>0$, $x\in S$ and $y\in S$.

Proposition 2.2. *Let* $\boldsymbol{X}=(W,X_t,P_x)$, $\{p(t,x,y)\}$, *and* m *be as in Theorem* 2.1 *and* $\boldsymbol{Y}^{t,y}$, *the* (t,y)*-conditional process of* $\boldsymbol{X}$ *for* $t>0$ *and* $y\in S$. *Suppose that* $\{p(t,x,y)\}$ *satisfies the condition* (p.4). *Then we have the following*:

(i) *For each* $0\leqq s<t$, $x\in S$, *and* $y\in S$

$$\{(Y^t_u)_{u\in(s,t)},\ Q^{t;y}_{s,x}\}\approx\{(Y^t_{s+t-u})_{u\in(s,t)},\ Q^{t;x}_{s,y}\}\,.$$

(ii) *For each* $0\leqq s<t$, $x\in S$, *and* $y\in S$

$$(2.7)\qquad Q^{t;y}_{s,x}(\lim_{u\uparrow t} Y^t_u \text{ exists and is equal to } y) = 1\,.$$

Proof. The first assertion follows from (2.5) and (p.4). To see (ii) note that $Q^{t;y}_{s,x}(\lim_{u\downarrow s} Y^t_u=Y^t_s=x)=1$ for each $0\leqq s<t, x\in S$, and $y\in S$. Hence by (i), we have

$$Q^{t;y}_{s,x}(\lim_{u\uparrow t} Y^t_u = y) = Q^{t;x}_{s,y}(\lim_{u\downarrow s} Y^t_u = y) = 1\,,$$

which completes the proof.

Since the process $\{W[0,t),\ (Y^t_u)_{u\in[s,t)},\ Q^{t;y}_{s,x}\}$ satisfies (2.7), we can replace the basic space $(W[0,t), \mathcal{F}^t_{[s,t)})$ by $(W, \mathcal{F}_{[s,t]})$. Precisely, we will construct a probability measure $P^{t;y}_{s;x}$ on $(W, \mathcal{F}_{[s,t]})$ such that $\{W, (X_u)_{u\in[s,t)}, P^{t;y}_{s;x}\} \approx \{W[0,t), (Y^t_u)_{u\in[s,t)}, Q^{t;y}_{s;x}\}$ and that $P^{t;y}_{s;x}(X_t=y)=1$. First we note that, for any bounded $\mathcal{F}_{[s,t]}$-measurable function $f(w)$ on W, there exists a countable dense subset $\{t_1, t_2, \cdots\}$ of $[s,t)$ and a measurable function $f^*(x_1, x_2, \cdots; x_0)$ on $S^\infty \times S$ such that $f(w)=f^*(w(t_1), w(t_2), \cdots; w(t))$ for all $w\in W$. Using this notation, we can define a probability measure $P^{t;y}_{s;x}$ on $(W, \mathcal{F}_{[s,t]})$ such that

$$\int_W f(w) P^{t;y}_{s;x}(dw) = \int_{\pi_t(W)} f^*(w'(t_1), w'(t_2), \cdots; y) Q^{t;y}_{s;x}(dw')$$

for every bounded $\mathcal{F}_{[s,t]}$-measurable function $f(w)$ on W since (2.7) implies $Q^{t;y}_{s;x}(\pi_t(W))=1$. One can immediately see that the probability measure $P^{t;y}_{s;x}$ is the desired one.

DEFINITION 2.2. Let $0\leqq s<t$, $x\in S$, and $y\in S$ be fixed. We call the above process $\{W, (X_u)_{u\in[s,t]}, P^{t;y}_{s;x}\}$ the *conditional process of* $\boldsymbol{X}$ *(corresponding to* $\{p(t,x,y)\}$ *and m) starting from x at time s and terminating in y at time t* or simply the $(s,x:t,y)$-*pinned process* of $\boldsymbol{X}$.

We shall collect the properties of the pinned processes for the future reference.

Theorem 2.2. *Let* $\boldsymbol{X}=(W, X_t, P_x)$ *be a Hunt process on S with a transition density function* $\{p(t,x,y)\}$ *relative to a positive Radon measure m satisfying* (p.1)$\sim$(p.4). *Then the pinned processes* $\{W, (X_u)_{u\in[s,t]}, P^{t;y}_{s;x}\}$ *of* $\boldsymbol{X}$, $0\leqq s<t$, $x\in S$, *and* $y\in S$, *satisfy the following*:

(i) $(x,y)\to P^{t;y}_{s;x}(B)$ *is* $\mathcal{B}(S)\times\mathcal{B}(S)$-*measurable for each* $B\in\mathcal{F}_{[s,t]}$.

(ii) $\boldsymbol{X}^{t,y}=(W, (X_u)_{u\in[s,t]}, P^{t;y}_{s;x}: s\in[0,t), x\in S)$ *is a time-inhomogeneous Markov process.*

(iii) *If* $0<u<t$, *then, for each* $B\in\mathcal{F}_{[0,t]}$,

$$P^{t;y}_{0;x}(B) = p(t,x,y)^{-1} E_x[p(t-u, X_u, y); B].$$

(iv) *For each* $B\in\mathcal{F}_{[0,t]}$ *and each* $E\in\mathcal{B}(S)$

$$P_x(B\cap\{X_t\in E\}) = \int_E P^{t;y}_{0;x}(B) p(t,x,y) m(dy).$$

(v) *For each* $0\leqq s<t$, $x\in S$, *and* $y\in S$

$$\{(X_u)_{u\in[s,t]}, P^{t;y}_{s;x}\} \approx \{(X_{u-s})_{u\in[s,t]}, P^{t-s,y}_{0,x}\}.$$

(vi) *For each* $t>0$, $x\in S$, *and* $y\in S$

$$\{(X_u)_{u\in[0,t]}, P^{t;y}_{0;x}\} \approx \{(X_{t-u})_{u\in[0,t]}, P^{t;x}_{0;y}\}.$$

3. Some preliminary facts on symmetric Lévy processes

Let $\{\Pi_t\}_{t>0}$ be a convolution semigroup of symmetric probability measures on $\boldsymbol{R}^d$. Then, by the Lévy-Khintchin formula, we have the following:

$$\int_{\boldsymbol{R}^d} e^{i\langle\xi,x\rangle}\Pi_t(dx) = \exp\{-tQ(\xi)\}\,, \tag{3.1}$$

$$Q(\xi) = \sum_{i,j=1}^d a_{ij}\xi^i\xi^j + \int_{\boldsymbol{R}^d-\{0\}}(1-\cos\langle\xi,y\rangle)n(dy)\,, \tag{3.2}$$

where $(a_{ij})_{1\leqq i,j\leqq d}$ is a symmetric non-negative definite constant matrix and $n(dy)$, a symmetric measure on $\boldsymbol{R}^d-\{0\}$ satisfying

$$\int_{\boldsymbol{R}^d-\{0\}}|y|^2(1+|y|^2)^{-1}n(dy)<\infty\,.$$

There is a one-to-one correspondence between the family of all such semigroups $\{\Pi_t\}_{t>0}$ and the family of all functions $Q(\xi)$ defined by (3.2).

Let $P(t,x,E)=\Pi_t(E-x)$, $t>0$, $x\in\boldsymbol{R}^d$, $E\in\mathscr{B}(\boldsymbol{R}^d)$, where $\mathscr{B}(\boldsymbol{R}^d)$ denotes the topological Borel field of $\boldsymbol{R}^d$. It is well known that there exists a Hunt process $\boldsymbol{X}=(W, X_t, P_x)$ on $\boldsymbol{R}^d$ having $\{P(t,x,E)\}$ as its transition function. We shall call this process $\boldsymbol{X}$ a d-dimensional *symmetric Lévy* process, and the corresponding $Q(\xi)$, the *exponent* of the process $\boldsymbol{X}$. In particular the process $\boldsymbol{X}$ is said to be a *symmetric stable* process of *order* $\alpha(0<\alpha\leqq 2)$ if its exponent is of the form

$$\begin{aligned} Q(\xi) &= \int_0^\infty\int_{S^{d-1}}(1-\cos\langle\xi,rs\rangle)\tilde{n}(ds)r^{-\alpha-1}dr && \text{if } 0<\alpha<2\,, \\ &= \sum_{i,j=1}^d a_{ij}\xi^i\xi^j && \text{if } \alpha=2\,, \end{aligned} \tag{3.3}$$

where $\tilde{n}(ds)$ is a symmetric finite measure on the unit sphere S^{d-1}, and $(a_{ij})_{1\leqq i,j\leqq d}$ is a symmetric non-negative definite matrix. The process $\boldsymbol{X}$ is said to be a *spherically symmetric stable* process of order $\alpha(0<\alpha\leqq 2)$ if

$$Q(\xi) = c|\xi|^\alpha\,, \tag{3.4}$$

where c is a positive constant.

In the following we shall make use of those terminologies and general results on Dirichlet forms and symmetric processes (see Fukushima [4]).

Let $L^2(\boldsymbol{R}^d)$ denote the real L^2-space with the usual inner product $(\ ,\ )$ and m, the d-dimensional Lebesgue measure. Since the symmetric Lévy process $\boldsymbol{X}$ is m-symmetric, the transition function $\{P(t,x,E)\}$ determines uniquely a strongly continuous Markovian semigroup $(T_t)_{t>0}$ of symmetric operators on $L^2(\boldsymbol{R}^d)$. In the present case, the Dirichlet form $(\mathcal{E},\mathcal{D}[\mathcal{E}])$ of $\boldsymbol{X}$, which is generated by the semigroup $(T_t)_{t>0}$, is given as follows:

$$\begin{aligned} &\mathcal{E}(u,v) = \int_{\boldsymbol{R}^d}Q(\xi)\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi, \quad u,v\in\mathcal{D}[\mathcal{E}]\,, \\ &\mathcal{D}[\mathcal{E}] = \{u\in L^2(\boldsymbol{R}^d);\ \int Q(\xi)|\hat{u}(\xi)|^2d\xi<\infty\}\,, \end{aligned} \tag{3.5}$$

where $\hat{u}(\xi)=(2\pi)^{-d/2}\int e^{i<\xi,x>}u(x)dx$ for $u\in L^2(\boldsymbol{R}^d)$ (see [4; Example 1.4.1]). Since the Dirichlet form $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ is regular, there exists a quasi-continuous version of u, denoted by $\tilde{u}$, for every $u\in\mathcal{D}[\mathcal{E}]$.

The infinitesimal generator $(L, \mathcal{D}(L))$ of the semigroup $(T_t)_{t>0}$ is given by

$$\widehat{Lu}(\xi) = -Q(\xi)\hat{u}(\xi), \quad u\in\mathcal{D}(L)\,, \tag{3.6}$$
$$\mathcal{D}(L) = \{u\in L^2(\boldsymbol{R}^d); \int |Q(\xi)\hat{u}(\xi)|^2 d\xi<\infty\}\,.$$

We also call $(L, \mathcal{D}(L))$ the generator of the process $\boldsymbol{X}$.

From now on we assume that the following condition is satisfied:

$$\exp\{-tQ(\xi)\}\in L^1(\boldsymbol{R}^d) \quad \text{for every} \quad t>0\,. \tag{Q}$$

Obviously, if $Q(\xi)$ is of the form (3.4), then the assumption (Q) is satisfied. If $Q(\xi)$ is of the form (3.3), then it is shown that the assumption (Q) is equivalent to the following nondegeneracy assumption: for $0<\alpha<2$, the support S_0 of $\tilde{n}(ds)$ spans $\boldsymbol{R}^d$ as a vector space; for $\alpha=2$, (a_{ij}) is positive definite.

