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A DECOMPOSITION OF THE SPACE $\mathcal{M}$ OF RIEMANNIAN METRICS ON A MANIFOLD

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0. Introduction

Let M be a compact C^∞-manifold. We denote by $\mathcal{M}$, $\mathcal{D}$ and $\mathcal{F}$ the space of all riemannian metrics on M, the diffeomorphism group of M, and the space of all positive functions on M, respectively. Then the group $\mathcal{D}$ and $\mathcal{F}$ acts on $\mathcal{M}$ by pull back and multiplication, respectively. D. Ebin and N. Koiso establish Slice theorem [4, Theorem 2.2] on the action of $\mathcal{D}$.

In this paper, we shall give a decomposition theorem on the action of $\mathcal{F}$ (Theorem 2.5). That is, there is a local diffeomorphism from $\mathcal{F}\times\bar{C}$ into $\mathcal{M}$ where $\bar{C}$ is a subspace of $\mathcal{M}$ of riemannian metrics with volume 1 and of constant scalar curvature τ_g such that $\tau_g=0$ or $\tau_g/(n-1)$ is not an eigenvalue of Δ_g. Combining the above theorems, we get the following decomposition of a deformation (Corollary 2.9). Let $g\in\bar{C}$ and $g(t)$ be a deformation of g. Then there are a curve $f(t)$ in $\mathcal{F}$, a curve $\gamma(t)$ in $\mathcal{D}$ and a curve $\bar{g}(t)$ in $\bar{C}$ such that $\delta\bar{g}'(0)=0$, which satisfy the equation $g(t)=f(t)\gamma(t)^*\bar{g}(t)$. (For the operator δ, see **1.**)

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1. Preliminaries

First, we introduce notation and definitions which will be used throughout this paper. Let M be an n-dimensional, connected and compact C^∞-manifold, and we always assmue $n\geqq 2$. For a vector bundle T over M, we *denote by* $H^r(T)$ the space of all H^r-sections, where H^r means an object which has derivatives defined almost everywhere up to order r and such that each partial derivative is square integrable. Then $H^r(T)$ is isomorphic to a Hilbert space and *the space* $C^\infty(T)$ of all C^∞-sections becomes an inverse limit of $\{H^r(T)\}_{r=1,2,\cdots}$. Therefore such a space is said to be *an ILH-space*. If a topological space $\mathcal{X}$ is isomorphic to an ILH-space locally, $\mathcal{X}$ is said to be *an ILH-manifold*. For details, see [5].

Let g be an H^s-metric on M. We consider the riemannian connection and use the following notations:

v_g; the volume element with respect to g,

R; the curvature tensor,

ρ; the Ricci tensor,

(For the standard sphere with orthnormal basis, $R_{121}{}^2=R_{1212}<0$ and $\rho_{11}<0$.)

τ; the scalar curvature,

$(\, , \,)$; the inner product in fibres of a tensor bundle defined by g,

$\langle \, , \, \rangle$; the global inner product for sections of a tensor bundle over M, i.e., $\langle \, , \, \rangle=\int_M (\, , \,) v_g$,

S^2; the symmetric covariant 2-tensor bundle over M,

$H^r(M)$; the Hilbert space of all H^r-functions,

$H^r_g(M)$; the Hilbert space of all H^r-functions f such that $\int_M f v_g=0$,

$H^r_g(S^2)$; the Hilbert space of all symmetric bilinear H^r-forms h such that $\langle h, g\rangle=0$,

∇; the covariant derivation,

δ; the formal adjoint of ∇ with respect to $\langle \, , \, \rangle$,

δ^*; the formal adjoint of $\delta | H^r(S^2)$,

$\Delta=\delta d$; the Laplacian operating on the space $H^r(M)$,

$\bar{\Delta}=\delta\nabla$; the rough Laplacian operating on the space $H^r(T^p_q)$,

Hess$=\nabla d$; the Hessian on the space $H^r(M)$,

$\mathcal{F}$; the ILH-manifold of all positive C^∞-functions on M,

$\mathcal{F}^r$; the Hilbert manifold of all positive H^r-functions on M,

$\mathcal{M}$; the ILH-manifold of all C^∞-metrics on M,

$\mathcal{M}^r$; the Hilbert manifold of all H^r-metrics on M,

$\mathcal{M}_1$; the ILH-manifold of all C^∞-metrics with volume 1,

$\mathcal{M}^r_1$; the Hilbert manifold of all H^r-metrics with volume 1.

