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CLASSIFICATION OF COMPACT TRANSFORMATION GROUPS ON COHOMOLOGY QUATERNION PROJECTIVE SPACES WITH CODIMENSION ONE ORBITS

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0. Introduction

Let M be an orientable closed $4n$-dimensional smooth manifold, whose rational cohomology algebra is isomorphic to that of a quaternion projective n-space $P_n(\boldsymbol{H})$. We call such a manifold M a rational cohomology quaternion projective n-space.

Let (G, M) be a pair of a compact connected Lie group G and a simply connected rational cohomology quaternion projective n-space M, on which G acts smoothly with a codimension one orbit G/K. We say that (G, M) is isomorphic to (G', M'), if there exist a Lie group isomorphism $h: G \to G'$ and a diffeomorphism $f: M \to M'$ satisfying

$$f(gx) = h(g)f(x)\,,$$

for every $g \in G$ and for every $x \in M$.

When G acts on M, $H = \bigcap_{x \in M} G_x$ (the intersection of all isotropy groups) is a closed normal subgroup of G. Since H acts on M trivially, the G-action on M induces an effective G/H-action on M. We say that (G, M) is essentially isomorphic to (G', M'), if there exists an isomorphism between the pairs with effective actions $(G/H, M)$ and $(G'/H', M')$.

The purpose of the present paper is to give a complete classification of such pairs (G, M) up to essential isomorphism. We shall show

Main Theorem. *Such a pair (G, M) is essentially isomorphic to one of the pairs listed in the next table.*

n	(G, M)	action	
$n\geqq 1$	$(G, P_n(\boldsymbol{H}))$	natural	(1)
$n_1+n_2+1,\ n_s>0$	$(Sp(n_1+1)\times Sp(n_2+1), P_n(\boldsymbol{H}))$	natural	
$n\geqq 1$	$(U(n+1), P_n(\boldsymbol{H}))$	natural	
$n\geqq 2$	$(SU(n+1), P_n(\boldsymbol{H}))$	natural	
2	$(SU(3), \boldsymbol{G}_2/SO(4))$	natural	
1	$(SO(3), S^4)$		(2)

(1) $G=Sp(n)$, $Sp(n)\times U(1)$, *or* $Sp(n)\times Sp(1)\subset Sp(n+1)$.

(2) $SO(3)$ *acts on* S^4 *by a* 5-*dimensional irreducible real representation.*

When $n=1$, M is a homotopy 4-sphere. Our main theorem for $n=1$ complements a lack of Wang's theorem [8], in which he classified (G, M) when M is a homotopy k-sphere for even $k\neq 4$ and for odd $k>31$. F. Uchida [7] gave a classification of pairs (G, M) when M is a rational cohomology complex projective space. To show our theorem, we follow a similar procedure. First, we recall in Section 1 some necessary facts on compact Lie groups, homogeneous spaces and G-manifolds with codimension one orbits. Section 2 is devoted to cohomological consideration. Our aim is to prove Theorem 2.1.4, which gives necessary conditions for (G, M) to be a pair of our problem. In Section 3, for each (G, M) appeared in our main theorem, we investigate the orbit types of M. Finally in the last three sections, we prove that there exists no pair (G, M) which is not essentially isomorphic to one of the pairs listed in our main theorem.

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1. Compact Lie groups and manifolds with group actions

1.1. We give here some definitions and propositions which are necessary for the subsequent discussion.

Let $G_1, \cdots, G_k$ be compact Lie groups. If G is a factor group of $G_1\times\cdots\times G_k$ by a finite normal subgroup, then we say that G is an essentially direct product of $G_1, \cdots, G_k$, and denote

$$G=G_1\circ\cdots\circ G_k\,.$$

As is well known,

(1.1.1) Every compact connected Lie group G has the form

$$G=T_0\circ G_1\circ\cdots\circ G_k\,,$$

where T_0 is a toral subgroup of G and G_s, $s=1, \cdots, k$, are closed connected simple normal subgroups of G.

The next two propositions will be used in the later sections without mention.

(1.1.2) Let G be a compact connected Lie group and G_1 its closed connected normal subgroup. Then there exists a closed connected normal subgroup G_2 of G, satisfying $G=G_1 \circ G_2$.

(1.1.3) Let G be an essentially direct product of compact connected Lie groups $G_1, \cdots, G_k$ and U a closed connected subgroup of G with rank U=rank G. Then, for every s, $s=1, \cdots, k$, there exists a closed connected subgroup U_s of G_s such that

$$\operatorname{rank} U_s = \operatorname{rank} G_s\,, \quad U = U_1 \circ \cdots \circ U_k\,,$$
$$G/U = G_1/U_1 \times \cdots \times G_k/U_k\,.$$

1.2. Let (G, U) be a pair of a compact connected simple Lie group G and its closed connected subgroup U, with rank U=rank G and let $p\colon \tilde{G} \rightarrow G$ be the universal covering. We say that such two pairs (G_1, U_1) and (G_2, U_2) are pairwise locally isomorphic if there exists an isomorphism $h\colon \tilde{G}_1 \rightarrow \tilde{G}_2$ such that $hp_1^{-1}(U_1)=p_2^{-1}(U_2)$. In the following propositions, we list the homogeneous spaces of simple Lie groups with certain Poincaré polynomials. They can be shown by a similar argument to Section 4 of [7].

