KNOTS OF UNKNOTTING NUMBER I AND THEIR ALEXANDER POLYNOMIALS

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In 1934 [3], H. Seifert gave a characterization of the Alexander polynomial of knots, but it has been unknown whether there are some relations between the Alexander polynomial and the unknotting number or not.

In this paper, after establishing normal forms of knots with unknotting number $\leq n$, we prove that the Alexander polynomial of any knot is able to be obtained from knots with unkotting number 1. This result shows that there is no relation between the Alexander polynomial and the unknotting number.

1. Normal form of a knot of unknotting number 1

Let axb be an arbitrary arc of the given knot k and let D be a disk such that

(i) $\dot{D} \cap k = axb$, and

(ii) $\overset{\circ}{D} \cap k = \{d\}$, d is a point of k,

where $\dot{D}$ and $\overset{\circ}{D}$ denote respectively the boundary and the interior of D. D will be called an *unknotting disk*. The operation of exchanging the arc axb by its complementary arc ayb with respect to $\dot{D}$ will then be called an *unknotting operation*.
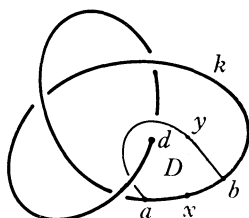

Fig. 1

Any knot can clearly be shown to be deformable into a trivial knot by a finite number of repetition of suitable unknotting operations. The minimum number of operations which are required in order to deform the given knot k into a trivial knot will be called the *unknotting number* of k and denoted by $u(k)$. The unknotting number of a trivial knot will naturally be defined to be 0.

We are now going to establish the normal form of a knot of unknotting number 1 due to H. Terasaka [5].

Let k be a knot with $u(k)=1$. Then there exists by definition an unknotting disk D such that

$$\dot{D} = axb \cup ayb, \ axb \cap ayb = \{a, b\}, \ \mathring{D} \cap k = \{d\} \,,$$

and that

$$k_1 = (k-axb) \cup ayb$$

is a trivial knot. Then there exists an ambient isotopy $\{f_t\}_{0 \leq t \leq 1}$ of the 3-dimensional euclidean space E^3 such that $f_1(k_1)$ is a circle C in the plane E^2.

We denote the images by f_1 of the points a, b, x, y, d, and the disk D respectively by a', b', x', y', d' and by D'.
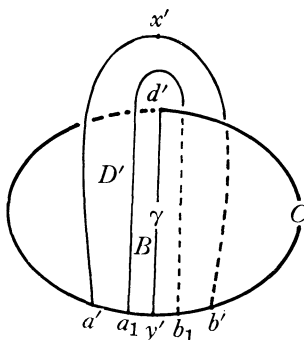

Fig. 2

Let γ be an arc connecting d' and y' inside D'. Let B be a band, that is a narrow, elongate disk, containing γ and contained in D' such that $\dot{B} \cap \dot{D}'$ is an arc $a_1 y' b_1$ of $a'y'b'$. Thus, if $a_1 x_1 b_1$ is the complementary arc of $a_1 y' b_1$ with respect to $\dot{B}$, then

$$k_0 = (C - a_1 y' b_1) \cup a_1 x_1 b_1$$

is of the same type of the original knot k. By a suitable deformation of the band B, k_0 will then be led to a normal form of Seifert's type (Fig. 4).

Thus we have proved:

Theorem 1. *Let C be a circle and let B be a band, i.e. an elongate disk,*

such that C and the boundary $\dot{B}$ of B has an arc ab in common and that C and the interior $\overset{\circ}{B}$ of B has a single point in common. Any knot of unknotting number 1 is then of the type of the knot

$$k_0 = (C \cup \dot{B}) - \textit{the interior of the arc } ab.$$

The band B in the above theorem will be called an *unknotting band*. Then there is no essential difficulty in proving the following generalization of Theorem 1.

