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THIN PATCHES AND SEMIPRIME FGC-RINGS

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Throughout this paper all rings are commutative with identity and all modules are unitary. A ring is called *FGC* (*NFGC*) provided every finitely generated (finitely generated nonsingular) module over the ring is a direct sum of cyclic submodules. For a given ring R, we denote the set of all prime ideals of R by spec(R). For a subset X of spec(R), we use $\min X$ and $\max X$ to denote the set of all minimal elements of X and the set of all maximal elements of X, respectively. X is said to be a thin patch if it coincides with the (patch) closure of $\min X$ in spec(R) ([10]).

In this paper, we show the following result, which seems to be a generalization of R. S. Pierce [7, Proposition 20.1]: Let R be a semiprime *NFGC*-ring and X the (patch) closure of minspec(R) in spec(R). Then

(1) $X=\min X\cup\max X$, and
(2) X has no 3-points.

Using this result, we can guarantee the following conjecture*) raised by T. Shores and R. Wiegand ([10]) is indeed true: Every *FGC*-ring has only finitely many minimal prime ideals. Thus, as was point out in [10], we should note that the solution for this conjecture allows us to remove the hypothesis "with Noetherian maximal ideal spectrum" from S. Wiegand [13, Corollary]. Consequently, the structure of a semiprime *FGC*-ring R is completely settled as follows: R is a finite direct product of h-local Bezout domains and each localization of R is an almost maximal valuation ring. The reader is referred to [8]–[12] for the study of *FGC*-rings.

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Let R be a ring. We denote its maximal ring of quotients by $Q(R)$. An R-module is said to be non-singular if every non-zero element of the ring is not annihilated by an essential ideal of R.

For a subset I of R, we put $V(I)=\{x\in\text{spec}(R)\,|\,x\supseteq I\}$ and $D(I)=\text{spec}(R)-$

*) After writing this paper, I was informed by R. Wiegand that he had already solved this, independently. His proof can be found in [11] or [12].

$V(I)$. It is showed in M. Hochster [3] that the family of all sets $V(a)\cap D(b_1)\cap\cdots\cap D(b_n)$, where $a, b_1, \cdots, b_n\in R$, forms an open basis for a topology on spec (R), and spec (R) becomes a Boolean space (that is, a compact, Hausdorff totally disconnected space) with this topology. This topology is called the patch topology. Note that the patch topology is stronger than the usual Zariski topology. In this paper, unless otherwise stated, all topological notions on spec (R) refer to the patch topology.

For a subset X of spec (R), by X^p, min X and max X we denote the closure of X in spec (R), the set of all minimal elements (by inclusion) of X and the set of all maximal elements of X, respectively. Following T. Shores and R. Wiegand [10], X is called thin patch provided $X=(\min X)^p$.

For a ring extension Q of R with the same identity, by $\lambda(Q; R)$ we denote the canonical mapping from spec (Q) to spec (R) given by $y\to y\cap R$. Clearly $\lambda(Q; R)$ is a continuous mapping. Moreover it is a closed mapping, since both spec (Q) and spec (R) are compact Hausdorff spaces.

Let X be a topological space, $x\in X$ and α a cardinal number. x is called an α-point if x lies simultaneously in the closure of each member of a pairwise disjoint family of α open subsets of X which do not contain x ([7]). Let N be the discrete space of natural number, and let βN be the Stone-Čech compactification of N (see [1] or [14]). In [7], R. S. Pierce showed assuming the continuum hypothesis that $\beta N-N$ has a 3-point, and he then asked if the existence of a 3-point can be shown without using the continuum hypothesis. N. Hindman [2] answered this in the affirmative. Using the existence of a 3-point in $\beta N-N$, R. S. Pierce also proved the following remarkable result ([7, Lemma 21.5]):

(Pierce's lemma) *Any infinite Boolean space contains a closed subset which has a* 3-*point* (*relative to the topology of the space*).

For latter use, we shall review an outline of the Pierce's proof of this: Let X be an infinite Boolean space. Then we can choose a countably infinite family $\{P_n \mid n\in N\}$ of pairwise disjoint non-empty open-closed subsets of X. Let $x_n\in P_n$, and let Z be the closure of $\{x_n \mid n\in N\}$. When Z has no 3-points, Pierce elegantly showed the fact that Z contains a closed subset which is homeomorphic to $\beta N-N$.

Now let us start with the following result.

Proposition 1. *If R is a semiprime ring, then* $(\text{minspec}\,(R))^p=\lambda(\text{spec}\,(Q(R)))$, *where* $\lambda=\lambda(Q(R); R)$.