(1.2.1) If $P(G/U; t)=1+t^4+\cdots+t^{4a}$, then (G, U) is pairwise locally isomorphic to

$$(Sp(a+1),\ Sp(a) \times Sp(1))\,,$$

or

$$(G_2,\ SO(4)) \qquad \text{in case } a=2\,.$$

(1.2.2) If $P(G/U; t)=1+t^2+\cdots+t^{2b}$, then (G, U) is pairwise locally isomorphic to one of the following:

$(SU(b+1),\ S(U(b) \times U(1)))$,
$(SO(2s+1),\ SO(2s-1) \times SO(2))$, where $s=(b+1)/2$,
$(Sp(s),\ Sp(s-1) \times U(1))$, where $s=(b+1)/2$,
(G_2, H), where H is locally isomorphic to $U(2)$, in case $b=5$.

(1.2.3) If $P(G/U; t)=(1+t^{2a})(1+t^4+\cdots+t^{4b})$, $a \geqq 2$, then (G, U) is pairwise locally isomorphic to one of the following:

$$(SO(2a+3),\ SO(3) \times SO(2a))\,, \qquad \text{in case } a=b \geqq 2\,,$$

$(Sp(3),\ Sp(1)\times Sp(1)\times Sp(1))$, in case $a=2$, $b=2$,
$(Sp(4),\ Sp(2)\times Sp(2))$, in case $a=4$, $b=2$,
$(Sp(5),\ Sp(2)\times Sp(3))$, in case $a=4$, $b=4$,
$(\boldsymbol{F}_4, H)$, where H is locally isomorphic to $SU(2)\times Sp(3)$, in case $a=4$, $b=5$.

1.3. We give a summary of some results on compact Lie group actions on spheres, proved by Montgomery-Samelson [5], Borel [1] and Nagano [6]. See, also W.C. Hsiang and W.Y. Hsiang [3].

(1.3.1) ([5]) Let G_1 and G_2 be compact connected Lie groups such that the product $G_1\times G_2$ acts on a homotopy sphere $\sum$ transitively. Then, one of two groups G_1, G_2 acts already on $\sum$ transitively.

Or, more precisely,

(1.3.2) Let G be a compact connected Lie group which acts on a homotopy sphere $\sum$ effectively and transitively with the isotropy subgroup H. Then there exists a closed simple normal subgroup G_1 of G such that $G_1/G_1\cap H=\sum$.

(1.3.3) ([5], [1] and [6]) Let G_1 be a compact connected simple Lie group which acts on a homotopy n-sphere $\sum$ effectively and transitively with the isotropy subgroup H_1. Then,

(i) if n is even, $(G_1, H_1)\cong(SO(n+1), SO(n))$ or $(\boldsymbol{G}_2, SU(3))$ in case $n=6$.

(ii) if $n=2s-1$ and s is odd, $(G_1, H_1)\cong(SO(n+1), SO(n))$ or $(SU(s), SU(s-1))$.

(iii) if $n=2s-1$ and s is even, $(G_1, H_1)\cong(SO(n+1), SO(n))$, $(SU(s), SU(s-1))$, $(Sp(s/2), Sp(s/2-1))$, $(Spin(9), Spin(7))$ in case $n=15$, or $(Spin(7), \boldsymbol{G}_2)$ in case $n=7$.

In each case, $\sum$ is the standard n-sphere and the action of G_1 on $\sum=S^n$ is linear.

1.4. We refer to some results due to Uchida, concerning manifolds with Lie group actions. For the proofs, see Sections 1 and 5 of [7].

Assume that G is a compact connected Lie group.

(1.4.1) Let M be a compact connected smooth manifold without boundary on which G acts smoothly with an orbit $G(x)$ of codimension one, satisfying

$$H^1(M; \boldsymbol{Z}_2)=0\,.$$

Then $G(x)=G/K$ is a principal orbit and there exist just two singular orbits $G(x_1)=G/K_1$ and $G(x_2)=G/K_2$, and we can assume $K\subset K_1\cap K_2$. Moreover, for each $G(x_s)$, $s=1, 2$, there is a closed invariant tubular neighborhood X_s

such that

$$M = X_1 \cup X_2 \quad \text{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2 = G/K$$

as G-manifolds.

X_s is a compact connected smooth manifold on which G acts smoothly and has the form

$$X_s = G \underset{K_s}{\times} D^{k_s},$$

Here, k_s is the codimension of $G(x_s)$ in M and K_s acts on k_s-dimensional disk D^{k_s} via the slice representation σ_s: $K_s \to O(k_s)$. This K_s-action is transitive on the (k_s-1)-sphere ∂D^{k_s}. Since $\partial X_s = G/K$, as G-manifolds,

(1.4.2) The fibre bundle $K_s/K \to G/K \to G/K_s$ is a (k_s-1)-sphere bundle.

In the above situation, assume that $M(f) = X_1 \underset{f}{\cup} X_2$ is obtained from X_1 and X_2 by identifying their boundaries under a G-equivariant diffeomorphism f: $\partial X_1 \to \partial X_2$. The following propositions will be used to classify the pairs (G, M) and to construct a representative example of each essential isomorphism class.

(1.4.3) Let f, f': $\partial X_1 \to \partial X_2$ be G-equivariant diffeomorphisms. Then, $M(f)$ is equivariantly diffeomorphic to $M(f')$ as G-manifolds, if one of the following conditions is satisfied:

(i) f is G-diffeotopic to f',

(ii) $f^{-1}f'$ is extendable to a G-equivariant diffeomorphism on X_1,

(iii) $f'f^{-1}$ is extendable to a G-equivariant diffeomorphism on X_2.