Theorem 2. *Let C be a circle and let $B_1, B_2, \cdots, B_n$ be n pieces of mutually disjoint unknotting bands. Any knot of unknotting number at most n is then of some type of the knot k_0:*

$$k_0 = (C \cup \bigcup_{i=1}^{n} \dot{B}_i) - \bigcup_{i=1}^{n} (\textit{Interior of } \dot{B}_i \cap C).$$
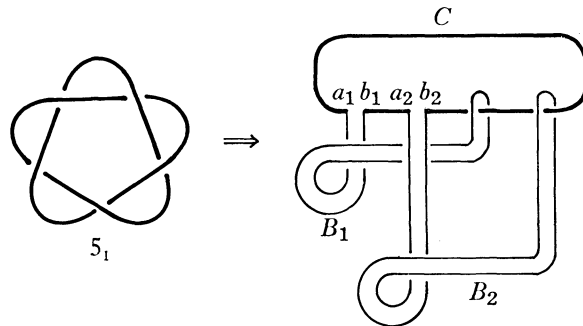
Example.

Fig. 3

N.B.   The knot 5_1 has been shown to be of unknotting number 2 by Murasugi [2].

2. Proof of main theorem

We shall be concerned in this section with some special kind of knots k of unknotting number 1 of the following type:

Let C be a circle and let B be an unknotting band starting first from an arc ab of C outside C. After entangling with itself, B comes inside C passing under C, and then goes out of C passing over C. After entangling with itself again outside C, the band B comes inside C passing under C and goes out of it directly passing over it. The repetition of these entangling, passing under and passing over will end at the final stage when $\overset{\circ}{B}$ intersect C in a point d. We refer the reader to the following Fig. 4.
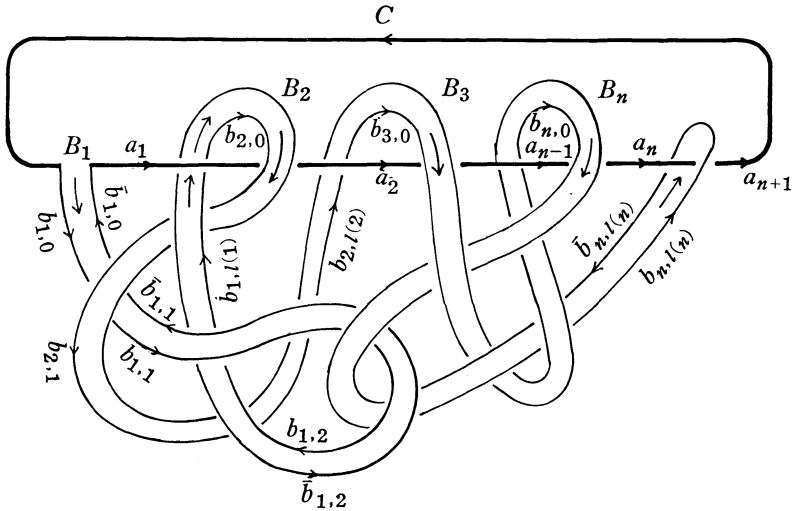
Fig. 4

Let us now begin by computing the Alexander polynomial of this type of knot k. For this purpose, we follow [4] for notation, deviating if necessary, and make full use of the Alexander equations as well as results obtained there. Thus, for example, the part of the band lying between ab and the location where the band passes for the first time under C, are all denoted by the same notation B_1, even if that part of the band is cut to pieces by another part of the whole band passing eventually over there. Especially of use are the following equations[1]:

(1) $\lambda_{i,1}(x-1)B_1 + \lambda_{i,2}(x-1)B_2 + \cdots$
$$+\lambda_{i,n}(x-1)B_n + b_{i,0} - b_{i,l(i)} = 0, \ (i = 1, 2, \cdots, n).$$
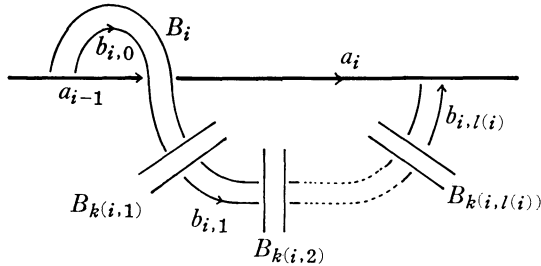

Fig. 5 (1)

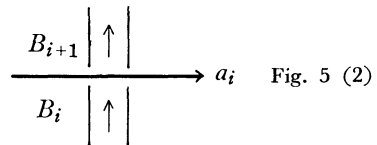(2) $xB_{i+1} - B_i = 0, \ (i = 1, 2, \cdots, n-1)$.