Under the assumption (Q), $\Pi_t(dx)$ has the continuous density

$$p(t,x) = (2\pi)^{-d}\int_{\boldsymbol{R}^d} e^{-i<\xi,x>}e^{-tQ(\xi)}d\xi$$

relative to m. Let $p(t,x,y)=p(t,y-x)$, $t>0$, $x\in\boldsymbol{R}^d$, $y\in\boldsymbol{R}^d$. Then $\{p(t,x,y)\}$ is the transition density function of $\boldsymbol{X}$ relative to m.

We now show the existence of the pinned processes of $\boldsymbol{X}$.

Proposition 3.1. *Under the assumption* (Q) *the transition density function* $\{p(t,x,y)\}$ *satisfies* (p.1)~(p.4) *of* §2 *and*

$$p(t,x,y)\leqq p(t,0,0)<\infty \quad \textit{for all} \quad t>0,\ x\in\boldsymbol{R}^d,\ \textit{and}\ y\in\boldsymbol{R}^d\,. \tag{3.7}$$

Proof. One can immediately check the conditions (p.1), (p.3), (p.4), and (3.7). All we have to show is that $p(t,x)>0$ for all $t>0$ and $x\in\boldsymbol{R}^d$.

Note that (i) $x\to p(t,x)$ is continuous for each $t>0$, (ii) $p(t,x)\geqq 0$ for all $t>0$ and $x\in\boldsymbol{R}^d$, (iii) $p(s,0)>0$ for all $s>0$, and (iv) $t\to p(t,x)$ is analytic on $(0,\infty)$ for each $x\in\boldsymbol{R}^d$. Let $x\in\boldsymbol{R}^d$ be such that $p(t,x)=0$ for some $t>0$. Then we have, for each $s\in(0,t)$,

$$\int_{\boldsymbol{R}^d} p(t-s,x-y)p(s,y)dy = p(t,x) = 0\,.$$

Thus it follows from (ii) and (iii) that $p(t-s,x)=0$ for each $s\in(0,t)$, hence $p(s,x)=0$ for each $s>0$ by (iv). Therefore

$$\begin{aligned} N \equiv \{x;\, p(t,x) = 0 \quad &\text{for some} \quad t>0\} \\ = \{x;\, p(t,x) = 0 \quad &\text{for all} \quad t>0\}\,. \end{aligned}$$

It follows from (i) and (iii) that N^c ($=\{x;\ p(t,x)>0$ for some $t>0\}$) is an open subset of $\boldsymbol{R}^d$ containing 0. To prove that $N=\phi$ it suffices to show that N is also an open subset of $\boldsymbol{R}^d$ since $\boldsymbol{R}^d$ is connected. To this end let $x\in N$. Since

$$\int_{\boldsymbol{R}^d} p(1,x-y)p(1,y)dy = p(2,x) = 0$$

and $p(1,y)>0$ for all $y\in N^c$, we have $p(1,z)=0$ for all $z\in x-N^c\equiv\{x-y;\ y\in N^c\}$, that is, $x-N^c\subset N$. This proves that N is open since $x-N^c$ is a neighbourhood of x. This completes the proof.

Proposition 3.2. *Let $\boldsymbol{X}=(W,\ X_t,\ P_x)$ be a d-dimensional symmetric Lévy process satisfying the condition* (Q) *and let $\{W,\ (X_u)_{u\in[s,t]},\ P^{t;y}_{s;x}\}$ be the $(s,x:t,y)$-pinned process of $\boldsymbol{X}$. Then we have the following*:
(i) $\{(X_u)_{u\in[0,t]},\ P^{t;y}_{0;x}\}\approx\{(x+X_u)_{u\in[0,t]},\ P^{t;y-x}_{0;0}\}$ *for each $t>0$, $x\in\boldsymbol{R}^d$, and $y\in\boldsymbol{R}^d$.*
(ii) *In particular, if the process $\boldsymbol{X}$ is a symmetric stable process of order $\alpha(0<\alpha\leqq 2)$, then, for each $\lambda>0$, $t>0$, and $y\in\boldsymbol{R}^d$,*

$$\{(X_{\lambda u})_{u\in[0,t]},\ P^{\lambda t;y}_{0;0}\}\approx\{(\lambda^{1/\alpha}X_u)_{u\in[0,t]},\ P^{t;z}_{0;0}\}\ ,$$

where $z=\lambda^{-1/\alpha}y$.

The first assertion follows from the spatial homogeneity of the Lévy process $\boldsymbol{X}$ and the second, from the scaling property of the symmetric stable process $\boldsymbol{X}$ of order α:

$$\{(X_{\lambda t})_{t\geqq 0},\ P_0\}\approx\{(\lambda^{1/\alpha}X_t)_{t\geqq 0},\ P_0\}\quad\text{for each }\ \lambda>0\,, \tag{3.8}$$

$$p(\lambda t,0,x) = \lambda^{-d/\alpha}p(t,0,\lambda^{-1/\alpha}x)\quad\text{for each }\ \lambda>0,\ t>0,\text{ and } x\in\boldsymbol{R}^d. \tag{3.9}$$

We omit the details of the proof.

4. The eigenvalue problem for $-L+q(x)$ on a bounded domain with the Dirichlet condition

Let $\boldsymbol{X}=(W,\ X_t,\ P_x)$ be a d-dimensional symmetric Lévy process satisfying (Q). As in §3, L and $(\mathcal{E},\ \mathcal{D}[\mathcal{E}])$ denote the generator and the Dirichlet form of $\boldsymbol{X}$ respectively and m, the d-dimensional Lebesgue measure.

Let G be an open domain of $\boldsymbol{R}^d$ and let $q(x)$ be a (non-random) real valued Borel function on $\boldsymbol{R}^d$ which is bounded on G. In this section we will consider the eigenvalue problem $(-L+q)u=\lambda u$ with the Dirichlet condition $u=0$ on G^c, when $m(G)<\infty$. But we do not assume that $m(G)<\infty$ for a while.

Let $\tau_G=\inf\{t>0; X_t\in G^c\}$ and define a Markov transition function $p_G(t,x,dy)$ on $(G,\ \mathcal{B}^*(G))$ by

$$p_G(t,x,E) = P_x(X_t \in E,\ t<\tau_G),\ t>0,\ x\in G,\ E\in \mathcal{B}^*(G)\,. \tag{4.1}$$

Here $\mathcal{B}^*(G)$ denotes the family of all universally measurable sets of G. Let $L^2(G)$ denote the real L^2-space on G with the usual inner product $(\ ,\ )_G$. It is known that the transition function $\{p_G(t,x,E)\}$ is m-symmetric (see Fukushima [4; Lemma 4.2.3]) and the associated semigroup $(T_{G,t})_{t>0}$ on $L^2(G)$ is strongly continuous. Let $(\mathcal{E}_G, \mathcal{D}[\mathcal{E}_G])$ be the Dirichlet form on $L^2(G)$ determined by the semigroup $(T_{G,t})_{t>0}$. Given a function u of $L^2(G)$, we will denote by u' an element of $L^2(\boldsymbol{R}^d)$ which is identical with u on G and vanishes on G^c. It is known [4; Theorem 4.4.2] that the Dirichlet form $(\mathcal{E}_G, \mathcal{D}[\mathcal{E}_G])$ is the part of $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ on G, *i.e.*,

$$\begin{aligned} \mathcal{D}[\mathcal{E}_G] &= \{u\in L^2(G);\ u'\in \mathcal{D}[\mathcal{E}],\ \tilde{u}'=0 \text{ quasi-everywhere on } G^c\}\,, \\ \mathcal{E}_G(u,v) &= \mathcal{E}(u',v'),\quad u,v\in\mathcal{D}[\mathcal{E}_G]\,, \end{aligned} \tag{4.2}$$

where $\tilde{u}'$ denotes a quasi-continuous version of $u'\in\mathcal{D}[\mathcal{E}]$ in the restricted sense.

Let $(L_G, \mathcal{D}(L_G))$ be the infinitesimal generator of the semigroup $(T_{G,t})_{t>0}$.

Proposition 4.1. *If $u\in\mathcal{D}[\mathcal{E}_G]$ and $u'\in\mathcal{D}(L)$, then $u\in\mathcal{D}(L_G)$ and $L_G u=(Lu')|_G$.*

Proof. Let $u\in\mathcal{D}[\mathcal{E}_G]$ and $u'\in\mathcal{D}(L)$. It then follows from (4.2) that, for each $v\in\mathcal{D}[\mathcal{E}_G]$,

$$\begin{aligned} \mathcal{E}_G(u,v) &= \mathcal{E}(u',v') \\ &= -(Lu',v') \\ &= -((Lu')|_G, v)_G\,. \end{aligned}$$

This means that $u\in\mathcal{D}(L_G)$ and $L_G u=(Lu')|_G$.

In the following we will investigate the eigenvalue problem $(-L_G+q)u=\lambda u$, $u\in\mathcal{D}(L_G)$ which we regard as a realization of the eigenvalue problem $(-L+q)u=\lambda u$ with the Dirichlet condition $u=0$ on G^c.

Let $(T^q_{G,t})_{t>0}$ be the semigroup generated by L_G-q. We show that each $T^q_{G,t}$ has an integral kernel which is represented by the pinned processes of $\boldsymbol{X}$.

We start with the following proposition.

Proposition 4.2. *Let $\{W, (X_u)_{u\in[s,t]}, P^{t;y}_{s;x}\}$ be the $(s,x:t,y)$-pinned process of $\boldsymbol{X}$. The expectation with respect to $P^{t;y}_{s;x}$ is denoted by $E^{t;y}_{s;x}[\cdot]$. Let $K^q_G(t,x,y)$ and $p_G(t,x,y)$, $t>0$, $x\in G$, $y\in G$ be defined by*

$$K^q_G(t,x,y) = E^{t;y}_{0;x}[\exp\{-\int_0^t q(X_s)ds\}\,;\ t<\tau_G]\,p(t,x,y)\,, \tag{4.3}$$

$$p_G(t,x,y) = P^{t;y}_{0;x}(t<\tau_G)\,p(t,x,y)\,. \tag{4.4}$$

Then, for each $s>0$, $t>0$, $x\in G$, and $y\in G$, we have the following relations:

(i) $K_G^q(t,x,y)\leqq\exp(t\|q^-\|_\infty)p(t,x,y)\leqq\exp(t\|q^-\|_\infty)p(t,0,0)$, *where* $q^-=\max(-q,0)$ *and* $\|\cdot\|_\infty$ *denotes the supremum norm.*

(ii) $E_x[f(X_t)\exp\{-\int_0^t q(X_s)ds\}\,;\,t<\tau_G]=\int_G f(z)K_G^q(t,x,z)dz$ *for each bounded Lebesgue measurable function* f.

(iii) $K_G^q(s+t,x,y)=\int_G K_G^q(s,x,z)K_G^q(t,z,y)dz$.

(iv) $K_G^q(t,x,y)=K_G^q(t,y,x)$.

(v) $p_G(t,x,y)-K_G^q(t,x,y)=\int_0^t du\int_G p_G(u,x,z)q(z)K_G^q(t-u,z,y)dz$.

The proof will be deferred till the end of this section.

REMARK. Note that if $q(x)\equiv 0$, then $K_G^q(t,x,y)\equiv p_G(t,x,y)$.