(1.4.4) The set of all G-equivariant diffeomorphisms of ∂X_1 onto ∂X_2 is naturally identified with the factor group $N(K; G)/K$. That is, we have a group isomorphism

$$\mathrm{Diff}^G(G/K, G/K) \approx N(K; G)/K.$$

Here, $N(K; G)$ is the normalizer of K in G.

(1.4.5) Let K_s and K be closed subgroups of G and $K \subset K_s$. Then there exists a natural G-equivariant diffeomorphism

$$f\colon G \underset{K_s}{\times} K_s/K \to G/K,$$

defined by $f([g, hK]) = ghK$ for $g \in G$ and $h \in K_s$. And we have, for each $x \in N(K; K_s)$, the following commutative diagram

$$\begin{array}{ccc} G\times_{K_s} K_s/K & \xrightarrow{f} & G/K \\ {\scriptstyle 1\times R_x}\downarrow & & \downarrow{\scriptstyle R_x} \\ G\times_{K_s} K_s/K & \xrightarrow{f} & G/K\,, \end{array}$$

where R_x is an equivariant diffeomorphism given by a right translation.

To classify (G, M) up to essential isomorphism, we can assume that G acts almost effectively on M, that is, $H=\bigcap_{x\in M} G_x$ is a finite group. Then G acts almost effectively on the principal orbit G/K and hence

(1.4.6) K does not contain any positive dimensional closed normal subgroup of G.

2. Cohomology of orbits

2.1. From now on, we assume that G is a compact connected Lie group and M is a simply connected rational cohomology quaternion projective n-space on which G acts smoothly with a codimension one orbit G/K. Then, by (1.4.1) there exist just two singular orbits G/K_1 and G/K_2 and we can assume that $K\subset K_1\cap K_2$. Let us denote by u a generator of $H^*(M; \boldsymbol{Q})$, that is,

$$H^*(M; \boldsymbol{Q}) = \boldsymbol{Q}[u]/u^{n+1}\,, \qquad \deg u = 4\,,$$

and let

$$f_s^*\colon H^*(M; \boldsymbol{Q}) \to H^*(G/K_s; \boldsymbol{Q}) \qquad (s = 1, 2)$$

be the homomorphism induced by the inclusion $f_s\colon G/K_s\subset M$. Then we can show the following proposition, in a similar way as in [7, Lemma 2.1.1].

(2.1.1) Let n_s be a non-negative integer such that

$$f_s^*(u^{n_s})\neq 0 \quad \text{and} \quad f_s^*(u^{n_s+1}) = 0\,.$$

Then we have $n=n_1+n_2+1$.

We denote by k_s the codimension of G/K_s in M, that is,

$$k_s = 4n-\dim G/K_s\,,$$

for $s=1$ and 2. Then we have

(2.1.2) $2\leqq k_s\leqq 4(n-n_s)$.

We notice the following:

(2.1.3) ([7, Lemma 2.2.3]) If $k_2>2$, then G/K_1 is simply connected and K_1 is connected.

Now we shall prove

Theorem 2.1.4.

(A) *Assume that* G/K_1 *and* G/K_2 *are orientable.*

(i) *If* k_1-k_2 *is even, then each* G/K_s *is a rational cohomology quaternion projective* n_s*-space and* $k_s=4(n-n_s)$, *for* $s=1, 2$.

(ii) *If* k_1 *is even and* k_2 *is odd, then* $k_1+k_2=2n+3$ *and there are two cases:*

(a) $n_1=n_2$ *and*

$$P(G/K_1;\, t) = (1+t^{k_2-1})(1+t^4+\cdots+t^{4n_1}),$$
$$P(G/K_2;\, t) = (1+t^{k_1-1})(1+t^4+\cdots+t^{4n_2}).$$

(b) $k_1=4n_2+4$, $k_2=2(n_1-n_2)+1$ *and*

$$P(G/K_1;\, t) = 1+t^4+\cdots+t^{4n_1}+t^{k_2-1}(1+t^4+\cdots+t^{4n_2}),$$
$$P(G/K_2;\, t) = (1+t^{2n+1})(1+t^4+\cdots+t^{4n_2}).$$

(B) *The case that* G/K_1 *is orientable and* G/K_2 *is non-orientable does not happen.*

(C) *Assume that* G/K_1 *and* G/K_2 *are non-orientable. Then* $n=1$ *and*

$$P(G/K_s;\, t) = 1,$$
$$P(G/K_s^0;\, t) = 1+t^2,$$

for $s=1, 2$. *Here* K_s^0 *is the identity component of* K_s.

This theorem can be proved by a completely analogous discussion to the proof of Theorem 2.2.2 in [7]. Therefore, we shall give only an outline of the proof in the remainder of this section.

2.2. As is seen in 1.4, there is a closed invariant tubular neighborhood X_s of G/K_s in M, such that

$$M = X_1 \cup X_2 \quad \text{and} \quad X_1 \cap X_2 = \partial X_1 = \partial X_2.$$

By the Poincaré duality for X_s, we obtain

$$P(X_s,\, \partial X_s;\, t) = t^{4n} P(X_s;\, t^{-1}).$$

Moreover, if G/K_s is orientable, we have

$$P(X_s,\, \partial X_s;\, t) = t^{k_s} P(G/K_s;\, t),$$

by Thom isomorphism.