Fig. 5 (2)

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[1] (1) corresponds in [4] to (2.9) with $\eta = \lambda$, (2) to (2.5) with $\delta_i = 1$, (3) to (2.8) with $\varepsilon_i = 1$, $B = B_i$, $\xi_{i+1} = a_i$ and $\xi_i = a_{i-1}$, (4) to (2.11) with $a_{1,0} = a_1$, (5) to (2.6) with $\delta_i = 1$ and $\xi = a_i$.

(3) $(1-x)B_i+a_i-a_{i-1}=0, (i=2, \cdots, n)$.
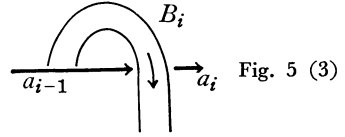
Fig. 5 (3)

(4) $a_1-b_{1\,0}+B_1=0$.
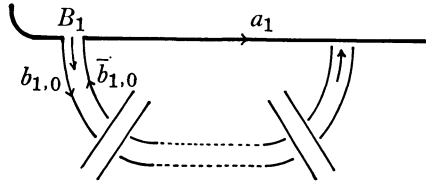
Fig. 5 (4)

(5) $(x-1)a_i-xb_{i+1,0}+b_{i,l(i)}=0, (i=1, 2, \cdots, n-1)$.
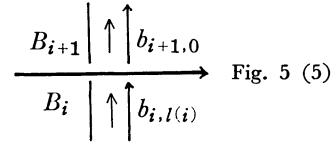
Fig. 5 (5)

The following equations (6) and (7) should be newly introduced: the Alexander equations at the crossing points α and β near the point d run respectively, by (1.12) in [4] with $\varepsilon=-1$.

(α) $(x-1)b_{n,l(n)}+a_{n+1}-xa_n=0$,
(β) $(x-1)a_n+\bar{b}_{n,l(n)}-xb_{n,l(n)}=0$.

Fig. 5 (6)

Since

$$b_{n,l(n)}-\bar{b}_{n,l(n)}=B_n,$$

we have, adding together (α) and (β),

$$-B_n-a_n+a_{n+1}=0.$$

This equation together with (α) gives at once, considering the relation

$$a_{n+1}=b_{1,0},$$

and so the following:

(6) $-B_n-a_n+b_{1,0}=0$,
(7) $(x-1)b_{n,l(n)}+b_{1,0}-xa_n=0$.

Thus, the Alexander matrix for k (cf. Fig. 4) runs as given in Table 1 (cf. [4], p.104).

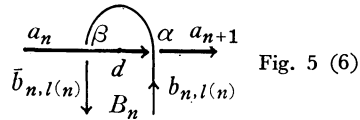|	B_1 B_2 $\cdots$ B_n	$b_{1,0}^{**} b_{2,0}$ $\cdots$ $b_{n,0} b_{1,l(1)}$ $\cdots$ $b_{n,l(n)}$	a_1 a_2 $\cdots$ a_n
(1)	$\lambda_{11}(x-1)\ \lambda_{12}(x-1)\ \cdots\ \lambda_{1n}(x-1)$ $(\lambda_{ij}=\lambda_{ji})$ $\lambda_{1n}(x-1)\ \lambda_{2n}(x-1)\ \cdots\ \lambda_{nn}(x-1)$	$\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{matrix}$ $\begin{matrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{matrix}$	0
(2)	$\begin{matrix} -1 & & x & & \\ & -1 & & x & 0 \\ & & \ddots & & \ddots \\ 0 & & -1 & & x \end{matrix}$	0	0
(3)	$\begin{matrix} 0 & 1-x & & & \\ & 0 & 1-x & & 0 \\ 0 & & \ddots & & \ddots \\ & & 0 & & 1-x \end{matrix}$	0	$\begin{matrix} -1 & 1 & & \\ & -1 & 1 & \\ 0 & & \ddots & \ddots \\ & & & -1 & 1 \end{matrix}$
(6)	$0 \quad \cdots\cdots\cdots \quad 0 \quad -1$	$1 \quad 0 \quad \cdots\cdots \quad 0$	$0 \cdots 0 \ -1$
*(4)	$1 \qquad 0 \quad \cdots\cdots \quad 0$	$-1 \quad 0 \quad \cdots\cdots \quad 0$	$1 \ 0 \ \cdots \ 0$
(5)	0	$\begin{matrix} -x & & 0 \\ & \ddots & \\ 0 & & -x \end{matrix}$ $\begin{matrix} 1 & & 0 \\ & \ddots & \vdots \\ 0 & & 1 & 0 \end{matrix}$	$\begin{matrix} x-1 & & 0 \\ & \ddots & \vdots \\ 0 & & x-1 & 0 \end{matrix}$
(7)	0	$1 \quad 0 \quad \cdots\cdots \quad 0 \qquad 0 \cdots 0 \ x-1$	$0 \cdots 0 \ -x$