Proposition 4.3. *Let $(T_{G,t}^q)_{t>0}$ be the semigroup generated by L_G-q. We then have*

$$T_{G,t}^q f(x)=\int_G f(y)K_G^q(t,x,y)dy \quad \text{for } f\in L^2(G). \tag{4.5}$$

Proof. Define $\tilde{T}_{G,t}^q f(x)$ for $f\in L^2(G)$ by the right hand side of (4.5). It follows from Proposition 4.2 that $(\tilde{T}_{G,t}^q)_{t>0}$ is a symmetric semigroup on $L^2(G)$. To prove $T_{G,t}^q=\tilde{T}_{G,t}^q$ it suffices to show that $(\tilde{T}_{G,t}^q)_{t>0}$ is strongly continuous and has L_G-q as its infinitesimal generator. The proof will be complete if the following relations are established: For each $f\in L^2(G)$,

(i) $T_{G,t}f-\tilde{T}_{G,t}^q f\to 0$ as $t\downarrow 0$ in $L^2(G)$,

(ii) $t^{-1}(T_{G,t}f-\tilde{T}_{G,t}^q f)\to q\cdot f$ as $t\downarrow 0$ in $L^2(G)$.

To prove (i) and (ii) we first note that the following inequality holds for every $f\in L^2(G)$:

$$\|\tilde{T}_{G,t}^q f\|_G\leqq\exp(t\|q\|_\infty)\|f\|_G,\quad t>0, \tag{4.6}$$

where $\|\cdot\|_G$ denotes the usual L^2-norm of $L^2(G)$. We can obtain this inequality by making use of Proposition 4.2 (i), Schwarz's inequality, and the symmetry of $p(t,\cdot,\cdot)$. Let $B(G)$ denote the space of all real bounded Borel functions on G. By Proposition 4.2 (v) and Fubini's theorem, we have, for every $f\in B(G)\cap L^2(G)$,

$$\begin{aligned} T_{G,t}f(x)-\tilde{T}_{G,t}^q f(x) &= \int_G f(y)\,(p_G(t,x,y)-K_G^q(t,x,y))dy \\ &= \int_0^t du\; T_{G,u}(q\cdot\tilde{T}_{G,t-u}^q f)(x), \end{aligned} \tag{4.7}$$

from which we get

$$\begin{aligned}||T_{G,t}f-\tilde{T}^q_{G,t}f||_G &\leqq \int_0^t ||T_{G,u}(q\cdot \tilde{T}^q_{G,t-u}f)||_G du \\ &\leqq ||q||_\infty \int_0^t ||\tilde{T}^q_{G,t-u}f||_G du .\end{aligned}$$

Thus, from (4.6), we have

$$||T_{G,t}f-\tilde{T}^q_{G,t}f||_G \leqq \{\exp(t||q||_\infty)-1\}\,||f||_G, \quad t>0\,, \tag{4.8}$$

for every $f\in B(G)\cap L^2(G)$. Since $B(G)\cap L^2(G)$ is dense in $L^2(G)$, the inequality (4.8) holds for every $f\in L^2(G)$. This proves the assertion (i).

Next let $S_t f(x)=t^{-1}(T_{G,t}f(x)-\tilde{T}^q_{G,t}f(x))-q(x)f(x)$ for $t>0$, $x\in G$, and $f\in B(G)\cap L^2(G)$. Then, from (4.7), we get

$$S_t f(x) = t^{-1}\int_0^t [T_{G,u}(q\cdot \tilde{T}^q_{G,t-u}f)-q\cdot f]\,(x)du$$

for each $t>0$ and $f\in B(G)\cap L^2(G)$. Thus we have

$$\begin{aligned}||S_t f||_G &\leqq t^{-1}\int_0^t ||T_{G,u}(q\cdot \tilde{T}^q_{G,t-u}f)-q\cdot f||_G du \\ &\leqq t^{-1}\int_0^t ||T_{G,u}(q\cdot(\tilde{T}^q_{G,t-u}-I)f)||_G du \\ &\quad +t^{-1}\int_0^t ||(T_{G,u}-I)\,(q\cdot f)||_G du\,,\end{aligned}$$

where I denotes the identity operator of $L^2(G)$. Hence the following inequality holds for every $f\in B(G)\cap L^2(G)$:

$$\begin{aligned}||t^{-1}(T_{G,t}f-\tilde{T}^q_{G,t}f)-q\cdot f||_G &\leqq ||q||_\infty \sup_{0\leqq u\leqq t} ||(\tilde{T}^q_{G,t-u}-I)f||_G \\ &\quad + \sup_{0\leqq u\leqq t} ||(T_{G,u}-I)\,(q\cdot f)||_G, \quad t>0\,.\end{aligned} \tag{4.9}$$

Since $B(G)\cap L^2(G)$ is dense in $L^2(G)$, this inequality holds for every $f\in L^2(G)$. Since $(\tilde{T}^q_{G,t})_{t>0}$ is strongly continuous by (i), the assertion (ii) has been proved.

REMARK. This proposition will be referred later in the case of $G=\boldsymbol{R}^d$. Note that $L_G=L$ in this case.

Assume that $m(G)<\infty$. Then each $T^q_{G,t}$ is a compact operator on $L^2(G)$ since, by Proposition 4.2 (i), the associated kernel $K^q_G(t,x,y)$ is of the Hilbert-Schmidt type. Therefore the spectrum of $-L_G+q$ consists only of eigenvalues of finite multiplicity having no accumulation point in $\boldsymbol{R}^1$. Thus they can be ordered as

$$\lambda^q_{G,1}\leqq\lambda^q_{G,2}\leqq \cdots .$$

Define the normalized distribution function $\rho^q_G(\lambda)$ of $\{\lambda^q_{G,i}\}_{i=1}^\infty$ by

$$\rho^q_G(\lambda) = m(G)^{-1}\sum_{\lambda^q_{G,i}\leqq\lambda}1, \quad \lambda\in\boldsymbol{R}^1\,. \tag{4.10}$$

Theorem 4.1. *We have, for each* $t>0$,

$$\int_{-\infty}^{\infty} e^{-t\lambda} d\rho_G^q(\lambda) = \frac{p(t,0,0)}{m(G)} \int_G E_{0,x}^{t,x}[\exp\{-\int_0^t q(X_s)ds\}\,;\, t<\tau_G]dx\,.$$

Proof. First we have, by (4.10),

$$\int_{-\infty}^{\infty} e^{-t\lambda} d\rho_G^q(\lambda) = m(G)^{-1} \sum_{i=1}^{\infty} \exp(-t\lambda_{G,i}^q), \quad t>0\,.$$

On the other hand, since the totality of the eigenvalues of $T_{G,t}^q$ is $\{\exp(-t\lambda_{G,i}^q)\}_{i=1}^{\infty}$, one can easily see that, for each $t>0$,

$$\iint_{G\times G} K_G^q(t/2,x,y)^2 dxdy = \sum_{i=1}^{\infty} \exp(-t\lambda_{G,i}^q)\,.$$

But, by Proposition 4.2, the left hand side of this equality is represented as

$$\begin{aligned}
&\iint_{G\times G} K_G^q(t/2,x,y) K_G^q(t/2,y,x) dxdy \\
&\quad = \int_G K_G^q(t,x,x)dx \\
&\quad = p(t,0,0) \int_G E_{0,x}^{t,x}[\exp\{-\int_0^t q(X_s)ds\}\,;\, t<\tau_G]dx\,.
\end{aligned}$$

This completes the proof.

Proof of Proposition 4.2. The proof depends on Theorem 2.2. However, since G is open, $\{t<\tau_G\}$ is not measurable relative to $\mathcal{F}_{[0,t]}$. Therefore to prove the proposition we have to show that Theorem 2.2 is valid for a family of σ-fields relative to which τ_G is a stopping time. This problem is settled by a standard completion argument. Before discussing the completion we show how the properties of $K_G^q(t,x,y)$ and $p_G(t,x,y)$ are derived from Theorem 2.2 (or rather its generalization).

One can easily obtain the inequality (i). Assertions (ii) and (iv) follow from Theorem 2.2 (iv) and (vi) respectively. Define $\tau_G^s = \inf\{t>s;\, X_t \notin G\}$ for $s \geqq 0$. Then we have the relation $\{s+t<\tau_G\} = \{s<\tau_G\} \cap \{s+t<\tau_G^s\}$ for each $s\geqq 0$ and $t>0$. Using this relation and Theorem 2.2, we have

$$\begin{aligned}
&K_G^q(s+t,x,y) \\
&\quad = E_{0,x}^{s+t,y}[\exp\{-\int_0^{s+t} q(X_u)du\}\,;\, s+t<\tau_G]p(s+t,x,y) \\
&\quad = E_{0,x}^{s+t,y}[\exp\{-\int_0^{s} q(X_u)du\}(E_{s,X_s}^{s+t,y}[\exp\{-\int_s^{s+t} q(X_u)du\}\,; \\
&\qquad s+t<\tau_G^s])\,;\, s<\tau_G]p(s+t,x,y) \\
&\quad = E_x[p(t,X_s,y)\exp\{-\int_0^{s} q(X_u)du\}(E_{0,X_s}^{t,y}[\exp\{-\int_0^{t} q(X_u)dy\}\,;
\end{aligned}$$

$$t<\tau_G]);\ s<\tau_G]$$
$$=E_x[\exp\{-\int_0^s q(X_u)du\}K_G^q(t,X_s,y);\ s<\tau_G]\,.$$

This, combined with (ii), gives (iii). Similarly the following calculation gives us (v):

$$\begin{aligned}
&p_G(t,x,y)-K_G^q(t,x,y)\\
&=E_{0,x}^{t,y}[(1-\exp\{-\int_0^t q(X_s)ds\});\ t<\tau_G]p(t,x,y)\\
&=\int_0^t du\, E_{0,x}^{t,y}[q(X_u)\exp\{-\int_u^t q(X_s)ds\};\ t<\tau_G]p(t,x,y)\\
&=\int_0^t du\, E_{0,x}^{t,y}[q(X_u)E_{u,X_u}^{t,y}[\exp\{-\int_u^t q(X_s)ds\},\ t<\tau_G^u];\\
&\quad u<\tau_G]p(t,x,y)\\
&=\int_0^t du\, E_x[q(X_u)E_{0,X_u}^{t-u,y}[\exp\{-\int_0^{t-u} q(X_s)ds\};\ t-u<\tau_G]\\
&\quad\times p(t-u,X_u,y);\ u<\tau_G]\\
&=\int_0^t du\, E_x[q(X_u)K_G^q(t-u,X_u,y);\ u<\tau_G]\,.
\end{aligned}$$

We have used the relation

$$1-\exp\{-\int_0^t q(X_s)ds\}=\int_0^t q(X_u)\exp\{-\int_u^t q(X_s)ds\}du$$

and Fubini's theorem for the second equality.

Finally we shall outline the completion problem. Define, for each $0\leqq s<u\leqq t$, $\mathcal{M}_{[s,u]}^t=\cap P_{s,\mu}^{t,\nu}$-completion of $\mathcal{F}_{[s,u]}$, where $P_{s,\mu}^{t,\nu}=\iint\mu(dx)\nu(dy)P_{s,x}^{t,y}$, and μ and ν run all finite Radon measures on $S\equiv\boldsymbol{R}^d$. It is known that if $0\leqq s<u\leqq t$, then $\{u<\tau_G^s\}\in\mathcal{M}_{[s,u]}^t$, and note that $\tau_G=\tau_G^0$. By a routine argument it is shown that Theorem 2.2 is valid when we replace the σ-fields $\mathcal{F}_{[0,t]}$, $\mathcal{B}(S)$, and $\mathcal{B}(S)\times\mathcal{B}(S)$ by $\mathcal{M}_{[0,t]}^t$, $\mathcal{B}^*(S)$ and $\overline{\mathcal{B}(S)\times\mathcal{B}(S)}$ respectively, where $\overline{\mathcal{B}(S)\times\mathcal{B}(S)}=\cap\,\mu\times\nu$-completion of $\mathcal{B}(S)\times\mathcal{B}(S)$ with μ and ν running all finite Radon measures on S. Further the process $\boldsymbol{X}^{t,y}=(X_u,P_{s,x}^{t,y})$ has the Markov property with respect to $(\mathcal{M}_{[s,u]}^t)$, that is, if $0\leqq s<u\leqq v\leqq t$ and $E\in\mathcal{B}^*(S)$, then $P_{s,x}^{t,y}(X_v\in E\,|\,\mathcal{M}_{[s,u]}^t)=P_{u,X_u}^{t,y}(X_v\in E)$, $P_{s,x}^{t,y}$-*a.s.*. This completes the proof.