Proof. Since R is a semiprime ring, $Q(R)$ is a (Von Neumann) regular ring and hence the patch topology on spec $(Q(R))$ is just the Zariski topology.

By [5, Proposition 2.2], we have minspec $(R) \subseteq \lambda$ (spec (QR))) and hence (minspec $(R))^p \subseteq \lambda$ (spec $(Q(R))$). Putting $Y = \lambda^{-1}$ ((minspec $(R))^p$), Y is a closed subset of spec $(Q(R))$ and $\bigcap_{y \in Y} (y \cap R) = \{0\}$. Since $(\bigcap_{y \in Y} y) \cap R = \{0\}$, we have $\bigcap_{y \in Y} y = \{0\}$. Thus, inasmuch as Y is a Zariski closed subset of spec $(Q(R))$ and $\bigcap_{y \in Y} y = \{0\}$, we see that $Y =$ spec $(Q(R))$. Consequently λ(spec $((Q(R))) = \lambda(Y) =$ (minspec $(R))^p$.

Proposition 2. *Let R be a ring and X a thin patch of* spec (R), *and let $I = \bigcap_{x \in X} x$. Then X is homeomorphic to* (minspec $(R/I))^p$.

Proof. Let ϕ be a canonical mapping: $X \rightarrow$ spec (R/I) given by $x \rightarrow x+I$. Then ϕ is a one to one continuous mapping. Since both X and spec (R/I) are compact Hausdorff spaces, X is homeomorphic to $\phi(X)$ by ϕ. Thus, to show the proposition, we may show $\phi(X) =$ (minspec $(R/I))^p$.

Let Z be the Zariski closure of X in spec(R). Then $I = \bigcap_{x \in Z} x$, $Z = \{x \in \text{spec}(R) \mid x \supseteq I\}$ and minspec $(R/I) = \{x+I \mid x \in \min Z\}$. Since X is a closed set, the first corollary of [3, Theorem 1] shows that every element of Z contains an element of X; whence $\min Z = \min X$. Thus minspec $(R/I) = \{x+I \mid x \in \min Z\} = \{x+I \mid x \in \min X\} = \phi(\min X)$, and hence (minspec$(R/I))^p = (\phi(\min X))^p = \phi(\min (X)^p) = \phi(X)$ as desired.

In [6], the author proved the following result: If R is a (Von Neumann) regular ring such that every finitely generated R-submodule of $Q(R)$ is a direct sum of cyclic submodules, then

$$2 \geq |\lambda^{-1}(x)|, \quad \text{the number of} \quad \lambda^{-1}(x)$$

for all $x \in$ spec (R), where $\lambda = \lambda(Q(R); R)$. This result suggests the following lemma.

Lemma 3 *Let R be a semiprime ring, and assume that* spec $(Q(R))$ *contains three points y_1, y_2 and y_3 such that*

$$\lambda(y_1) \subseteq \lambda(y_2) \quad \text{and} \quad \lambda(y_3) \subseteq \lambda(y_2)$$

where $\lambda = \lambda(Q(R); R)$. Then there exists a finitely generated R-submodule of $Q(R)$ which is not a direct sum of cyclic submodules.

Proof. Since R is a semiprime ring, $Q(R)$ is a regular ring. So, we can easily choose pairwise orthogonal idempotents e, f and g in $Q(R)$ such that

(1) $e \equiv 1 \pmod{y_1}$, $f \equiv 1 \pmod{y_2}$ and $g \equiv 1 \pmod{y_3}$.

Putting

$$A = R(e+f)+R(f+g)$$

we claim that A is not a direct sum of cyclic submodules.

First suppose, to the contrary, that A is a cyclic R-module. Then

$$A = R(re+(r+s)f+sg)$$

for some r, $s\in R$. In $A=R(re+(r+s)f+sg)$, express $e+f$ and $f+g$ as

$$e+f = \alpha re+\alpha(r+s)f+\alpha sg\,,$$
$$f+g = \beta re+\beta(r+s)f+\beta sg$$

where α, $\beta\in R$. Since e, f and g are pairwise orthogonal idempotents, it follows that

$$e \equiv \alpha re \,(\mathrm{mod}\, y_1)\,,$$
$$0 \equiv \beta re \,(\mathrm{mod}\, y_1) \quad \text{and} \quad g \equiv \beta sg \,(\mathrm{mod}\, y_3)\,.$$