Table 1

$$M=$$

$\lambda_{11}x^{n-1}(x-1)+\lambda_{12}x^{n-2}(x-1)+\cdots+\lambda_{1n}(x-1)$ $(\lambda_{ij}=\lambda_{ji})$ $\lambda_{1n}x^{n-1}(x-1)+\lambda_{2n}x^{n-2}(x-1)+\cdots+\lambda_{nn}(x-1)$	$\begin{matrix} 0 & \cdots & 0 \\ 1 & & \\ & \ddots & 0 \\ 0 & \ddots & \\ & & 1 \end{matrix}$ $\begin{matrix} -1 & & \\ & \ddots & 0 \\ 0 & & \ddots \\ & & -1 \end{matrix}$	0
$\begin{matrix} -x^{n-1}+x^{n-2} \\ -x^{n-2}+x^{n-3} \\ \vdots \\ -x^2+x \\ -x+1 \end{matrix}$	$0 \qquad\qquad 0$	$\begin{matrix} -1 & 1 & & \\ & -1 & 1 & 0 \\ & & \ddots & \ddots \\ 0 & & & -1 & 1 \end{matrix}$
-1	$0 \qquad\qquad 0$	$0 \qquad -1$
0 $\vdots$ 0	$\begin{matrix} -x & & \\ & \ddots & 0 \\ 0 & & -x \end{matrix}$ $\begin{matrix} 1 & & 0 \\ & \ddots & \vdots \\ 0 & & 1 & 0 \end{matrix}$	$\begin{matrix} x-1 & & 0 \\ & \ddots & \vdots \\ 0 & & x-1 \end{matrix}$
0	$0 \qquad\qquad 0 \quad x-1$	$0 \qquad -x$

Now, since the row (4) (marked with *) can be obtained by adding together the rows (2), (3) and (6), we delete it; and since $b_{1,0}$ can be set equal to 0 (cf. [4], p.105), we delete the column under $b_{1,0}$ (marked with **). Then the determinant of the remaining matrix is just the required Alexander polynomial $\Delta_k(x)$ of k.

To compute the determinant, multiply first the column under B_1 by x and add it to the column under B_2, and multiply the new column under B_2 by x and add it to the column under $B_3, \cdots$, and multiply the new column under B_{n-1} by x and add it to the column under B_n. Then, we have immediately:
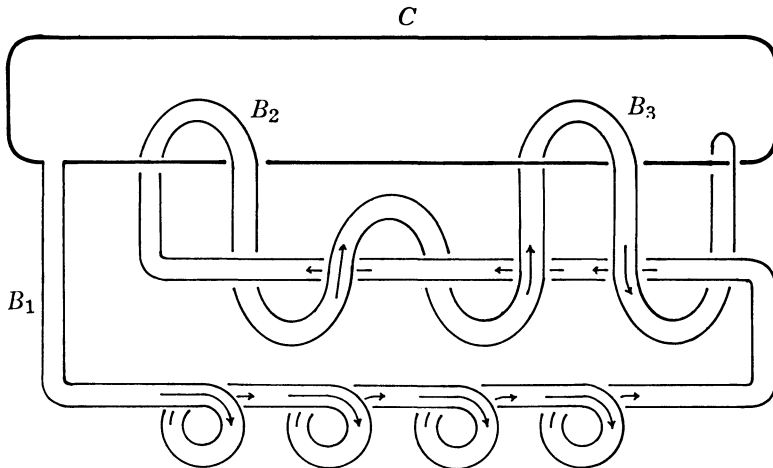
$$\Delta_k(x) = \det(M)$$

We find that

$$\Delta_k(x) = (\lambda_{11}x^{n-1} + \lambda_{12}x^{n-2} + \cdots + \lambda_{1n})(x-1)^2$$
$$+ (\lambda_{12}x^{n-1} + \lambda_{22}x^{n-2} + \cdots + \lambda_{2n})x(x-1)^2$$
$$+ \cdots\cdots$$
$$+ (\lambda_{1n}x^{n-1} + \lambda_{2n}x^{n-2} + \cdots + \lambda_{nn})x^{n-1}(x-1)^2$$
$$- (n-1)x^{n-1}(x-1)^2 + x^n.$$