First, we assume that both G/K_1 and G/K_2 are orientable. Then, in consideration of the rational cohomology exact sequence for $(M,\, X_s)$, we obtain

$$P(G/K_1;\, t) = t^{k_2-1}P(G/K_2;\, t) + 1 + t^4 + \cdots + t^{4n_1} - t^{-1}(t^{4n_1+4} + \cdots + t^{4n})$$

and

$$(2.2.1)\quad P(G/K_2;\, t) = t^{k_1-1}P(G/K_1;\, t) + 1 + t^4 + \cdots + t^{4n_2} - t^{-1}(t^{4n_2+4} + \cdots + t^{4n}).$$

Using these equations, we can prove the part (A) of Theorem 2.1.4 in the same way as in [7, 2.3]. Notice that to get (2.2.1) we require only the orientability of G/K_1.

Next, we assume that G/K_2 is non-orientable. Then, by (2.1.2) and (2.1.3), we have $k_1=2$. We can show by the argument due to Uchida [7, 2.4～2.6] (see also Wang [8]),

$$(2.2.2)\quad \begin{aligned} P(G/K_2^0;\, t) &= (1+t^{k_2})P(G/K_2;\, t)\,, \\ P(G/K^0;\, t) &= (1+t^{2k_2-1})P(G/K_2;\, t) - P(n_1, n_2;\, t)\,, \end{aligned}$$

where

$$P(n_1, n_2;\, t) = \begin{cases} (1+t^{-1})(t^{4n_1+4} + \cdots + t^{4n_2}), & \text{if } n_1 < n_2\,, \\ 0\,, & \text{if } n_1 \geqq n_2\,. \end{cases}$$

If G/K_1 is orientable, then by the use of the Poincaré duality for G/K_1, we obtain from (2.2.1),

$$t^{4n-1}P(G/K_2;\, t^{-1}) = P(G/K_1;\, t) + t^{4n_1+3}(1 + t^4 + \cdots + t^{4n_2}) - (1 + t^4 + \cdots + t^{4n_1})\,.$$

By the Poincaré duality for G/K_2^0 and (2.2.2), we have

$$t^{4n}P(G/K_2;\, t^{-1}) = t^{2k_2}P(G/K_2;\, t)\,.$$

From these two equations and (2.2.1), it follows

$$\begin{aligned}(1-t^{2k_2})P(G/K_1;\, t) = {} & (1-t^{2k_2+4n_2+2})(1 + t^4 + \cdots + t^{4n_1}) \\ & + (t^{2k_2-1} - t^{4n_1+3})(1 + t^4 + \cdots + t^{4n_2})\,.\end{aligned}$$

The both sides of this equation are divisible by $1-t^2$ and we have $\chi(G/K_1) = P(G/K_1;\, -1) \neq 0$. Hence $P(G/K_1;\, t)$ is an even function. Therefore, $k_2 = 2n_1+2$ and we have

$$(1-t^{4n_1+4})P(G/K_1;\, t) = (1-t^{4n+2})(1 + t^4 + \cdots + t^{4n_1})\,.$$

It follows

$$(1+t^2)P(G/K_1;\, t) = 1 + t^2 + t^4 + \cdots + t^{4n}\,.$$

This is impossible. Hence, the case that G/K_1 is orientable and G/K_2 is non-orientable can not occur, and (B) of Theorem 2.1.4 is proved.

Finally, we assume that G/K_1 and G/K_2 are both non-orientable. Then $k_1=k_2=2$. As in [7, 2.7], we have $n_1=n_2$ and

$$(2.2.3)\qquad \begin{aligned} P(G/K_s^0;\, t) &= (1+t^2)P(G/K_s;\, t)\,,\\ P(G/K^0;\, t) &= (1+t^3)P(G/K_s;\, t)\,, \end{aligned}$$

for $s=1, 2$. Considering the Mayer-Vietoris cohomology sequence for $(M;\, X_1,\, X_2)$, we have

$$(2.2.4)\qquad (1-t^3)P(G/K_1;\, t) = (1-t^{4n_1+3})(1+t^4+\cdots+t^{4n_1})\,.$$

Therefore, $\chi(G/K_1)\neq 0$ and $P(G/K_1;\, t)$ is an even function. Hence by (2.2.4) we have $n_1=0$ and $P(G/K_1;\, t)=1$. By (2.2.3), we obtain $P(G/K_s;\, t)=1$ and $P(G/K_s^0;\, t)=1+t^2$, for $s=1$ and 2. Thus (C) of Theorem 2.1.4 is proved.

3. The representative examples of the pairs $(G,\, M)$

3.1. Here we give some examples of pairs (G, M), each of which consists of a compact connected Lie group G and a simply connected rational cohomology quaternion projective space M on which G acts smoothly with a 1-codimensional orbit.

Let $n=n_1+n_2+1$. In case $n_2=0$, we choose $Sp(n)\times 1$, $Sp(n)\times U(1)$, or $Sp(n)\times Sp(1)$ for G. And in case $n_1>0$, $n_2>0$, we set simply $G=Sp(n_1+1)\times Sp(n_2+1)$. Then, the natural action of G on $P_n(\boldsymbol{H})=P(\boldsymbol{H}^{n_1+1}\oplus\boldsymbol{H}^{n_2+1})$ is transitive on a $(4n-1)$-dimensional submanifold

$$X=\{(u,\, v)\mid |u|^2=|v|^2,\quad u\in\boldsymbol{H}^{n_1+1},\, v\in\boldsymbol{H}^{n_2+1}\}\,,$$

and has two singular orbits $P_{n_1}(\boldsymbol{H})$ and $P_{n_2}(\boldsymbol{H})$. This gives an example of (G, M) of the type (A) (i) in Theorem 2.1.4.