When we consider the metric space $\mathcal{M}^s$, the covariant derivation, the curvature tensor and the Ricci tensor with respect to an element g of $\mathcal{M}^s$ will be *denoted by* ∇_g, R_g or ρ_g. By *a deformation of* g we mean a C^∞-curve $g(t)\colon I\to\mathcal{M}$ such that $g(0)=g$, where I is an open interval. The differential $g'(0)$ is called *an infinitesimal deformation,* or simply *an i-deformation.* If there is a 1-parameter family $\gamma(t)$ of diffeomorphisms such that $g(t)=\gamma(t)^*g$ then the deformation $g(t)$ is said to be *trival.* If there is a 1-form ξ such that $h=\delta^*\xi$, then the i-deformation h is said to be *trival.* On the other hand, an i-deformation h is said to be *essential* if $\delta h=0$.

Now, we give some fundamental propositions.

Lemma 1.1 [6,11.3]. *Let E and F be vector bundles over M and $f\colon E\to F$ be a fiber preserving C^∞-map. If $s>\frac{n}{2}$, then the map $\phi\colon H^s(E)\to H^s(F)$ which is defined by $\phi(\alpha)=f\circ\alpha$ is C^∞.*

Proposition 1.2. *If $s>\frac{n}{2}$, then the map D: $\mathcal{M}^{s+1}\times H^{s+1}(T^p_q)\to H^s(T^p_{q+1})$ which is defined by $D(g, \xi)=\nabla_g\xi$ is C^∞.*

Proof. Let g_0 be a fixed C^∞-metric on M. We define the tensor field $T(g)$ by $T(g)(X, Y)=(\nabla_g)_X Y-(\nabla_{g_0})_X Y$ for an H^s-metric g on M. Then we get

$$(T(g))^k{}_{ij}=\frac{1}{2}g^{kl}\{(\nabla_{g_0})_i g_{lj}+(\nabla_{g_0})_j g_{li}-(\nabla_{g_0})_l g_{ij}\},$$

and

$$\begin{aligned}(D(g, \xi))^{i_1\cdots i_p}{}_{j_0\cdots j_q}&-(D(g_0, \xi))^{i_1\cdots i_p}{}_{j_0\cdots j_q}\\&=-\sum_{a=1}^{k}(T(g))^l{}_{j_0 j_a}\,\xi^{i_1\cdots i_p}{}_{j_1\cdots j_{a-1}lj_{a+1}\cdots j_q}\\&\quad+\sum_{b=1}^{p}(T(g))^{i_b}{}_{j_0 k}\,\xi^{i_1\cdots i_{b-1}ki_{b+1}\cdots i_p}{}_{j_1\cdots j_q}.\end{aligned}$$

By the definition of the H^s-topology, we know that the map $:g\to(\nabla_{g_0})g$ is a C^∞-map from $\mathcal{M}^{s+1}$ to $H^s(T^0_3)$. Hence Lemma 1.1 implies that the map: $g\to T(g)$ is a C^∞-map from $\mathcal{M}^{s+1}$ to $H^s(T^1_2)$. Applying Lemma 1.1 to the above formula, we see that the map : $(T(g), \xi)\to D(g, \xi)-D(g_0, \xi)$ is a C^∞-map from $H^s(T^1_2)\times H^{s+1}(T^p_q)$ to $H^s(T^p_{q+1})$. But the map : $\xi\to D(g_0, \xi)$ is a continuous linear map from $H^{s+1}(T^p_q)$ to $H^s(T^p_{q+1})$, hence the map: $(T(g), \xi)\to D(g, \xi)$ is C^∞. Thus we see that the map D is a composition of C^∞-maps, and so is C^∞.