5. Existence of the spectral distribution function

Let $(\Omega,\mathcal{B},P)$ be a probability space and $q(x,\omega)$, a real valued function defined on $\boldsymbol{R}^d\times\Omega$ such that $q(x,\omega)$ is measurable in ω for each $x\in\boldsymbol{R}^d$. We say that $\{q(x,\omega):\ x\in\boldsymbol{R}^d,\ \omega\in\Omega\}$ is a *stationary random field* defined on $(\Omega,\mathcal{B},P)$ if the law of $\{q(\cdot+x,\omega)\}$ is identical with that of $\{q(\cdot,\omega)\}$ for each

$x\in \boldsymbol{R}^d$. In the following we assume that almost all sample functions $x\to q(x,\omega)$ are continuous.

Let L be the generator of a d-dimensional symmetric Lévy process $\boldsymbol{X}=(W, X_t, P_x)$ with the exponent $Q(\xi)$ satisfying the condition (Q) of §3. Let V be a rectangle of the form

$$V=\{x=(x^1,\cdots,x^d);\ -a_i<x^i<b_i,\ i=1,\cdots,d\}\ ,$$

where $a_i,\ b_i>0,\ i=1,\cdots,d$. Since $x\to q(x,\omega)$ is bounded on V for each $\omega\in\Omega_0 \equiv\{\omega\in\Omega;\ x\to q(x,\omega)$ is continuous$\}$, the totality of eigenvalues of the self-adjoint operator $(-L_V+q(x,\omega),\ \mathcal{D}(L_V))$ can be ordered as

$$\lambda^{\omega}_{V,1}\leqq\lambda^{\omega}_{V,2}\leqq\ \cdots\ ,$$

and we define

$$\rho^{\omega}_V(\lambda)=m(V)^{-1}\textstyle\sum_{\lambda^{\omega}_{V,i}\leqq\lambda}1,\quad \lambda\in\boldsymbol{R}^1\ ,$$

where m denotes the d-dimensional Lebesgue measure. Let $(W, (X_u)_{u\in[0,t]}, P^{t;y}_{0;x})$ denote the $(0,x\colon t,y)$-pinned process of the process $\boldsymbol{X}$. Then Theorem 4.1 gives the following relation:

$$\int_{-\infty}^{\infty}e^{-t\lambda}d\rho^{\omega}_V(\lambda)=\frac{p(t,0,0)}{m(V)}\int_V E^{t;x}_{0;x}[\exp\{-\int_0^t q(X_s,\omega)ds\}\ ;\ t<\tau_V]dx \tag{5.1}$$
$$\text{for every } t>0\ .$$

Note that the Stieltjes inversion formula assures the measurability of $\rho^{\omega}_V(\lambda)$ in $\omega\in\Omega_0$.

Theorem 5.1. *Suppose that the following two conditions are satisfied:*
(A) $\exp\{-tQ(\xi)^{1/2}\}\in L^1(\boldsymbol{R}^d)$ *for every* $t>0$ (*this condition implies the condition* (Q)).
(B) *There exists a constant* $r>2$ *such that*

$$\exp\{\int_0^t q^-(X_s,\omega)ds\}\in L^r(P\times P_0)$$

for every $t>0$, *where* $q^-=\max(-q,\ 0)$ *and* $P\times P_0$ *denotes the product measure of* P *and* P_0.
Then there exists a right continuous nondecreasing function $\rho(\lambda)$ *defined on* $\boldsymbol{R}^1$ *with* $\rho(-\infty)=0$ *such that, for each continuity point* λ *of* $\rho(\lambda)$,

$$\lim_{V\to\infty}E[\rho^{\omega}_V(\lambda)]=\rho(\lambda)\ ,$$

where "$V\to\infty$*" means that* $a_i,\ b_i\to\infty$ *for* $i=1,\cdots,d$. *Moreover we have*

$$\int_{-\infty}^{\infty}e^{-t\lambda}d\rho(\lambda)=p(t,0,0)E\times E^{t;0}_{0;0}[\exp\{-\int_0^t q(X_s,\omega)ds\}] \tag{5.2}$$
$$\textit{for every } t>0\ ,$$

where $E\times E_{0;0}^{t;0}$ *denotes the expectation with respect to the product measure* $P\times P_{0;0}^{t;0}$.

If, in particular, $q(x,\omega)$ *is non-negative for all* $x\in \boldsymbol{R}^d$ *and* $\omega\in\Omega$, *then condition* (B) *is satisfied and we have* $\rho(0+)=0$.

DEFINITION 5.1. We call $\rho(\lambda)$ the *spectral distribution function* of $\{-L+q(x,\omega)\colon \omega\in\Omega\}$.

The proof of the above theorem goes along the same line as the proof in Nakao [6]. We will show how Nakao's proof is carried out for the present case.

Lemma 5.1. *Let* Φ *be the set of all non-negative right continuous nondecreasing functions on* $\boldsymbol{R}^1$. *Then we have the following*:

(i) *If* F *is in* Φ *and* $F(-\infty)=0$, *then*

$$\int_{-\infty}^{\infty} e^{-t\lambda}dF(\lambda) = t\int_{-\infty}^{\infty} e^{-t\lambda}F(\lambda)d\lambda \quad \textit{for every} \quad t>0\,.$$

(ii) *Let* $f(t)$ *be a non-negative measurable function defined on* $(0,1]$ *and* $g(t)$, *a finite nondecreasing function defined on* $[1,\infty)$. *Let* $\Phi(f,g)$ *be the set of all* $F\in\Phi$ *satisfying* $t\int_{-\infty}^{\infty} e^{-t\lambda}F(\lambda)d\lambda\leqq f(t)$ *for all* $t\in(0,1]$, *and* $t\int_{-\infty}^{\infty} e^{-t\lambda}F(\lambda)d\lambda\leqq g(t)$ *for all* $t\in[1,\infty)$. *Further suppose that* $f(t)$ *satisfies the following condition*:

$$\int_1^{\infty} e^{-s\lambda}f(1/\lambda)d\lambda<\infty \quad \textit{for every} \quad s>0\,. \tag{5.3}$$

Then there exists a non-negative measurable function $G(\lambda)$ *on* $\boldsymbol{R}^1$ *such that*

$$\int_{-\infty}^{\infty} e^{-t\lambda}G(\lambda)d\lambda<\infty \quad \textit{for every} \quad t>0 \tag{5.4}$$

and $F(\lambda)\leqq G(\lambda)$ *for every* $F\in\Phi(f,g)$ *and* $\lambda\in\boldsymbol{R}^1$.

Proof. Fubini's theorem gives (i). One can choose a strictly increasing continuous function $\hat{g}(t)$ defined on $[1,\infty)$ such that $g(t)\leqq\hat{g}(t)$ for all $t\geqq 1$ and $\hat{g}(\infty)=\infty$. Let $h(\lambda)$, $\lambda\in[\hat{g}(1),\infty)$ be the inverse function of $\hat{g}(t)$, $t\geqq 1$. Since $\Phi(f,g)\subset\Phi(f,\hat{g})$, we have

$$\hat{g}(t)\geqq t\int_{-\hat{g}(t)}^{\infty} e^{-t\lambda}F(\lambda)d\lambda\geqq F(-\hat{g}(t))\exp(t\hat{g}(t))$$

for each $F\in\Phi(f,g)$ and $t\geqq 1$. This implies that $F(\lambda)=F(-\hat{g}(h(-\lambda)))\leqq -\lambda\exp(h(-\lambda)\lambda)$ for $\lambda\leqq-\hat{g}(1)$. On the other hand we have, for each $F\in\Phi(f,g)$ and $t\in(0,1]$,

$$f(t)\geqq t\int_{1/t}^{\infty} e^{-t\lambda}F(\lambda)d\lambda\geqq F(1/t)\int_{1/t}^{\infty} te^{-t\lambda}d\lambda = F(1/t)e^{-1}\,,$$

which implies that $F(\lambda)\leqq ef(1/\lambda)$ for $\lambda\geqq 1$. Define a function $G(\lambda)$ on $\boldsymbol{R}^1$ as follows: $G(\lambda)=ef(1/\lambda)$ for $\lambda\geqq 1$, $G(\lambda)=ef(1)$ for $-\hat{g}(1)<\lambda<1$, and $G(\lambda)$

$=-\lambda\exp(h(-\lambda)\lambda)$ for $\lambda\leqq-\hat{g}(1)$. Then we have $F(\lambda)\leqq G(\lambda)$ for all $\lambda\in\boldsymbol{R}^1$ and $F\in\Phi(f,g)$. It only remains to check (5.4). But (5.4) follows from (5.3) and the fact that $h(\infty)=\infty$. This completes the proof.

Lemma 5.2. *We have, for each $t>0$ and $x\in\boldsymbol{R}^d$,*

$$p(t,0,0)E_{0,x}^{t;x}[\exp\{-\int_0^t q(X_s,\omega)ds\}]$$
$$\leqq p(t/2,0,0)E_x[\exp\{-2\int_0^{t/2}q(X_s,\omega)ds\}]\,.$$

Proof. Using Schwarz's inequality, Theorem 2.2, and (3.7), we have

$$\begin{aligned}
&E_{0,x}^{t;x}[\exp\{-\int_0^t q(X_s,\omega)ds\}]\\
&\leqq E_{0,x}^{t;x}[\exp\{-2\int_0^{t/2}q(X_s,\omega)ds\}]^{1/2}E_{0,x}^{t;x}[\exp\{-2\int_{t/2}^t q(X_s,\omega)ds\}]^{1/2}\\
&=E_{0,x}^{t;x}[\exp\{-2\int_0^{t/2}q(X_s,\omega)ds\}]\\
&=p(t,x,x)^{-1}E_x[p(t-t/2,X_{t/2},x)\exp\{-2\int_0^{t/2}q(X_s,\omega)ds\}]\\
&\leqq p(t,0,0)^{-1}p(t/2,0,0)E_x[\exp\{-2\int_0^{t/2}q(X_s,\omega)ds\}]\,.
\end{aligned}$$

Lemma 5.3. *Suppose that the condition* (Q) *is satisfied. Then we have the following*:

(i) $p(t,0,0)\to 0$ *as* $t\to\infty$.