3.2. We shall consider the natural action of $U(n+1)$ on $P_n(\boldsymbol{H})=Sp(n+1)/Sp(n)\times Sp(1)$. Let $(u_0, u_1, \cdots, u_n)$ be the homogeneous coordinate of a point of $P_n(\boldsymbol{H})$ with the identification $(u_0q, u_1q, \cdots, u_nq)\sim(u_0, u_1, \cdots, u_n)$, for $q\in\boldsymbol{H}$ and $q\neq 0$. Consider the orbit $G(u(t))$ of a point $u(t)=(0, \cdots, 0, t, j)\in P_n(\boldsymbol{H})$. Here, t is a real number with $0\leqq t\leqq 1$ and j is the element in the standard basis $\{1, i, j, k\}$ of $\boldsymbol{H}$. Then, it is easy to see that

$$\begin{aligned} G(u(0)) &= G/U(n)\times U(1)\,,\\ G(u(1)) &= G/U(n-1)\times SU(2) \end{aligned}$$

are singular orbits and for every t, $0<t<1$,

$$G(u(t)) = G/U(n-1)\times S(U(1)\times U(1))$$

is principal. This gives an example of (G, M) of the type (A) (ii) in Theorem 2.1.4. Note that when $n\geqq 2$ we can take $SU(n+1)$ as G instead of $U(n+1)$.

3.3. There is an another example of (G, M) of the above type in case $n=2$.

Let $\boldsymbol{Cay}$ be the division algebra of Cayley numbers. It is an 8-dimensional real vector space with a basis $\{e_0, e_1, \cdots, e_7\}$ and its non-associative algebra structure is given as follows:

$$\begin{aligned} &e_0 = 1, \quad e_i^2 = -1 \quad (1 \leqq i \leqq 7), \\ &e_i e_j = -e_j e_i \quad (i \neq j;\ 1 \leqq i, j \leqq 7), \\ &e_1 e_2 = e_3, \quad e_1 e_4 = e_5, \quad e_1 e_6 = e_7, \quad e_3 e_4 = e_7, \\ &e_3 e_5 = e_6, \quad e_2 e_5 = e_7, \quad e_2 e_6 = e_4, \end{aligned}$$

and

$$a(ab) = a^2 b, \quad (ab)b = ab^2, \qquad \text{for } a, b \in \boldsymbol{Cay}.$$

The group of automorphisms of $\boldsymbol{Cay}$ is the exceptional Lie group $\boldsymbol{G}_2$. Since every element of $\boldsymbol{G}_2$ induces an orthogonal transformation on the linear subspace $\boldsymbol{R}^7$ of $\boldsymbol{Cay}$ spanned by $\{e_1, \cdots, e_7\}$, there exists the canonical inclusion $\boldsymbol{G}_2 \subset SO(7)$, via which $\boldsymbol{G}_2$ acts on S^6 transitively. The isotropy group $(\boldsymbol{G}_2)_{e_1}$ at e_1 is isomorphic to $SU(3)$, and $(\boldsymbol{G}_2)_{e_1} \cap (\boldsymbol{G}_2)_{e_2}$ is isomorphic to $SU(2)$. The canonical inclusions

$$(\boldsymbol{G}_2)_{e_1} \cap (\boldsymbol{G}_2)_{e_2} \subset (\boldsymbol{G}_2)_{e_1} \subset SO(7)$$

correspond to the inclusions

$$SU(2) \subset SU(3) \subset SO(7),$$

which are defined by

$$\begin{pmatrix} a+bi & -c+di \\ c+di & a-bi \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & a+bi & -c+di \\ 0 & c+di & a-bi \end{pmatrix} \mapsto \left(\begin{array}{ccc|cccc} 1 & & & & & & \\ & 1 & & & 0 & & \\ & & 1 & & & & \\ \hline & & & a & -b & -c & -d \\ & & & b & a & d & -c \\ & 0 & & c & -d & a & b \\ & & & d & c & -b & a \end{array}\right).$$

Here, a, b, c and d are real numbers with $a^2+b^2+c^2+d^2=1$. Let

$$S = \{x \in \boldsymbol{Cay} \mid x = ae_0 + be_1 + ce_2 + de_3,\ a^2+b^2+c^2+d^2 = 1\}.$$

For $x = ae_0 + be_1 + ce_2 + de_3 \in S$, we define an $\boldsymbol{R}$-linear homomorphism

$$h_x \colon \boldsymbol{Cay} \to \boldsymbol{Cay}$$

by

$$h_x(e_i) = \begin{cases} (xe_i)\bar{x} & (i = 0, 1, 2, 3). \\ \bar{x} e_i & (i = 4, 5, 6, 7). \end{cases}$$

Then, h_x is represented by

$$A_x=\left(\begin{array}{ccc|cccc} a^2+b^2-c^2-d^2 & 2(bc-ad) & 2(ac+bd) & & & & \\ 2(ad+bc) & a^2-b^2+c^2-d^2 & 2(cd-ab) & & 0 & & \\ 2(bd-ac) & 2(ab+cd) & a^2-b^2-c^2+d^2 & & & & \\ \hline & & & a & b & -c & d \\ & 0 & & -b & a & d & c \\ & & & c & -d & a & b \\ & & & -d & -c & -b & a \end{array}\right).$$