Corollary 1.3. *If $s>\frac{n}{2}$, then the map : $(g, f)\to\nabla_g f$ is a C^∞-map from $\mathcal{M}^{s+1}\times H^{s+2}(M)$ to $H^s(M)$.*

Proof. We apply Proposition 1.2 to the formula ; $\Delta_g f=-g^{ij}\nabla_i d_j f$.

Corollary 1.4. *If $s>\frac{n}{2}$, then the maps : $g\to R, \rho, \tau$ are C^∞-maps from $\mathcal{M}^{s+2}$ to $H^s(T^1_3)$, $H^s(S^2)$ and $H^s(M)$, respectively.*

Proof. The smoothness of the map : $g\to R$ completes the proof. By easy computation, we get the next formula :

$$\begin{aligned}R(g)_{ijk}{}^l-R(g_0)_{ijk}{}^l&=(\nabla_{g_0})_i(T(g))^l{}_{jk}-(\nabla_{g_0})_j(T(g))^l{}_{ik}\\&\quad+(T(g))^l{}_{im}(T(g))^m{}_{jk}-(T(g))^l{}_{jm}(T(g))^m{}_{ik}.\end{aligned}$$

Thus, applying Proposition 1.2, we see that the map : $g\to R$ is C^∞.

Lemma 1.5 [9,(19.5); 1,(2.11) (2.12)]. *Let $g(t)$ be a deformation of g. If we set $h=g'(0)$, then we have the following formulae;*

$$\frac{d}{dt}\Big|_0\tau_{g(t)}=\Delta\mathrm{tr}\,h+\delta\delta h-(h, \rho), \tag{1.5.1}$$

$$\frac{d}{dt}\Big|_0\rho_{g(t)}=\frac{1}{2}\{\bar{\Delta}h+2Qh+2Lh-2\delta^*\delta h-\mathrm{Hess}\ \mathrm{tr}\ h,\} \tag{1.5.2}$$

where $2(Qh)_{ij}=\rho_i{}^k h_{kj}+\rho_j{}^k h_{ik}$ and $(Lh)_{ij}=R_{ikjl}h^{kl}$.

2. A decomposition of the space $\mathcal{M}$

We *denote by* C^r the space of all H^r-metrics with constant scalar curvature and with volume 1. Fix a C^∞-metric $g_0 \in \mathcal{M}_1$. For an integer $r > \frac{n}{2}+4$ and $g \in \mathcal{M}_1^r$, we define a C^∞-map

$$\sigma_g^r : H_{g_0}^r(M) \to H_{g_0}^{r-4}(M)$$

by $\sigma_g^r(f) = (n-1)(\Delta_g)^2 f - \tau_g \Delta_g f - \int \{(n-1)(\Delta_g)^2 f - \tau_g \Delta_g f\} v_{g_0}$.
In fact the map: $(g, f) \to \sigma_g^r(f)$ is a C^∞-map from $\mathcal{M}_1^r \times H_{g_0}^r(M)$ to $H_{g_0}^{r-4}(M)$ owing to Corollary 1.3 and Corollary 1.4. First we show some lemmas.

Lemma 2.1. *If we denote by K^r the subset of $\mathcal{M}_1^r$ of all metrics $g \in \mathcal{M}_1^r$ such that σ_g^r is an isomorphism, then K^r is open in $\mathcal{M}_1^r$.*

Proof. The map : $g \to \sigma_g^r$ is a C^∞-map from $\mathcal{M}_1^r$ to the space $L(H_{g_0}^r(M), H_{g_0}^{r-4}(M))$ of all continuous linear maps from $H_{g_0}^r(M)$ to $H_{g_0}^{r-4}(M)$. On the other hand the set of all isomorphisms is open in $L(H_{g_0}^r(M), H_{g_0}^{r-4}(M))$, hence K^r is open $\mathcal{M}_1^r$.