(ii) *The condition* (A) *holds if and only if*

$$\int_0^\infty e^{-t\lambda}p(1/\lambda,0,0)d\lambda<\infty \quad \textit{for every}\ \ t>0\,. \tag{5.6}$$

Proof. By the condition (Q), we get

$$\infty>\int_{\boldsymbol{R}^d}\exp\{-tQ(\xi)\}d\xi\geqq\int_{\{Q(\xi)\leqq\eta\}}\exp\{-tQ(\xi)\}d\xi$$
$$\geqq e^{-t\eta}m(\{\xi;\,Q(\xi)\leqq\eta\})$$

for each $\eta\in\boldsymbol{R}^1$. Thus we can define a right continuous nondecreasing function $F^Q(\eta)\equiv(2\pi)^{-d}m(\{\xi;\,Q(\xi)\leqq\eta\})$, $\eta\in\boldsymbol{R}^1$. But since $\exp\{-tQ(\xi)\}=\int\cos\langle\xi,x\rangle p(t,x)dx<1$ for $\xi\neq0$, that is, $Q(\xi)>0$ for $\xi\neq0$, we have $F^Q(0)=0$. Hence we obtain

$$p(t,0,0)=\int_0^\infty e^{-t\eta}dF^Q(\eta),\quad t>0\,. \tag{5.7}$$

The assertion (i) immediately follows from this relation and the dominated convergence theorem. To see (ii), using (5.7), we have

$$
\begin{aligned}
(\infty \geqq)\, I &\equiv \int_0^\infty e^{-t\lambda} p(1/\lambda, 0, 0)\, d\lambda \\
&= \int_0^\infty e^{-t\lambda} d\lambda \int_0^\infty e^{-\eta/\lambda} dF^Q(\eta) \\
&= \int_0^\infty \Big[\int_0^\infty \exp(-t\lambda - \eta/\lambda)\, d\lambda\Big] dF^Q(\eta) \\
&= \int_0^t J\, dF^Q(\eta) + \int_t^\infty J\, dF^Q(\eta) \\
&\equiv I_1 + I_2\,,
\end{aligned}
$$

where $J = \int_0^\infty \exp(-t\lambda - \eta/\lambda)\, d\lambda$. But since

$$
J = \sqrt{\eta/t} \int_0^\infty \exp\{-\sqrt{t\eta}(\mu + 1/\mu)\}\, d\mu \;(\lambda = \sqrt{\eta/t})\,,
$$

we get

$$
\begin{aligned}
I_1 &\leqq \int_0^t \Big[t^{-1} \int_0^\infty \sqrt{t\eta} \exp(-\sqrt{t\eta}\, \mu)\, d\mu\Big] dF^Q(\eta) \\
&= t^{-1} F^Q(t) < \infty\,.
\end{aligned}
$$

On the other hand, noting that

$$
\begin{aligned}
&\int_1^\infty \exp\{-\sqrt{t\eta}(\mu + 1/\mu)\}\, (1 - \mu^{-2})\, d\mu \\
&= \int_0^\infty \exp\{-\sqrt{t\eta}(s + 2)\}\, ds \quad (s + 2 = \mu + 1/\mu) \\
&= 1/\sqrt{t\eta}\, \exp(-2\sqrt{t\eta})
\end{aligned}
$$

and

$$
\int_0^1 \exp\{-\sqrt{t\eta}(\mu + 1/\mu)\}\, d\mu = \int_1^\infty \exp\{-\sqrt{t\eta}(\mu + 1/\mu)\}\, \mu^{-2} d\mu\,,
$$

we have

$$
\begin{aligned}
(\infty \geqq) I_2 &= \int_t^\infty \Big[\sqrt{\eta/t} \int_1^\infty \exp\{-\sqrt{t\eta}(\mu + 1/\mu)\}\, (1 + \mu^{-2})\, d\mu\Big] dF^Q(\eta) \\
&= \int_t^\infty t^{-1} \exp(-2\sqrt{t\eta})\, dF^Q(\eta) \\
&\quad + 2 \int_t^\infty \Big[\sqrt{\eta/t} \int_1^\infty \exp\{-\sqrt{t\eta}(\mu + 1/\mu)\}\, \mu^{-2} d\mu\Big] dF^Q(\eta) \\
&\equiv I_3 + I_4\,.
\end{aligned}
$$

Since

$$
I_3 = t^{-1} \int_{\{Q(\xi) > t\}} \exp\{-2t^{1/2} Q(\xi)^{1/2}\}\, d\xi
$$

and $I_3 \leqq I$, "if" part of (ii) is established. To prove "only if" part, we assume

(A). Then we have $I_3<\infty$ and, noting that $\mu+1/\mu\geqq 2$ for $\mu>0$,

$$I_4\leqq 2t^{-1/2}\int_t^\infty[\sqrt{\eta}\exp(-2\sqrt{t\eta})\int_1^\infty \mu^{-2}d\mu]dF^Q(\eta)$$
$$\leqq 2t^{-1/2}\sup_{\eta>t}\{\sqrt{\eta}\exp(-\sqrt{t\eta})\}\int_{\{Q(\xi)>t\}}\exp\{-t^{1/2}Q(\xi)^{1/2}\}d\xi<\infty .$$

Thus we have $I=I_1+I_3+I_4<\infty$, which completes the proof.

Lemma 5.4. *Under the assumption of Theorem* 5.1, *there exists a measurable function* $G(\lambda)$ *on* $\boldsymbol{R}^1$ *such that* $E[\rho_V^\omega(\lambda)]\leqq G(\lambda)$ *for every rectangle* V *and* $\lambda\in\boldsymbol{R}^1$ *and that* $\int_{-\infty}^\infty e^{-t\lambda}G(\lambda)d\lambda<\infty$ *for every* $t>0$.

Proof. It follows from (5.1), Lemma 5.1 (i), and Lemma 5.2 that

$$t\int_{-\infty}^\infty e^{-t\lambda}E[\rho_V^\omega(\lambda)]d\lambda\leqq p(t/2,0,0)E\times E_0[\exp\{2\int_0^{t/2}q^-(X_s,\omega)ds\}] .$$

Define

$$f(t)=p(t/2,0,0)E\times E_0[\exp\{2\int_0^{1/2}q^-(X_s,\omega)ds\}] \quad \text{for} \quad t\in(0,1]$$

and

$$g(t)=p(1/2,0,0)E\times E_0[\exp\{2\int_0^{t/2}q^-(X_s,\omega)ds\}] \quad \text{for} \quad t\in[1,\infty),$$

where $E\times E_0$ denotes the expectation with respect to the product measure $P\times P_0$. Then $E[\rho_V^\omega(\cdot)]\in\Phi(f,g)$ for every rectangle V. Since $f(t)$ satisfies (5.3) by Lemma 5.3 (ii), the lemma is an immediate consequence of Lemma 5.1 (ii).

Proof of Theorem 5.1. As in Nakao [6; Theorem 4.1], using Hölder's inequality, condition (B), and the fact that

$$\lim_{V\to\infty} m(V)^{-1}\int_V P_{0;x}^{t;x}(t\geqq\tau_V)dx=0 \quad \text{for} \quad t>0,$$

we have

$$\lim_{V\to\infty} t\int_{-\infty}^\infty e^{-t\lambda}E[\rho_V^\omega(\lambda)]d\lambda=p(t,0,0)E\times E_{0;0}^{t;0}[\exp\{-\int_0^t q(X_s,\omega)ds\}]$$

for every $t>0$. Thus Lemma 5.4, Helly's selection theorem, and the uniqueness theorem of the Laplace transforms gives us the theorem except for the last assertion. But this follows from the fact that, when $q(x,\omega)$ is non-negative, the right hand side of (5.2) tends to zero as $t\to\infty$ by Lemma 5.3 (i). This completes the proof.

We next mention the ergodic theorem for the spectral distribution function. Let $\tilde{\Omega}$ denote the space of all real functions on $\boldsymbol{R}^d$ and $\tilde{\mathscr{B}}$, the smallest σ-algebra with respect to which all the coordinate functions $\tilde{\Omega}\ni\tilde{\omega}\rightarrow\tilde{\omega}(x)$ ($x\in\boldsymbol{R}^d$) are measurable. A stationary random fieid $\{q(x,\omega)\}$ over $(\Omega, \mathcal{B}, P)$ induces a probability measure $\tilde{P}$ on $(\tilde{\Omega}, \tilde{\mathcal{B}})$ and the shift operator T_x on $(\tilde{\Omega}, \tilde{\mathcal{B}})$ defined by $T_x\tilde{\omega}=\tilde{\omega}(\cdot+x)$ makes a measure preserving transformation for each $x\in\boldsymbol{R}^d$. We say that $\{q(x,\omega)\}$ is *ergodic* if the associated family $\{T_x: x\in\boldsymbol{R}^d\}$ is ergodic, *i.e.*, $P(\Lambda)$ is equal to 0 or 1 for every $\{T_x\}$-invariant set $\Lambda\in\tilde{\mathcal{B}}$.

Theorem 5.2. *Suppose that the conditions* (A) *and* (B) *of Theorem* 5.1 *are satisfied. Further suppose that* $\{q(x,\omega): x\in\boldsymbol{R}^d, \omega\in\Omega\}$ *is ergodic. Then there exists a subset* Ω_1 *of* Ω *with* $P(\Omega_1)=1$ *such that, for each* $\omega\in\Omega_1$,

$$\lim_{V\to\infty}\rho_V^\omega(\lambda)=\rho(\lambda) \quad \textit{for every continuity point } \lambda \textit{ of } \rho(\lambda),$$

where $\rho(\lambda)$ *is the spectral distribution function of* $\{-L+q(x,\omega): \omega\in\Omega\}$.

The proof is omitted since it is the same as in Nakao [6; Theorem 4.2].

6. Asymptotic behaviour of the spectral distribution functions near the origin for the random fields induced by a Poisson random measure

In this section we will be concerned with a special class of non-negative random fields $\{q(x, \omega)\}$ (described below) and investigate the asymptotic behaviour of $\rho(\lambda)$ when $\lambda\downarrow 0$.

Let ν be a positive constant and m, the d-dimensional Lebesgue measure. A family of measures $\{p^\omega(dy): \omega\in\Omega\}$ on $\boldsymbol{R}^d$ is said to be a *Poisson random measure with characteristic measure* $\nu\cdot m$ over a probability space $(\Omega, \mathcal{B}, P)$ if (i) for each $\omega\in\Omega$, $p^\omega(dy)$ is a Radon measure of the form $\sum_{i=1}^\infty\delta_{x_i}$, where δ_{x_i} is the Diract measure at point $x_i\in\boldsymbol{R}^d$, (ii) all mappings $\omega\rightarrow p^\omega(A)$ ($A\in\mathcal{B}(\boldsymbol{R}^d)$) are measurable, and (iii) for each finite disjoint family $\{A_i; i=1,\cdots,k\}\subset\mathcal{B}(\boldsymbol{R}^d)$ and each sequence of non-negative integers $\{n_i;\ i=1,\cdots,k\}$

$$\begin{aligned}P(\omega: p^\omega(A_i)=n_i,\ i=1,\cdots,k)&\\ =\textstyle\prod_{i=1}^k\exp(-\nu m(A_i))\,(\nu m(A_i))^{n_i}/n_i!&\end{aligned}$$

with the convention that $\exp(-\infty)\times\infty=0$.

Let $\varphi(x)$ be a non-negative continuous function on $\boldsymbol{R}^d$ satisfying $\varphi(x)=O(|x|^{-(d+\varepsilon)})$ ($|x|\rightarrow\infty$) for some positive constant ε. We can define a random field $\{q(x,\omega): x\in\boldsymbol{R}^d, \omega\in\Omega\}$ by

$$q(x,\omega)=\int_{\boldsymbol{R}^d}\varphi(x-y)p^\omega(dy). \tag{6.1}$$

It is well known that $\{q(x, \omega)\}$ is an ergodic stationary random field having continuous sample functions.

Let $\boldsymbol{X}$ be a d-dimensional symmetric Lévy process with exponent $Q(\xi)$

satisfying the condition (A) and let L be the generator of $\boldsymbol{X}$. Let $\{(X_s)_{s\in[0,t]}, P_{0,x}^{t,y}\}$ denote the $(0,x:t,y)$-pinned process of $\boldsymbol{X}$ and $E_{0,x}^{t,y}$, the expectation with respect to $P_{0,x}^{t,y}$. Since the random field $\{q(x,\omega)\}$ under consideration is non-negative, Theorem 5.1 assures the existence of the spectral distribution function $\rho(\lambda)$ of $\{-L+q(x,\omega): \omega\in\Omega\}$ with $\rho(0+)=0$ and the equality

$$(6.2) \qquad \int_0^\infty e^{-t\lambda}d\rho(\lambda)=p(t,0,0)E\times E_{0,0}^{t,0}[\exp\{-\int_0^t q(X_s,\omega)ds\}]$$

holds for each $t>0$.