Since $\det A_x=1$ and

$$h_x(e_i)h_x(e_j)=h_x(e_ie_j),$$

for every i and j ($0\leqq i, j\leqq 7$), we have $h_x\in \boldsymbol{G}_2$. Moreover, for $x, y\in S$, we define h_xh_y by

$$(h_xh_y)(u)=h_x(h_y(u)),$$

for all $u\in \boldsymbol{Cay}$. Then we can show $h_xh_y=h_{xy}$. Since h_{e_0} is the identity automorphism of $\boldsymbol{Cay}$, it follows that

$$A=\{A_x\,|\,x\in S\}$$

can be considered as a subgroup of $\boldsymbol{G}_2$. As above, we identify $SU(2)$ with a subgroup of $\boldsymbol{G}_2$. Then, $A\cap SU(2)\cong \boldsymbol{Z}_2$ and A is the identity component of the centralizer of $SU(2)$ in $\boldsymbol{G}_2$. Define

$$H=A\circ SU(2)=A\times SU(2)/\boldsymbol{Z}_2\,.$$

By (1.2.1), the homogeneous space $M=\boldsymbol{G}_2/H$ is an 8-dimensional rational cohomology quaternion projective 2-space. Let us consider the $SU(3)$-action on M defined by the canonical inclusion $SU(3)\cong(\boldsymbol{G}_2)_{e_1}\subset \boldsymbol{G}_2$. Define

$$x_t=(1-t)e_1+te_7\in \boldsymbol{Cay}$$

for t, $0\leqq t\leqq 1$. Since $\boldsymbol{G}_2$-action on S^6 is transitive, there exist an element $g_t\in \boldsymbol{G}_2$ and a positive number r_t such that

$$g_tx_t=r_te_1\,.$$

Define

$$H_t=\{h\in H\,|\,hx_t=x_t\}\,.$$

Then we can see that the isotropy group at $g_tH\in M$ is $g_tH_tg_t^{-1}$. Since

$$H_0=(\boldsymbol{G}_2)_{e_1}\cap H\cong S(U(2)\times U(1))\,,$$
$$H_1=\{h\in H\,|\,he_7=e_7\}\cong SO(3)\,,$$

we have two singular orbits $SU(3)/S(U(2)\cap U(1))$ and $SU(3)/SO(3)$. Moreover, we have $\dim H_t=1$ for $0<t<1$, since an element h of H is in H_t if and only if $he_1=e_1$ and $he_7=e_7$. Hence the orbit through g_tH $(0<t<1)$ is of codimension 1 and principal.

3.4. Consider the space V of all symmetric 3×3 real matrices with trace 0. This is a real vector space of dimension 5. We introduce an inner product $\langle\ ,\ \rangle$ in V by

$$\langle X,\ Y\rangle = \text{trace}\ XY\,,$$

for $X,\ Y\in V$. Define an $SO(3)$-action ρ on V as follows: For each $A\in SO(3)$, set $\rho_A\colon V\to V$ by

$$\rho_A(X) = AXA^{-1}\,, \qquad X\in V.$$

It is easy to see that ρ_A is well-defined and that $\langle\ ,\ \rangle$ is ρ_A-invariant. Now we restrict ρ on the unit sphere $S(V)$ in V. Define

$$X_t = \frac{1}{\sqrt{6}}\begin{pmatrix} 2\cos\frac{\pi}{3}t & 0 & 0 \\ 0 & -\cos\frac{\pi}{3}t+\sqrt{3}\sin\frac{\pi}{3}t & 0 \\ 0 & 0 & -\cos\frac{\pi}{3}t-\sqrt{3}\sin\frac{\pi}{3}t \end{pmatrix}$$

for t, $0\leqq t\leqq 1$. Then the isotropy group at X_t is the group

$$\left\{\begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_1\varepsilon_2 \end{pmatrix} \middle|\ \varepsilon_1 = \pm 1,\ \varepsilon_2 = \pm 1\right\}$$

for $0<t<1$, and hence the orbit through X_t is of codimension 1 for $0<t<1$. The isotropy groups at X_0 and X_1 are

$$S(O(1)\times O(2)) \quad \text{and} \quad S(O(2)\times O(1))$$

respectively. The corresponding orbits are real projective planes. This gives an example of the type (C) of Theorem 2.1.4.

4. Classification of (G, M) with orientable singular orbits I

4.1. In this section, we classify (G, M) of the type (A) (i) in Theorem 2.1.4. We assume that the two singular orbits G/K_1, G/K_2 are orientable and even-dimensional. Then, by Theorem 2.1.4 and (2.1.1), G/K_s $(s=1, 2)$ is a rational cohomology quaternion projective n_s-space and $n=n_1+n_2+1$. We shall prove

Theorem 4.1.1. *Under the above assumption, (G, M) is essentially isomorphic to*

$$(Sp(n)\times H', P_n(\boldsymbol{H})),\ H' = \{1\},\ U(1) \text{ or } Sp(1), \qquad \text{in case } n_1n_2=0\,,$$

or

$$(Sp(n_1+1)\times Sp(n_2+1),\ P_n(\boldsymbol{H})), \qquad \text{in case } n_1n_2 \neq 0\,.$$

Here, in both cases, the group acts naturally on $P_n(\boldsymbol{H})=Sp(n+1)/Sp(n)\times Sp(1)$ as a subgroup of $Sp(n+1)$.