Lemma 2.2. *Let $\bar{C}$ be the subset of $\mathcal{M}$ of all metrics g with constant scalar curvature τ_g such that $\tau_g = 0$ or $\tau_g/(n-1)$ is not an eigenvalue of Δ_g. Then $C^r \cap K^r \cap \mathcal{M} = \bar{C}$.*

Proof. Let $g \in \bar{C}$. Then $g \in C^r \cap \mathcal{M}$, and so it is sufficient to prove that $g \in K^r$. If $f \in \operatorname{Ker} \sigma_g^r$ then $(n-1)(\Delta_g)^2 f - \tau_g \Delta_g f$ is a constant. By integration we see

$$(n-1)(\Delta_g)^2 f - \tau_g \Delta_g f = 0 .$$

But here $\tau_g = 0$ or τ_g is not an eigenvalue of Δ_g. Hence $\Delta_g f$ is a constant, and so the assumption that $f \in H_{g_0}^r(M)$ implies $f = 0$. Thus we see σ_g^r is injective. On the other hand $\operatorname{Im}\{(n-1)(\Delta_g)^2 - \tau_g \Delta_g\} = H_g^{r-4}(M)$ implies σ_g^r is surjective. Therefore $\bar{C} \subset C^r \cap K^r \cap \mathcal{M}$, and by the definition of $\bar{C}$ and K^r we see $\bar{C} \supset C^r \cap K^r \cap \mathcal{M}$.

Lemma 2.3.(1) *$C^r \cap K^r$ is an submanifold of $\mathcal{M}_1^r$.*

Proof. We define a C^∞-map $\widetilde{\Delta\tau} : \mathcal{M}_1^r \to H_{g_0}^{r-4}(M)$ by

$$\widetilde{\Delta\tau}(g) = \Delta_g \tau_g - \int \Delta_g \tau_g v_{g_0} .$$

Then $C^r = (\widetilde{\Delta\tau})^{-1}(0)$. By differentiation we get

(1) A.E. Fischer and J.E. Marsden [8, Theorem 3] show that the space $\boldsymbol{R} \cdot \bar{C}$ becomes a submanifold of $\mathcal{M}$.

$$T_g(\widetilde{\Delta\tau})(h) = \Delta'_{(g,h)}\tau_g + \Delta_g\tau'_{(g,h)} - \int\{(\Delta'_{(g,h)} + \Delta_g\tau'_{(g,h)}\} v_{g_0} .$$

Let $g \in C^r$. Then we get

$$\Delta'_{(g,h)}\tau_g = \frac{d}{dt}|_0 \Delta_{g+th}\tau_g = 0 .$$

If h is conformal, i.e., there is $f \in H^r_g(M)$ such that $h=fg$, by substituting to the formula (1.5.1) we get

$$\tau'_{(g,fg)} = (n-1)\Delta_g f - \tau_g f .$$

Thus we get $T_g(\widetilde{\Delta\tau})\ (fg) = \sigma^r_g(f)$, and $T_g(\widetilde{\Delta\tau})$ is surjective. This implies, by implicit function theorem, $C^r \cap K^r$ is a submanifold of $\mathcal{M}^r_1$, and so of $\mathcal{M}^r$.