We now summarize the above results.

Theorem 6.1. *Let $\boldsymbol{X}$ be a d-dimensional symmetric Lévy process with exponent $Q(\xi)$ satisfying the condition* (A) *and let L be the generator of $\boldsymbol{X}$. Let $\varphi(x)$ be a non-negative continuous function on $\boldsymbol{R}^d$ such that $\varphi(x)=O(|x|^{-(d+\varepsilon)})$ $(|x|\to\infty)$ for some constant $\varepsilon>0$ and $\{p^\omega(dy): \omega\in\Omega\}$, a Poisson random measure with characteristic measure $\nu\cdot m$ over a probability space $(\Omega, \mathcal{B}, P)$. Let $\{q(x,\omega): x\in\boldsymbol{R}^d, \omega\in\Omega\}$ be the random field defined by* (6.1). *Then there exists the spactral distribution function $\rho(\lambda)$ of $\{-L+q(x,\omega): \omega\in\Omega\}$ with $\rho(0+)=0$ satisfying* (6.2) *for each $t>0$.*

From now on we shall evaluate the asymptotic behaviour of the spectral distribution function $\rho(\lambda)$ when $\lambda\downarrow 0$.

Let α be a constant such as $0<\alpha\leqq 2$. Let $Q^{(\alpha)}(\xi)$ denote the exponent of a d-dimensional symmetric stable process $\boldsymbol{X}^{(\alpha)}$ (see (3.3)). We will consider the following condition on the exponent $Q(\xi)$ of the Lévy process $\boldsymbol{X}$:
(C) (i) $Q(\xi)=Q^{(\alpha)}(\xi)+o(|\xi|^\alpha)$ $(|\xi|\downarrow 0)$ and (ii) the function

$$Q_*(\xi)=\inf_{t\geqq 1} t^\alpha Q(t^{-1}\xi)$$

satisfies the following summability condition for each $\delta>0$ and $r>0$:

$$\sum_\xi \exp(-\delta Q_*(\xi))<\infty \quad (\xi\in(r\boldsymbol{Z})^d),$$

where $(r\boldsymbol{Z})^d$ denotes the discrete subgroup of $\boldsymbol{R}^d$ consisting of vectors having for each coordinate an integral multiple of r.

REMARK. Note that the condition (C) is satisfied if $Q(\xi)=\sum_{i=1}^n Q^{(\alpha_i)}(\xi)$ and $\alpha=\min\alpha_i$, where $\{\alpha_i\}_{i=1}^n$ is a sequence of different numbers such that $0<\alpha_i\leqq 2$, $i=1,\cdots,n$.

Let $L^{(\alpha)}$ denote the generator of the process $\boldsymbol{X}^{(\alpha)}$. Given an open domain G of $\boldsymbol{R}^d$ with $m(G)<\infty$, let $\lambda(G)$ be the smallest eigenvalue of $-L_G^{(\alpha)}$ (see §4).

Theorem 6.2. *Let L, $\varphi(x)$, and $\{p^\omega(dy): \omega\in\Omega\}$ be those in Theorem* 6.1. *Let α be such as $0<\alpha\leqq 2$. Suppose that the condition* (C) *is satisfied and that*

(6.3) $$\varphi(x) = o(|x|^{-(d+\alpha)})\ (|x| \to \infty) \text{ and } \int_{R^d} \varphi(x)dx > 0 .$$

Then the spectral distribution function $\rho(\lambda)$ *of* $\{-L + \int_{R^d} \varphi(x-y)p^\omega(dy) : \omega \in \Omega\}$ *satisfies*

(6.4) $$\lim_{\lambda \downarrow 0} \lambda^{d/\alpha} \log \rho(\lambda) = -\nu(\lambda_\alpha)^{d/\alpha} ,$$

where $\lambda_\alpha = \inf_G \lambda(G)$ *with* G *running all open domains of* $m(G)=1$.

We will denote the left hand side of (6.2) by $k(t)$ and the right hand side of (6.2) by $I(t)$;

(6.5) $$k(t) = \int_0^\infty e^{-t\lambda} d\rho(\lambda) ,$$

(6.6) $$I(t) = p(t,0,0) E \times E_{0;0}^{t;0} [\exp\{-\int_0^t q(X_s, \omega) ds\}] .$$

Appealing to the Minlos-Povzner Tauberian theorem (Fukushima [3; Theorem 2.2]), Theorem 6.2 is reduced to showing that

$$\lim_{t \to \infty} t^{-d/(d+\alpha)} \log k(t) = -\nu^{\alpha/(d+\alpha)} \left(\frac{d+\alpha}{\alpha}\right) \left(\frac{\alpha\lambda_\alpha}{d}\right)^{d/(d+\alpha)} .$$

But since $k(t) = I(t)$, it suffices to show the following theorem:

Theorem 6.2′. *Under the assumption of Theorem* 6.2, *we have*

(6.7) $$\lim_{t \to \infty} t^{-d/(d+\alpha)} \log I(t) = -k(\nu, L^{(\alpha)}) ,$$

where $k(\nu, L^{(\alpha)}) = \nu^{\alpha/(d+\alpha)} \left(\frac{d+\alpha}{\alpha}\right) \left(\frac{\alpha\lambda_\alpha}{d}\right)^{d/(d+\alpha)}$.

We next consider another sort of estimate on the exponential decay of $\rho(\lambda)$ for $\lambda \downarrow 0$.

Theorem 6.3. *Let* $Q(\xi), L, \varphi(x)$, *and* $\{p^\omega(dy) : \omega \in \Omega\}$ *be as in Theorem* 6.1. *Let* $0 < \beta < \alpha \leqq 2$. *Suppose that*

(6.8) $$0 < K_1 = \varliminf_{|x| \to \infty} |x|^{d+\beta} \varphi(x) \leqq \varlimsup_{|x| \to \infty} |x|^{d+\beta} \varphi(x) = K_2 < \infty$$

and that

(D) $$Q(\xi) = O(|\xi|^\alpha)\ (|\xi| \downarrow 0) .$$

Then the spectral distribution function $\rho(\lambda)$ *of* $\{-L + \int_{R^d} \varphi(x-y)p^\omega(dy) : \omega \in \Omega\}$ *satisfies*

(6.9) $$-(x_2/x_1) C_1(\nu, \beta, K_1) \leqq \varliminf_{\lambda \downarrow 0} \lambda^{d/\beta} \log \rho(\lambda)$$

$$\leqq \overline{\lim_{\lambda \downarrow 0}}\, \lambda^{d/\beta} \log \rho(\lambda) \leqq -C_1(\nu, \beta, K_1),$$

where $C_1(\nu,\beta,K_1)=\frac{\beta}{d+\beta}\left(\frac{d}{d+\beta}\right)^{d/\beta}\left(\Gamma\left(\frac{\beta}{d+\beta}\right)\nu\Omega_d\right)^{(d+\beta)/\beta}K_1^{d/\beta}$. *Here* Ω_d *denotes the volume of the d-dimensional sphere of unit radius and* x_1 (*resp.* x_2), *the least* (*resp. largest*) *solution of the following equation*:

$$x^{-\beta/d}+x=\{(d/\beta)^{\beta/(d+\beta)}+(\beta/d)^{d/(d+\beta)}\}(K_2/K_1)^{d/(d+\beta)}.$$

If, in particular, $K_1=K_2$, *then* $x_1=x_2$ *and we have*

$$\lim_{\lambda \downarrow 0} \lambda^{d/\beta} \log \rho(\lambda) = -C_1(\nu,\beta,K_1). \tag{6.10}$$

REMARKS. It is known that $(x_2/x_1)C_1(\nu,\beta,K_1)$ increases to $C_2(\nu,\beta,K_2) =\left(\Gamma\left(\frac{\beta}{d+\beta}\right)\nu\Omega_d\right)^{(d+\beta)/\beta}K_2^{d/\beta}$ as $K_1 \downarrow 0$ and decreases to $C_1(\nu,\beta,K_2)$ as $K_1 \uparrow K_2$. The third inequality in (6.9) holds even if $K_2=\infty$ and the first inequality holds even if $K_1=0$ with $C_2(\nu,\beta,K_2)$ replacing $(x_2/x_1)C_1(\nu,\beta,K_1)$.

As before, Theorem 6.3 is reduced to the following theorem based on a Tauberian theorem of exponential type due to Y. Kasahara [5; Theorem 3].

Theorem 6.3′. *Under the assumption of Theorem* 6.3, *we have*

$$\begin{aligned} -\kappa(\nu,\beta,K_2) &\leqq \underline{\lim_{t\to\infty}}\, t^{-d/(d+\beta)} \log I(t) \\ &\leqq \overline{\lim_{t\to\infty}}\, t^{-d/(d+\beta)} \log I(t) \leqq -\kappa(\nu,\beta,K_1), \end{aligned} \tag{6.11}$$

where $\kappa(\nu,\beta,K_i)=\Gamma\left(\frac{\beta}{d+\beta}\right)\nu\Omega_d K_i^{d/(d+\beta)}, \quad i=1, 2.$

Finally we will consider the case when

$$0<\underline{\lim_{|x|\to\infty}}|x|^{d+\alpha}\varphi(x)\leqq \overline{\lim_{|x|\to\infty}}|x|^{d+\alpha}\varphi(x)<\infty \tag{6.12}$$

and $Q(\xi)=O(|\xi|^{\alpha})$ $(|\xi| \downarrow 0)$.

Theorem 6.4. *Let* L, $\varphi(x)$, *and* $\{p^{\omega}(dy)\colon \omega\in\Omega\}$ *be as in Theorem* 6.1. *Let* α *be such as* $0<\alpha\leqq 2$. *Suppose that the conditions* (6.12) *and* (D) *are satisfied. Then the spectral distribution function* $\rho(\lambda)$ *of* $\{-L+\int_{R^d}\varphi(x-y)p^{\omega}(dy)\colon \omega\in\Omega\}$ *satisfies*

$$-\infty<\underline{\lim_{\lambda\downarrow 0}}\,\lambda^{d/\alpha}\log\rho(\lambda)\leqq\overline{\lim_{\lambda\downarrow 0}}\,\lambda^{d/\alpha}\log\rho(\lambda)<0. \tag{6.13}$$

This theorem is reduced to the following theorem:

Theorem 6.4′. *Under the assumption of Theorem* 6.4, *we have*

$$-\infty<\varliminf_{t\to\infty} t^{-d/(d+\alpha)} \log I(t)\leqq\varlimsup_{t\to\infty} t^{-d/(d+\alpha)} \log I(t)<0\,. \tag{6.14}$$

The proofs of Theorems 6.2'∼6.4' will be given in the following sections: Theorem 6.2' follows from Lemmas 8.6 and 9.2; Theorems 6.3' and 6.4' follow from Lemmas 8.4 and 9.1.

7. An inequality due to L.A. Pastur

In this section we shall prove a theorem which is a generalization of the lemma of Pastur [8]. This theorem will play a fundamental role in the proof of the theorems of the preceding section.