Without loss of generality, we can suppose that G-action on M is almost effective and that $G=G_1\times T^h$, where G_1 is a simply connected compact Lie group and T^h is an h-dimensional toral group.

4.2. First, consider the case $n_1\geqq n_2=0$. Then, $k_2=4n$. Therefore $K_2=G$ and $G/K=K_2/K$ is a $(4n-1)$-sphere. It follows that G/K_1 is simply connected and the groups K_1 and K are connected. By (1.3.2), there exists a simple closed connected normal subgroup H of G, which acts transitively on the $(4n-1)$-sphere G/K, and we can write

$$G = H\times H'\,,$$

where H' is a connected closed normal subgroup of G. Note that H acts on G/K_1 transitively. Since rank K_1=rank G and G/K_1 is indecomposable (that is, G/K_1 cannot be a product of positive-dimensional manifolds), we have

$$K_1 = H_1\times H'\,,\quad G/K_1 = H/H_1\,,$$

where $H_1=H\cap K_1$. Hence H/H_1 is a rational cohomology quaternion projective $(n-1)$-space and by (1.2.1), (H, H_1) is pairwise locally isomorphic to $(Sp(n), Sp(n-1)\times Sp(1))$, or $(G_2, SO(4))$ in case $n=3$. But the latter case does not occur. For, non-transitively of G_2-action on S^{11} (see, (1.3.3)) contradicts the fact that H acts on $G/K=S^{11}$ transitively. Therefore, (H, H_1) is pairwise isomorphic to $(Sp(n), Sp(n-1)\times Sp(1))$. Since the G-action on M is almost effective by our assumption, G acts on G/K almost effectively. Therefore, H' acts on $G/K=Sp(n)/Sp(n-1)$ almost effectively and $Sp(n)$-equivariantly, and there exists a locally injective homomorphism

$$H' \to N(Sp(n-1);\ Sp(n))/Sp(n-1)\,.$$

Since $N(Sp(n-1);\ Sp(n))^0=Sp(n-1)\times Sp(1)$, we have

$$H' = \{1\},\ U(1) \text{ or } Sp(1)\,.$$

Now, we consider the slice representation

$$\sigma_1\colon K_1 = (Sp(n-1)\times Sp(1))\times H' \to O(4)\,.$$

Note that $k_1=4(n-n_1)=4$ in this case. Then $Sp(n-1)\subset\ker\sigma_1$, and $Sp(1)$ acts on $K_1/K=H_1/H_1\cap K=S^3$ via σ_1 transitively and freely. Since we can write

$$SO(4) = Sp(1)_L\circ Sp(1)_R$$

(where $Sp(1)_L$ resp. $Sp(1)_R$ denotes the multiplication by quaternions of norm 1 on the left resp. right), the $Sp(1)$-action on $K_1/K=S^3$ via σ_1 may be regarded as $Sp(1)_L$-action on S^3. Then there exists a representation ρ: $H'\to Sp(1)$ satisfying

$$\sigma_1(q, x)q' = qq'\rho(x)^{-1}, \tag{4.2.1}$$

for $q\in Sp(1)$, $x\in H'$ and $q'\in \boldsymbol{H}$. Hence, for each H', σ_1 is determined uniquely up to conjugation in $O(4)$. Let $G=Sp(n)\times H'$. Then, using (4.2.1), we can determine K as the isotropy group at $q'=1$, and we have

$$N(K; G)/K\cong Sp(1), \quad \text{in case } H'=\{1\}\ ,$$
$$N(K; G)/K\cong U(1), \quad \text{in case } H' = U(1)\ ,$$
$$N(K; G)^0 = K \quad \text{and} \quad N(K; G)/K\cong \boldsymbol{Z}_2, \quad \text{in case } H'=Sp(1)\ ,$$

where in the last formula, $\boldsymbol{Z}_2$ is generated by the class of the antipodal involution of $G/K=K_2/K=S^{4n-1}$. Therefore by (1.4.3) and (1.4.4), (G, M) is uniquely determined up to essential isomorphism in each of the above cases.

4.3. Next, we consider the case where $n_1>0$ and $n_2>0$. Since $k_1>2$ and $k_2>2$ in this case, G/K_1 and G/K_2 are simply connected. Hence K_1, K_2 and K are connected. Since G/K_1 and G/K_2 are indecomposable and rank $K_1=$ rank $K_2=$rank G, only the following two cases are possible:

$$\text{(I)} \qquad \begin{aligned} G &= H_1\times H_2\times G'\ ,\\ K_1 &= H_{(1)}\times H_2\times G'\ ,\\ K_2 &= H_1\times H_{(2)}\times G'\ ,\end{aligned}$$

where, for $s=1, 2$, H_s is a compact simply connected simple Lie group, $H_{(s)}$ is a closed connected subgroup of H_s and G' is a compact connected Lie group.

$$\text{(II)} \qquad \begin{aligned} G &= H\times G'\\ K_s &= H_s\times G' \qquad (s = 1, 2)\ ,\end{aligned}$$

where H is a compact simply connected simple Lie group, H_s is a closed connected subgroup of H, and G' is a compact connected Lie group. Note that by (1.2.1), $(H_s, H_{(s)})$ or (H, H_s) is pairwise locally isomorphic to one of the following:

$$\text{(4.3.1)} \qquad \begin{aligned} &(Sp(n_s+1),\ Sp(n_s)\times Sp(1))\ ,\\ &(G_2,\ SO(4)), \qquad \text{in case } n_s = 2\ .\end{aligned}$$