Therefore, using (1), we get

$$1 \equiv \alpha r \,(\mathrm{mod}\, \lambda(y_1))\,,$$
$$0 \equiv \beta r \,(\mathrm{mod}\, \lambda(y_1) \quad \text{and} \quad 1 \equiv \beta s \,(\mathrm{mod}\, \lambda(y_3)).$$

However, since $\lambda(y_1)\subseteq\lambda(y_2)$ and $\lambda(y_3)\subseteq\lambda(y_2)$, it follows

$$1 \equiv \alpha r \,(\mathrm{mod}\, \lambda(y_2))\,,$$
$$0 \equiv \beta r \,(\mathrm{mod}\, \lambda(y_2)) \quad \text{and} \quad 1 \equiv \beta s \,(\mathrm{mod}\, \lambda(y_2))$$

from which we obtain $1\in\lambda(y_2)$, a contradiction. Thus A is not a cyclic R-module.

Next assume that A is a direct sum of n (>1) cyclic submodules, say

$$A = R(r_1e+(r_1+s_1)f+s_1g)\oplus\cdots\oplus R(r_ne+(r_n+s_n)f+s_ng)\,.$$

Then we can verify that

$$QA = Q(r_1e+(r_1+s_1)f+s_1g)\oplus\cdots\oplus Q(r_ne+(r_n+s_n)f+s_ng)$$

where $Q=Q(R)$. Therefore we must have

$$(2) \quad (r_ie)(r_je) = 0, \quad ((r_i+s_i)f)((r_j+s_j)f) = 0 \quad \text{and} \quad (s_ig)(s_jg) = 0$$

for $i\neq j$. Since $e+f$ and $f+g\in A=\sum_{i=1}^{n}\oplus R(r_ie+(r_i+s_i)f+s_ig)$, there exist j, k and l such that

$$r_je\notin y_1\,, \quad (r_k+s_k)f\notin y_2 \quad \text{and} \quad s_lf\notin y_3\,.$$

Then, it follows from (2) that

(3) $\qquad r_i e \in y_1 \quad \text{if } i \neq i$,

(4) $\quad (r_i + s_i) f \in y_2 \quad \text{if } i \neq k$, and

(5) $\qquad s_i g \in y_3 \quad \text{if } i \neq l$.

Here, let us express $e+f$ and $f+g$ in $A = \sum_{i=1}^{n} \oplus R(r_i e + (r_i + s_i) f + s_i g)$ as follows:

$$e+f = \sum_{i=1}^{n} \alpha_i r_i e + \sum_{i=1}^{n} \alpha_i (r_i + s_i) f + \sum_{i=1}^{n} \alpha_i s_i g ,$$

$$f+g = \sum_{i=1}^{n} \beta_i r_i e + \sum_{i=1}^{n} \beta_i (r_i + s_i) f + \sum_{i=1}^{n} \beta_i s_i g ,$$

where α_i, $\beta_i \in R$, $i=1, \cdots, n$. Since e, f and g are pairwise orthogonal idempotents, we have then

$$e = \sum_{i=1}^{n} \alpha_i r_i e , \quad f = \sum_{i=1}^{n} \alpha_i (r_i + s_i) f , \quad 0 = \sum_{i=1}^{n} \alpha_i s_i g$$

$$0 = \sum_{i=1}^{n} \beta_i r_i e , \quad f = \sum_{i=1}^{n} \beta_i (r_i + s_i) f , \quad g = \sum_{i=1}^{n} \beta_i s_i g .$$

Hence according to (1), (3), (4) and (5), we infer that

$$1 \equiv \alpha_j r_j (\text{mod } \lambda(y_1)) , \quad 1 \equiv \alpha_k (r_k + s_k) (\text{mod } \lambda(y_2)) , \quad 0 \equiv \alpha_l s_l (\text{mod } \lambda(y_3)) ,$$
$$0 \equiv \beta_j r_j (\text{mod } \lambda(y_1)) , \quad 1 \equiv \beta_k (r_k + s_k) (\text{mod } \lambda(y_2)) , \quad 1 \equiv \beta_l s_l (\text{mod } \lambda(y_3)) .$$

Since $0 \equiv \beta_j r_j \equiv \beta_j r_j \alpha_j \equiv \beta_j (\text{mod } \lambda(y_1))$, we see that

$$0 \equiv \beta_j (\text{mod } \lambda(y_1)); \quad \text{similarly} \quad 0 \equiv \alpha_l (\text{mod } \lambda(y_3)) .$$