For simplicity let us now set all $\lambda_{ij} = \lambda_{ji}$ other than $\lambda_{1j} = \lambda_{j1}$ equal to 0, and set

$$\lambda_{11} = \alpha_{n-1}, \ \lambda_{12} = \alpha_{n-2}, \cdots, \lambda_{1n} = \alpha_0.$$

$$\lambda_{11} = 4, \ \lambda_{12} = 2, \ \lambda_{13} = -1$$

Fig. 6

Then we have for this knot k_0,

$$(8) \quad \Delta_{k_0}(x) = (\alpha_{n-1}x^{n-1}+\alpha_{n-2}x^{n-2}+ \cdots +\alpha_1 x+\alpha_0)\,(x-1)^2$$
$$+(\alpha_{n-2}x^n+\alpha_{n-3}x^{n+1}+ \cdots +\alpha_1 x^{2n-3}+\alpha_0 x^{2n-2})\,(x-1)^2$$
$$-(n-1)x^{n-1}(x-1)^2+x^n$$

or

$$\Delta_{k_0}(x) = \alpha_0(x^{2n}+1)+(-2\alpha_0+\alpha_1)\,(x^{2n-1}+x)$$
$$+(\alpha_0-2\alpha_1+\alpha_2)\,(x^{2n-2}+x^2)+ \cdots$$
$$+(\alpha_{i-2}-2\alpha_{i-1}+\alpha_i)\,(x^{2n-i}+x^i)+ \cdots$$
$$+(\alpha_{n-3}-2\alpha_{n-2}+\alpha_{n-1}-n+1)\,(x^{n+1}+x^{n-1})$$
$$+(2\alpha_{n-2}-2\alpha_{n-1}+2n-1)x^n .$$

Now, if

$$\Delta(x) = a_0(x^{2n}+1)+a_1(x^{2n-1}+x)+ \cdots +a_{n-1}(x^{n+1}+x^{n-1})+a_n x^n$$

with $\Delta(1)=1$ is the Alexander polynomial of a given knot, then the solution $\alpha_0, \alpha_1, \cdots, \alpha_{n-1}$ of the linear equations

$$(*) \quad \begin{cases} \alpha_0 = a_0 , \\ -2\alpha_0+\alpha_1 = a_1 , \\ \alpha_0-2\alpha_1+\alpha_2 = a_2 , \\ \cdots\cdots\cdots\cdots \\ \alpha_{i-2}-2\alpha_{i-1}+\alpha_i = a_i , \\ \cdots\cdots\cdots\cdots \\ \alpha_{n-3}-2\alpha_{n-2}+\alpha_{n-1}-n+1 = a_{n-1} \end{cases}$$

satisfies the equation

$$2\alpha_{n-2}-2\alpha_{n-1}+2n-1 = a_n$$

on account of

$$\Delta_{k_0}(1) = 1$$

by (8), which proves that $\Delta(x)$ coincides with some $\Delta_{k_0}(x)$.

We have thus proved our main theorem:

Theorem 3. *For a given Alexander polynomial $\Delta(x)$, there exists a knot k_0 of unknotting number 1 with the Alexander polynomial $\Delta(x)$. In fact, for*

$$\Delta(x) = a_0(x^{2n}+1)+a_1(x^{2n-1}+x)+ \cdots +a_{n-1}(X^{n+1}+x^{n-1})+a_n x^n$$

with $\Delta(1)=1$, let us set all $\lambda_{ij}=\lambda_{ji}=0$ other than $\lambda_{1j}=\lambda_{j1}$, and let

$$\lambda_{11} = \alpha_{n-1}, \quad \lambda_{12} = \alpha_{n-2}, \quad \cdots, \quad \lambda_{1n} = \alpha_0 ,$$

where $\{\alpha_0, \cdots, \alpha_{n-2}, \alpha_{n-1}\}$ *is a solution of* (*). *Then,* k_0 *is obtained as a normal form of a knot of unknotting number* 1 *with respect to the* $\{\lambda_{ij}\}$. *Here,* λ_{ij} *is the linking number of the band* B_j *and* B_i.

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