Theorem 7.1. *Let X be a d-dimensional symmetric Lévy process with the exponent $Q(\xi)$ satisfying* (Q) *and* ($\mathcal{E}$, $\mathcal{D}[\mathcal{E}]$), *the Dirichlet form of X. The expectation with respect to the $(0,x:t,y)$-pinned process of X is denoted by $E^{t:y}_{0:x}$. Let $\{q(x,\omega)\colon x\in \boldsymbol{R}^d,\ \omega\in\Omega\}$ be a stationary random field defined on a probability space $(\Omega, \mathcal{B}, P)$. Suppose that*

$$E[\exp(-tq(0, \omega))]<\infty \tag{7.1}$$

for each $t>0$, where E denotes the expectation with respect to P. Then it holds that, for each $f\in\mathcal{D}[\mathcal{E}]$ with $\|f\|=1$,

$$\begin{aligned}\|f\|_{L^1}^{-2}\exp\{-t\mathcal{E}(f,f)-\Phi_t(f)\}&\\ \leqq p(t,0,0)E\times E^{t:0}_{0:0}[\exp\{-\int_0^t q(X_s,\omega)ds\}]&\\ \leqq p(t,0,0)E[\exp(-tq(0,\omega))]\,,&\end{aligned} \tag{7.2}$$

where

$$\Phi_t(f) = -\log E[\exp\{-t\int_{\boldsymbol{R}^d} q(x,\omega)f(x)^2dx\}] \tag{7.3}$$

and $\|\cdot\|$ (resp. $\|\cdot\|_{L^1}$) denotes the usual norm of $L^2(\boldsymbol{R}^d)$ (resp. $L^1(\boldsymbol{R}^d)$).

Proof. By (7.1), one can assume that $q(x,\omega)$ is bounded uniformly for $x\in\boldsymbol{R}^d$ and $\omega\in\Omega$. The second inequality of (7.2) immediately follows from Jensen's inequality and the stationarity of $\{q(x,\omega)\}$. To show the first inequality we define, for each $\omega\in\Omega$, $t>0$, $x\in\boldsymbol{R}^d$, and $y\in\boldsymbol{R}^d$,

$$K^\omega(t,x,y) = p(t,x,y)E^{t:y}_{0:x}[\exp\{-\int_0^t q(X_s,\omega)ds\}]\,.$$

Let $(L,\mathcal{D}(L))$ denote the generator of X. Then from Proposition 4.3 and its remark it follows that, for each $\omega\in\Omega$, the selfadjoint operator $L^\omega\equiv L-q(x,\omega)$ with the domain $\mathcal{D}(L)$ generates a semigroup $(T^\omega_t)_{t>0}$ such that each operator T^ω_t has $K^\omega(t,x,y)$ as its integral kernel. Let $\{E^\omega_\lambda\colon \lambda\in\boldsymbol{R}^1\}$ be the resolution of the identity associated with $-L^\omega$, *i.e.*, $-L^\omega=\int_{-\infty}^\infty \lambda\, dE^\omega_\lambda$. Suppose that

$f\in\mathcal{D}[\mathcal{E}]$ satisfies $\|f\|=1$. Then, using Jensen's inequality, we get

$$\begin{aligned}E[(T_t^\omega f,f)] &= E[\int_{-\infty}^{\infty} e^{-t\lambda} d_\lambda(E_\lambda^\omega f,f)]\\ &\geqq E[\exp\{-t\int_{-\infty}^{\infty} \lambda\, d_\lambda(E_\lambda^\omega f,f)\}]\\ &= E[\exp\{-t\mathcal{E}(f,f)-t\int_{R^d} q(x,\omega)f(x)^2dx\}]\\ &= \exp\{-t\mathcal{E}(f,f)-\Phi_t(f)\}\,.\end{aligned}$$

It only remains to prove that

$$E[(T_t^\omega f,f)]\leqq \|f\|_{L^1}^2 E[K^\omega(t,0,0)]\,.$$

But since

$$E[(T_t^\omega f,f)]\leqq \int_{R^d}|f(x)|\,dx\int_{R^d}E[K^\omega(t,x,y)]\,|f(y)|\,dy\,,$$

it suffices to show that $E[K^\omega(t,x,y)]\leqq E[K^\omega(t,0,0)]$. Using Proposition 4.2 and Schwarz's inequality, we have

$$\begin{aligned}K^\omega(t,x,y) &= \int_{R^d} K^\omega(t/2,x,z)K^\omega(t/2,z,y)dz\\ &\leqq \left(\int_{R^d}K^\omega(t/2,x,z)^2dz\right)^{1/2}\left(\int_{R^d}K^\omega(t/2,z,y)^2dz\right)^{1/2}\\ &= K^\omega(t,x,x)^{1/2}K^\omega(t,y,y)^{1/2}\,.\end{aligned}$$

Take the expectation with respect to P. Then it follows from Schwarz's inequality, the homogeneity of the process X, and the stationarity of $\{q(x,\omega)\}$ that

$$\begin{aligned}E[K^\omega(t,x,y)] &\leqq E[K^\omega(t,x,x)]^{1/2}E[K^\omega(t,y,y)]^{1/2}\\ &= E[K^\omega(t,0,0)]\,,\end{aligned}$$

which completes the proof.

8. The lower estimates

In this section, we shall prove the lower estimates in Theorems 6.2'$\sim$6.4'. Recall that $\{q(x,\omega)\colon x\in\boldsymbol{R}^d,\ \omega\in\Omega\}$ is the random field defined by (6.1) and let

$$I(t)=p(t,0,0)E\times E_{0,0}^{t,0}[\exp\{-\int_0^t q(X_s,\omega)ds\}],\quad t>0\,. \tag{8.1}$$

It follows from Theorem 7.1 that

$$I(t)\geqq \|f\|_{L^1}^{-2}\exp\{-t\mathcal{E}(f,f)-\Phi_t(f)\} \tag{8.2}$$

holds for each $f\in C_0^\infty(\boldsymbol{R}^d)$ with $\|f\|=1$, where

$$\Phi_t(f) = \nu \int_{\boldsymbol{R}^d} \{1-\exp(-t\int_{\boldsymbol{R}^d} \varphi(y-x)f(y)^2 dy)\}\, dx\,. \tag{8.3}$$

Here $C_0^\infty(\boldsymbol{R}^d)$ denotes the space of all real C^∞-functions on $\boldsymbol{R}^d$ with compact support.

We start with the following lemma.

Lemma 8.1. *Suppose that the condition* (D) *is satisfied. Then there exists a positive constant c such that*

$$R^\alpha Q(R^{-1}\xi) \leqq c(|\xi|^\alpha + |\xi|^2) \quad \text{for all } \xi \in \boldsymbol{R}^d \text{ and } R \geqq 1\,. \tag{8.4}$$

Proof. By the condition (D), there exist constants $c_1 > 0$ and $\eta > 0$ such that $Q(\xi) \leqq c_1 |\xi|^\alpha$ whenever $|\xi| \leqq \eta$. On the other hand, one can easily check that there exists a constant $c_2 > 0$ such that $Q(\xi) \leqq c_2 |\xi|^2$ whenever $|\xi| \geqq \eta$ by the explicit form of (3.2). This completes the proof.

For each $\psi \in C_0^\infty(\boldsymbol{R}^d)$ with $\|\psi\| = 1$ and $R > 0$, we define

$$\psi_R(x) = R^{-d/2}\psi(R^{-1}x), \quad x \in \boldsymbol{R}^d. \tag{8.5}$$

Note that $\|\psi_R\| = 1$.

Lemma 8.2. *Let $\mathcal{E}$ and $\mathcal{E}^{(\alpha)}$ denote the Dirichlet forms of the processes $\boldsymbol{X}$ and $\boldsymbol{X}^{(\alpha)}$ respectively. Then we have the following*:

(i) *Condition* (D) *implies that $\mathcal{E}(\psi_R, \psi_R) = O(R^{-\alpha})$ $(R \to \infty)$ for each $\psi \in C_0^\infty(\boldsymbol{R}^d)$.*

(ii) *Condition* (C) *implies that $\mathcal{E}(\psi_R, \psi_R) = R^{-\alpha}\mathcal{E}^{(\alpha)}(\psi, \psi) + o(R^{-\alpha})$ $(R \to \infty)$ for each $\psi \in C_0^\infty(\boldsymbol{R}^d)$.*

Proof. By definition one can easily see that

$$\mathcal{E}(\psi_R, \psi_R) = \int Q(R^{-1}\xi)|\hat{\psi}(\xi)|^2 d\xi \quad \text{for each } \psi \in C_0^\infty(\boldsymbol{R}^d)\,. \tag{8.6}$$

Thus the lemma follows from the previous lemma, Fatou's lemma, and the dominated convergence theorem.

Lemma 8.3. *Let R_0 be a positive constant and let*

$$R(t) = R_0 t^{1/(d+\alpha)}, \quad t > 0\,. \tag{8.7}$$

Let $\psi \in C_0^\infty(\boldsymbol{R}^d)$ satisfy $\|\psi\| = 1$. Then we have the following:

(i) *If $\varphi(x) = o(|x|^{-(d+\alpha)})$ $(|x| \to \infty)$, then*

$$\Phi_t(\psi_{R(t)}) \leqq \nu R(t)^d m(G) + o(t^{d/(d+\alpha)}) \ (t \to \infty)\,, \tag{8.8}$$

where G denotes any open set containing the support of ψ.

(ii) *If $\varphi(x) = O(|x|^{-(d+\alpha)})$ $(|x| \to \infty)$, then*

$$\Phi_t(\psi_{R(t)}) = O(t^{d/(d+\alpha)}) \ (t \to \infty)\,. \tag{8.9}$$

(iii) *Let* $0<\beta<\alpha$. *If* $K_2=\overline{\lim}_{|x|\to\infty}|x|^{d+\beta}\varphi(x)<\infty$, *then*

$$\Phi_t(\psi_{R(t)})\leqq t^{d/(d+\beta)}\{\kappa(\nu,\beta,K_2)+o(1)\}\ (t\to\infty)\,. \tag{8.10}$$

Proof. Let E denote the support of ψ and let $\delta>0$ be so small that $E^\delta\subset G$, where $E^\delta=\{x\in\boldsymbol{R}^d;\ |x-y|<\delta \text{ for some } y\in E\}$. By change of variables, we get, for each $t>0$,

$$\Phi_t(\psi_{R(t)})=\nu R(t)^d\int\{1-\exp(-t\int\varphi(R(t)\,(y-x))\psi(y)^2dy)\}\,dx\,.$$

Since $\varphi(x)\geqq 0$, we have

$$\Phi_t(\psi_{R(t)})\leqq \nu R(t)^d(m(E^\delta)+I_1),\quad t>0\,, \tag{8.11}$$

where $I_1=\int_{(E^\delta)^c}\{1-\exp(-t\int\varphi(R(t)\,(y-x))\psi(y)^2dy)\}\,dx$. By an elementary calculation one can check that $I_1=o(1)$ $(t\to\infty)$ if $\varphi(x)=o(|x|^{-(d+\alpha)})$ and $I_1=O(1)$ $(t\to\infty)$ if $\varphi(x)=O(|x|^{-(d+\alpha)})$, proving (i) and (ii).

To prove (iii) we observe that

$$\begin{aligned}&\Phi_t(\psi_{R(t)})\\&\quad=\nu t^{d/(d+\beta)}\int\{1-\exp(-t\int\varphi(R(t)y-t^{1/(d+\beta)}x)\psi(y)^2dy)\}\,dx\,.\end{aligned}$$

Let $\varepsilon>0$ and $K'>K_2$ be arbitrarily fixed. Then an elementary calculation gives us

$$t\int\varphi(R(t)y-t^{1/(d+\beta)}x)\psi(y)^2dy\leqq K'(1+o(1))\,,$$

where $o(1)$ tends to zero as $t\to\infty$ uniformly for $|x|>\varepsilon$. Thus we have

$$\begin{aligned}&t^{-d/(d+\beta)}\Phi_t(\psi_{R(t)})\\&\quad\leqq \nu\varepsilon\Omega_d+\nu\int\{1-\exp(-K'(1+o(1)))\}\,dx\\&\quad=\nu\varepsilon\Omega_d+\Gamma\Big(\frac{\beta}{d+\beta}\Big)\nu\Omega_d\{K'(1+o(1))\}^{d/(d+\beta)}\ (t\to\infty)\,.\end{aligned}$$

This completes the proof.