Thus, noting $\lambda(y_1)$, $\lambda(y_3) \subseteq \lambda(y)_2$, we get

$$0 \equiv \beta_j \equiv \alpha_l (\text{mod } \lambda(y_2)) ,$$
$$1 \equiv \alpha_k (r_k + s_k) \equiv \beta_k (r_k + s_k) (\text{mod } \lambda(y_2)) .$$

From these relations, we must have $j \neq k$ and $l \neq k$. However (3) and (5) then show that $0 \equiv r_k (\text{mod } \lambda(y_1))$ and $0 \equiv s_k (\text{mod } \lambda(y_3))$; whence $0 \equiv r_k \equiv s_k (\text{mod } \lambda(y_2))$, from which $1 \equiv \alpha_k (r_k + s_k) \equiv 0 (\text{mod } \lambda(y_2))$, a contradiction.

Now, we are in a position to show the following theorem, which is a generalization of R. S. Pierce [7, Proposition 20.1].

Theorem 4. *Let R be a semiprime NFGC-ring and let X=*(minspec $(R))^p$. *Then the following conditions hold*:

(1) min X=minspec (R), *and hence X is a thin patch.*

(2) $X = \min X \cup \max X$.

(3) *X has no 3-points.*

Proof. (1) always holds.

(2) follows from Lemma 3.

(3). Put $\lambda=\lambda(Q(R); R)$. Then, by Proposition 1 and Lemma 3, $X=\lambda(\operatorname{spec}(Q(R)))$ and $|\lambda^{-1}(x)|\leq 2$ for all $x\in X$. Now, suppose that X has a 3-point, say x. Then there exist pairwise disjoint open subsets U_1, U_2 and U_3 of X such that $x\in U_i^p-U_i$, $i=1, 2, 3$. Put $W_i=(\lambda^{-1}(U_i))^p$, $i=1, 2, 3$. Then, inasmuch as $\operatorname{spec}(Q(R))$ is extremely disconnected space ([7, p. 102]), we see that W_1, W_2 and W_3 are also pairwise disjoint. Since λ is a closed mapping, $\lambda(W_i)$ contains U_i^p, $i=1, 2, 3$. It follows that $\lambda^{-1}(x)\cap W_i\neq\emptyset$, $i=1, 2, 3$ and hence $|\lambda^{-1}(x)|\geq 3$, a contradiction.

Theorem 5. *Let R be a ring and X a thin patch of* spec(R). *If $R/\bigcap_{x\in X'} x$ is NFGC for every subset $X'\subseteq X$, then X is a finite set.*

Proof. Since X is a closed subset of spec (R), X is also a Boolean space with the relative topology. By Proposition 2 and Theorem 4, we conclude that

(*) $X=\min X\cup\max X$, and

(**) every thin patch contained in X has no 3-points.

Now, we want to see that $\min X$ is a finite set. Thus, assume that $\min X$ is an infinite set. Since X is a Boolean space and $\min X$ is an infinite subset of X, we can easily take a set $\{P_n\mid n\in N\}$ of pairwise disjoint, non-empty open-closed subsets of X such that $P_n\subset\min X\neq\emptyset$ for each $n\in N$. Choose $y_n\in P_n\cap\min X$ and let $Z=\{y_n\mid n\in N\}^p$. Then $Z\subseteq X$ and Z is clearly a thin patch of spec (R). Hence, by (**), Z has no 3-points. Therefore, by the proof of the Pierce's lemma, Z contains a closed subspace V which is homeomorphic to $\beta N-N$. Since $V\approx\beta N-N$, [7, Corollary 21.3] says that V has a 3-point. So, V is not a thin patch of spec (R) by (**) and hence $V-(\min V)^p\neq\emptyset$. Furthermore $V-(\min V)^p\subseteq\max V$ by (*).

Let $x\in V-(\min V)^p$, and take an open-closed subset W of spec(R) such that $W\cap(\min V)^p=\emptyset$. Putting $T=V\cap W$, T is an open-closed subset of V which contains x. If T is a finite set, then x must be an isolated point in V. But this conflicts with the fact that $\beta N-N$ has no isolated points (see, e.g. [14, p. 74]). Therefore T must be an infinite closed subset of spec (R).

Since $T=\min T$, every closed subset of T is clearly a thin patch of spec (R) and hence, by (**), every closed subset of T has no 3-points. However, inasmuch as T is an infinite Boolean space, the Pierce's lemma says that T contains a closed subset which has a 3-point, a contradiction.

As an immediate corollary of Theorem 5, we have

Corollary 6. *Every FGC-ring has only finitely many minimal prime ideals.*

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