Lemma 2.4. *Define a C^∞-map $\chi^r : \mathcal{F}^r \times (C^r \cap K^r) \to \mathcal{M}^r$ by $\chi^r(f,g)=fg$. If $g \in \bar{C}$ then $T_{(f,g)}\chi^r$ is an isomorphism.*

Proof. Injectivity. We see

$$(T_{(f,g)}\chi^r)(\phi, h) = fh + \phi g .$$

If $fh+\phi g=0$, then $\tilde{\phi} g \in \mathrm{Ker}\ T_g(\widetilde{\Delta\tau})$, where $\tilde{\phi} = -\phi/f$. Hence

$$\Delta_g\ \mathrm{tr}_g(\tilde{\phi} g) + \delta_g\delta_g(\tilde{\phi} g) - (\tilde{\phi} g, \rho_g)_g = 0 ,$$

therefore $(n-1)\Delta_g\tilde{\phi} - \tau_g\tilde{\phi} = 0$.
But here $g \in \bar{C}$, which implies $\tilde{\phi}=0$, and so $h=0$, $\phi=0$.

Surjectivity. The equation $\mathrm{Im}\, T_{(f,g)}\chi^r = fT_g(C^r) + H^r(M)g$ shows that $\mathrm{Im} T_{(f,g)}\chi^r$ is closed in $H^r(S^2)$. Hence, if $T_{(f,g)}\chi^r$ is not surjective then there exists a non-zero element $\bar{h}$ in $H^r(S^2)$ orthogornal to $fT_g(C^r)$ and $H^r(M)g$. We set

$$K_g(h) = \Delta_g(\Delta_g \mathrm{tr}_g h + \delta_g\delta_g h - (h, \rho_g)_g) .$$

Then we get $T_g(C^r) = \mathrm{Ker}\ T_g(\widetilde{\Delta\tau}) = \mathrm{Ker}\ T_g(\Delta\tau) = \mathrm{Ker}\ K_g$. On the other hand K_g has surjective symbol. Hence [2, Corollary 6.9] implies that $H^r(S^2)$ has the decomposition

$$H^r(S^2) = \boldsymbol{R}g \oplus T_g(C^r) \oplus \mathrm{Im}\ K_g{}^* ,$$

where $K_g{}^*$ is the formal adjoint of K_g. $f\bar{h}$ is orthogonal to $T_g(C^r)$ and $H^r(M)g$, hence $f\bar{h} \in \mathrm{Im} K_g{}^*$. If we set $f\bar{h} = K_g{}^*(\psi)$, then we see

$$f\bar{h} = (\Delta_g)^2\psi + \nabla_g\nabla_g\Delta_g\psi - \Delta_g\psi\rho_g .$$

Since $f\bar{h}$ is orthogonal to $H^r(M)g$, we see

$$0=\mathrm{tr}_g(f\bar{h})=(n-1)(\Delta_g)^2\psi-\tau_g\Delta_g\psi\,.$$

By the assumption that $g\in\bar{C}$, we see $\Delta_g\psi=0$ and so $f\bar{h}=0$, which contradicts the assumption that $\bar{h}\neq 0$.

Theorem 2.5.(2) *The space $\bar{C}$ is an ILH-submanifold of $\mathcal{M}$ and the map $\chi:\mathcal{F}\times\bar{C}\to\mathcal{M}$ is a local ILH-diffeomorphism into $\mathcal{M}$, where χ is defined by $\chi(f,g)=fg$.*

(For the notation ILH, see [5, pp. 168–169].)

Remark 2.6. J.L. Kazdan and F.W. Warner [3, Theorem 1.1] show that $\bar{C}$ is not empty.

Remark 2.7. When $n=2$, this result is classical. That is, any metric g is conformal to some metric with constant scalar curvature.

Proof. We fix a sufficiantly large integer r. By Lemma 2.2, Lemma 2.4 and the inverse function theorem there is an open neighbourhood W^r of $\mathcal{F}\times\bar{C}$ in $\mathcal{F}^r\times(C^r\cap K^r)$ such that $\chi^r\,|\,W^r$ is a local diffeomorphism. We denote by $\bar{C}^r$ the set of all metrics $g\in C^r\cap K^r$ such that there is an H^r-function f such that $(f,g)\in W^r$. For an integer $s\geqq r$ we set $\bar{C}^s=\bar{C}^r\cap\bigcap_{i=r}^{s}(C^i\cap K^i)$. We easily see that $\bar{C}^s\supset\bar{C}^{s+1}$ and, by Lemma 2.1, that $\bar{C}^s$ is open in $C^s\cap K^s$. Moreover we see $\bigcap_{s=r}^{\infty}\bar{C}^s=\bar{C}$ by Lemma 2.2, and thus we can define an ILH-structure on $\bar{C}$ as $\bar{C}=\varprojlim\bar{C}^s$.