The following lemma, which follows from the above lemmas, proves the lower estimates in Theorems 6.3′ and 6.4′.

Lemma 8.4. *Suppose that the condition* (D) *is satisfied. Let* β *be a constant such as* $0<\beta\leqq\alpha$ *and let*

$$K_2=\overline{\lim_{|x|\to\infty}}\,|x|^{d+\beta}\varphi(x)\quad(0\leqq K_2\leqq\infty)\,. \tag{8.12}$$

If $\beta<\alpha$, *then we have*

$$-\kappa(\nu, \beta, K_2) \leqq \varliminf_{t\to\infty} t^{-d/(d+\beta)} \log I(t) \quad (\leqq 0)\,. \tag{8.13}$$

If $\beta=\alpha$ and $K_2<\infty$, then we have

$$-\infty < \varliminf_{t\to\infty} t^{-d/(d+\alpha)} \log I(t) \quad (\leqq 0)\,. \tag{8.14}$$

Proof. Let $\psi\in C_0^\infty(\boldsymbol{R}^d)$ satisfy $||\psi||=1$. Substitute $\psi_{R(t)}$, which is defined by (8.5) and (8.7), for f in (8.2). Then the lemma is immediate from Lemmas 8.2 and 8.3.

To prove the lower estimate in Theorem 6.2′ we prepare the following lemma.

Lemma 8.5. *Let A be a Borel subset of $\boldsymbol{R}^d$ with $m(A)>0$, and let $f\in\mathcal{D}[\mathcal{E}^{(\alpha)}]$ with $||f||=1$ satisfy $f=0$ a.e. on A^c. Let $\varepsilon>0$ be given. Then there exist a compact subset E of $\boldsymbol{R}^d$ and an element ψ of $\mathcal{D}[\mathcal{E}^{(\alpha)}]$ with $||\psi||=1$ such that $\psi=0$ a.e. on E^c, $m(E)\leqq m(A)+\varepsilon$, and $\mathcal{E}^{(\alpha)}(\psi,\psi)\leqq\mathcal{E}^{(\alpha)}(f,f)+\varepsilon$.*

For this lemma we refer to Lemmas 3.7 and 3.8 of Donsker and Varadhan [2]. Although these lemmas are stated in the terms of their I-functions, the relation between I-functions and Dirichlet forms is substantially given by them [2; p.533] (see also Fukushima [4; Theorem 1.3.1]). Thus one can easily see that the results of their lemmas are transferred to the present situation. So we omit the proof.

Now we prove the lower estimate in Theorem 6.2′.

Lemma 8.6. *Suppose that the condition* (C) *is satisfied and that*

$$\varphi(x) = o(|x|^{-(d+\alpha)}) \quad (|x|\to\infty)\,. \tag{8.15}$$

Then we have

$$-k(\nu, L^{(\alpha)}) \leqq \varliminf_{t\to\infty} t^{-d/(d+\alpha)} \log I(t) \leqq 0\,. \tag{8.16}$$

Proof. Let E be any compact set of $\boldsymbol{R}^d$ and let ψ be any element in $\mathcal{D}[\mathcal{E}^{(\alpha)}]$ with $||\psi||=1$ such that $\psi=0$ *a.e.* on E^c. Let $\{\rho_\delta\}_{\delta>0}\subset C_0^\infty(\boldsymbol{R}^d)$ be a family of mollifiers such that the support of each ρ_δ is contained in the open ball: $|x|<\delta$. Set $\psi^\delta=||\rho_\delta*\psi||^{-1}\rho_\delta*\psi$, where $*$ denotes the convolution. Then it follows that $\psi^\delta\in C_0^\infty(\boldsymbol{R}^d)$, $||\psi^\delta||=1$, and the support of ψ^δ is contained in E^δ. Define, for each $R>0$,

$$\psi_R^\delta(x) = R^{-d/2}\psi^\delta(R^{-1}x), \quad x\in\boldsymbol{R}^d\,,$$

and substitute ψ_R^δ for f in (8.2). Then, by Lemma 8.2 (ii), we have, for each $t>0$,

$$I(t) \geqq ||\psi_R^\delta||_{L^1}^{-2} \exp(-tR^{-\alpha}\{\mathcal{E}^{(\alpha)}(\psi^\delta, \psi^\delta)+o(1)\} \tag{8.17}$$

$$-\Phi_t(\psi_R^\delta)) \quad (R\to\infty).$$

One can easily check that there exists a constant $R_0>0$ such that, for each $t>0$, $R(t)\equiv R_0 t^{1/(d+\alpha)}$ minimizes the function $[tR^{-\alpha}\mathcal{E}^{(\alpha)}(\psi^\delta, \psi^\delta)+\nu R^d m(E^\delta)]$ of $R>0$ and the minimum value is equal to $t^{d/(d+\alpha)}\cdot k_\delta$, where

$$k_\delta = (\nu m(E^\delta))^{\alpha/(d+\alpha)}\left(\frac{d+\alpha}{\alpha}\right)\left\{\frac{\alpha}{d}\mathcal{E}^{(\alpha)}(\psi^\delta, \psi^\delta)\right\}^{d/(d+\alpha)}.$$

Substituting the above $R(t)$ for R in (8.17) and noting that $||\psi_R^\delta||_{L^1}\leqq R^{d/2}m(E^\delta)^{1/2}$, Lemma 8.3 (i) implies

$$I(t)\geqq R(t)^{-d}m(E^\delta)^{-1}\exp\{-t^{d/(d+\alpha)}(k_\delta+o(1))\} \quad (t\to\infty).$$

Hence we have

$$\varliminf_{t\to\infty} t^{-d/(d+\alpha)}\log I(t)\geqq -k_\delta.$$

Noting that $m(E^\delta)\downarrow m(E)$ and $\mathcal{E}^{(\alpha)}(\psi^\delta,\psi^\delta)\to\mathcal{E}^{(\alpha)}(\psi,\psi)$ as $\delta\downarrow 0$, we have

$$\text{(8.18)}\qquad \varliminf_{t\to\infty} t^{-d/(d+\alpha)}\log I(t) \geqq -\{\nu m(E)\}^{\alpha/(d+\alpha)}\left(\frac{d+\alpha}{\alpha}\right)\left\{\frac{\alpha}{d}\,\mathcal{E}^{(\alpha)}(\psi,\psi)\right\}^{d/(d+\alpha)}.$$

On the other hand, note that, for each open set G, there exists an element u in $\mathcal{D}(L_G^{(\alpha)})\subset\mathcal{D}[\mathcal{E}_G^{(\alpha)}]$ with $||u||_G=1$ such that $\lambda(G)=\mathcal{E}_G^{(\alpha)}(u,u)$ (see §3). Let $\varepsilon>0$ be given. By the definition of λ_α one can see that there exists an open subset G of $\boldsymbol{R}^d$ with $m(G)=1$ and an element g in $\mathcal{D}[\mathcal{E}^{(\alpha)}]$ with $||g||=1$ such that $g=0$ *a.e.* on G^c and $\mathcal{E}^{(\alpha)}(g,g)\leqq\lambda_\alpha+\varepsilon$. Further from Lemma 8.5 it follows that there exist a compact subset E of $\boldsymbol{R}^d$ and an element ψ in $\mathcal{D}[\mathcal{E}^{(\alpha)}]$ with $||\psi||=1$ such that $\psi=0$ *a.e.* on E^c, $m(E)\leqq 1+\varepsilon$, and $\mathcal{E}^{(\alpha)}(\psi,\psi)\leqq\lambda_\alpha+2\varepsilon$. Since the bound (8.18) holds for these E and ψ, we have

$$\varliminf_{t\to\infty} t^{-d/(d+\alpha)}\log I(t)\geqq -k(\nu, L^{(\alpha)})+o(1)\ (\varepsilon\downarrow 0).$$

This completes the proof.

9. The upper estimates

First we will prove the upper estimates in Theorems 6.3′ and 6.4′.

Lemma 9.1. *Let X, $\varphi(x)$, $\{p^\omega(dy)\colon \omega\in\Omega\}$, and $\{q(x,\omega)\colon x\in\boldsymbol{R}^d,\ \omega\in\Omega\}$ be as in Theorem 6.1. Let β be a positive constant and let*

$$\text{(9.1)}\qquad K_1=\varliminf_{|x|\to\infty}|x|^{d+\beta}\,\varphi(x)\quad (0\leqq K_1\leqq\infty).$$

Then the function $I(t)$ defined by (6.6) satisfies

(9.2) $$-\kappa(\nu,\beta,K_1)\geqq \varlimsup_{t\to\infty} t^{-d/(d+\beta)}\log I(t)\quad (\geqq -\infty)\,.$$

Proof. It follows from (6.2), (6.9), and the second inequality in (7.2) that, for each $t>0$,

$$\begin{aligned} k(t) &\leqq p(t,0,0)E[\exp\{-tq(0,\omega)\}] \\ &= p(t,0,0)\exp\{-\nu\int_{R^d}(1-e^{-t\varphi(x)})dx\}\,. \end{aligned}$$

By an elementary calculation one can see that (9.1) implies

$$\nu\int_{R^d}(1-e^{-t\varphi(x)})\,dx\geqq t^{d/(d+\beta)}\{\kappa(\nu,\beta,K_1)+o(1)\}\quad (t\to\infty)\,.$$

Noting that $p(t,0,0)$ is a decreasing function of $t>0$, we have the lemma.

It remains to prove the upper estimate for Theorem 6.2′.

Lemma 9.2. *Suppose that the condition* (C) *is satisfied and that* $\int_{R^d}\varphi(x)dx>0$. *Then the function* $I(t)$ *defined by* (6.6) *satisfies*

(9.3) $$\varlimsup_{t\to\infty} t^{-d/(d+\alpha)}\log I(t)\leqq -k(\nu,L^{(\alpha)})\,.$$

Proof. Calculating the expectation with respect to P in (6.6), we have

(9.4) $$I(t)=p(t,0,0)E_{0;0}^{t;0}[F(t)],\quad t>0\,,$$

where

$$F(t)=\exp(-\nu\int_{R^d}(1-\exp\{-\int_0^t\varphi(X_s-y)ds\})dy)\,.$$

We can prove the following estimate:

(9.5) $$\varlimsup_{t\to\infty} t^{-d/(d+\alpha)}\log E_0[F(t)]\leqq -k(\nu,L^{(\alpha)})\,.$$

This has already been substantially obtained by Donsker and Varadhan [2] in the case when the process $\boldsymbol{X}$ itself is a d-dimensional symmetric stable process. But modifying their proof, we can show that (9.5) holds in our general case (we will give a complete proof of (9.5) in a forthcoming paper).

Here is Nakao's trick which reduces (9.3) to (9.5). Note that if $0<a<1$, then $F(t)\leqq F(at)$ and $F(at)$ is $\mathcal{F}_{[0,at]}$-measureable in the notation of §2. Then it follows from (9.4), Theorem 2.2 (iii), and (3.7) that, for each $0<a<1$,

$$\begin{aligned} I(t) &\leqq p(t,0,0)E_{0;0}^{t;0}[F(at)] \\ &= E_0[p(t-at,X_{at},0)F(at)] \\ &\leqq p((1-a)t,0,0)E_0[F(at)]\,. \end{aligned}$$

Hence, by (9.5), we have

$$\overline{\lim_{t\to\infty}}\, t^{-d/(d+\alpha)} \log I(t) \leqq \overline{\lim_{t\to\infty}}\, t^{-d/(d+\alpha)} \log E_0[F(at)]$$
$$\leqq -a^{d/(d+\alpha)} k(\nu, L^{(\alpha)})$$

for $0<a<1$. Letting $a \uparrow 1$, we have the lemma.

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