Next we shall prove that the map $\chi^r\,|\,\mathcal{F}^s\times\bar{C}^s:\mathcal{F}^s\times\bar{C}^s\to\mathcal{M}^s$ is a local diffeomorphism. Lemma 1.1 implies the smoothness of this map. To prove the smoothness of the inverse map, we choose an open covering $\{W^r_\alpha\}$ of W^r such that $\chi^r\,|\,W^r_\alpha$ is a diffeomorphism. We apply the following lemma to $(\chi^r\,|\,W^r)^{-1}$.

Lemma 2.8 [4, Lemma 2.8]. *Let E and F be vector bundles over M associated to the frame bundle of M. Then there exists a cannonical linear map $\eta^*:H^0(E)\to H^0(E)$ for a diffeomorphism η of M. Let A be an open set of $H^r(E)$ and $\phi:A\to H^r(F)$ be a C^∞-map which commutes with any η^*. If we set $A^s=A\cap H^s(E)$ for $s\geqq r$, then $\phi(A^s)\subset H^s(F)$ and the map $\phi\,|\,A^s:A^s\to H^s(F)$ is C^∞.*

If we set $\mathrm{Im}(\chi^r\,|\,W^r_\alpha)=A$ and $(\chi^r\,|\,W^r_\alpha)^{-1}=\phi$, then ϕ is a C^∞-map from A into $H^r(M)\times H^r(S^2)$ which commutes with the action of the diffeomorphism group $\mathcal{D}$ of M. Hence Lemma 2.8 implies that the map

(2) J.P. Bourguignon [7, VIII. 8. Proposition] shows that $\tau:\mathcal{M}\to\mathcal{F}$ is a submersion around a metric $g\in\mathcal{M}$ such that τ_g is not non-negative constant.

$$(\chi^r | W_\alpha^r)^{-1} | A^s : A^s \to H^s(M) \times H^s(S^2)$$

is C^∞. But here $\mathcal{F}^s \times \bar{C}^s$ is a submanifold of $H^s(M) \times H^s(S^2)$, hence the map $(\chi^r | W^r)^{-1} | A^s : A^s \to \mathcal{F}^s \times \bar{C}^s$ is C^∞. Thus χ^s is a local diffeomorpnhism and $\chi = \varprojlim \chi^s$ is an ILH-diffeomorphism, which implies that $\bar{C}$ is an ILH-submanifold of $\mathcal{M}$.

Corollary 2.9. *Let $g=f\bar{g}$, where $f \in \mathcal{F}$ and $\bar{g} \in \bar{C}$. If $g(t)$ is a deformation of g with sufficiently small domain of t, then there exist a 1-parameter family of positive functions $f(t)$ on M, a 1-parameter family of diffeomorphisms $\gamma(t)$ of M and a deformation $\bar{g}(t)$ in $\bar{C}$ such that $f(0)=f$, $\delta\bar{g}'(0)=0$ and $g(t)=f(t)\gamma(t)^*\bar{g}(t)$.*

Proof. By Theorem 2.5, $g(t)$ is decomposed into $f(t)\tilde{g}(t)$, where $\tilde{g}(t)$ is a deformation in $\bar{C}$. Applying Slice theorem [4, Theorem 2.2] to $\tilde{g}(t)$, we get $\tilde{g}(t) = \gamma(t)^*\bar{g}(t)$, where $\bar{g}(t)$ is a deformation such that $\delta\bar{g}'(0)=0$. Also we easily see that $\bar{g}(t) \in \bar{C}$ for each t.

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