First, we consider the case (I). Since $K_1 \cap K_2 = H_{(1)} \times H_{(2)} \times G'$, we have $\dim G/(K_1 \cap K_2) = 4n-4$. Therefore, K is a subgroup of $K_1 \cap K_2$ with codimension 3, and we can see

$$(4.3.2) \qquad H_{(s)} \not\subset K \qquad (s = 1, 2).$$

For, if not so, then a sphere K_{3-s}/K becomes decomposable, which is impossible. Let N be a closed connected normal subgroup of $K_1 \cap K_2$ such that $(K_1 \cap K_2)/N$ acts on $(K_1 \cap K_2)/K$ almost effectively. Since $H_{(s)}$ is semi-simple, we can write

$$N = N_1 \times N_2 \times N',$$

where N_s $(s=1, 2)$ is a closed normal subgroup of $H_{(s)}$ and N' is a closed normal subgroup of G'. Note that by (4.3.2) $N_s \subsetneqq H_{(s)}$. Consider the group isomorphism

$$\frac{K_1 \cap K_2}{N} = \frac{H_{(1)}}{N_1} \times \frac{H_{(2)}}{N_2} \times \frac{G'}{N'}.$$

From $\dim (K_1 \cap K_2)/K = 3$, it follows $\dim (K_1 \cap K_2)/N \leqq 6$ (see, for example, [4, § 2].) On one hand, $\dim H_{(s)}/N_s \geqq 3$ by (4.3.1). Hence we have $\dim H_{(s)}/N_s = 3$, for $s=1, 2$, and $N' = G'$. Since G acts on G/K almost effectively and $G' = N'$ is a closed normal subgroup of K, we have $G' = \{1\}$ by (1.4.6). Thus $G = H_1 \times H_2$, and

(4.3.3) $H_{(s)} = U_s \circ N_s$ $(s=1, 2)$, where U_s is a closed connected simple subgroup of $H_{(s)}$ with $\dim U_s = 3$ and $N_1 \times N_2$ is a closed normal subgroup of K.

Now we shall show that H_s cannot be $\boldsymbol{G}_2$. Suppose, for example, that $H_2 = \boldsymbol{G}_2$. Then $n_2 = 2$ and $K_1/K = S^{11}$. Hence $K_1 = H_{(1)} \times \boldsymbol{G}_2$ acts transitively on S^{11}. By (1.3.3), $\boldsymbol{G}_2$ does not act transitively on S^{11}. Therefore, by (1.3.1) $H_{(1)}$ acts on S^{11} transitively and we can write $K_1 = H_{(1)}K$. Then, since $G = H_1K_1 = H_1H_{(1)}K = H_1K$, we see that H_1 acts on G/K transitively. It follows that H_1 acts on $G/K_2 = H_2/H_{(2)}$ transitively, which contradicts the assumption that H_1 is a normal subgroup of K_2. Thus, by (4.3.1) we have

(4.3.4) $(H_s, H_{(s)})$ is pairwise locally isomorphic to $(Sp(n_s+1), Sp(n_s) \times Sp(1))$, for $s=1, 2$.

Now by (4.3.3) and (4.3.2), we can assume that

$$K = \left\{ \left(\frac{*}{0} \middle| \frac{0}{q} \right) \times \left(\frac{*}{0} \middle| \frac{0}{q} \right) \in Sp(n_1+1) \times Sp(n_2+1) \,|\, q \in Sp(1) \right\}$$

up to conjugation by an element of

$$N(H_{(1)};\, H_1)\times N(H_{(2)};\, H_2) = N(K_1\cap K_2;\, G)\,.$$

Therefore, the slice representations

$$\sigma_1\colon K_1 = Sp(n_1)\times Sp(1)\times Sp(n_2+1) \to O(4n_2+4)\,,$$
$$\sigma_2\colon K_2 = Sp(n_1+1)\times Sp(n_2)\times Sp(1) \to O(4n_1+4)$$

are determined uniquely up to conjugation. Moreover, $N(K;\, G)/K\cong \boldsymbol{Z}_2$, which is generated by the class of the antipodal involution of K_s/K ($s=1, 2$). Therefore, in the case (I), (G, M) is uniquely determined up to essential isomorphism.

4.4. Next, we show that the case (II) does not occur. Suppose that

$$G = H\times G'\,,\quad K_s = H_s\times G'\,,\qquad s = 1, 2\,,$$

where H is a simply connected simple Lie group. By a similar argument to [7, (8.5.2)], we obtain the fact that H_s acts transitively on K_s/K, $s=1, 2$. From (1.2.1), it follows that $n_1=n_2$ and

$$H_s = Sp(n_s)\times Sp(1)\,,$$

or

$$H_s = Sp(1)\circ Sp(1)\,,\qquad \text{for } n_s = 2$$

and

$$K_s/K = S^{4n_s+3}\,.$$

On one hand, by (1.3.3), $Sp(n_s)$ cannot act transitively on S^{4n_s+3} and $Sp(1)$ cannot act transitively on S^{11}.

Thus in consideration of the examples given in 3.1, the proof of Theorem 4.1.1 is completed.

5. Classification of (G, M) with orientable singular orbits II

5.1. In this section, we classify (G, M) of the type (A) (ii) of Theorem 2.1.4. That is, we suppose that the singular orbits G/K_1 and G/K_2 are orientable and

(5.1.1) for $k_s=4n-\dim G/K_s$, $s=1, 2$,

$$k_1\equiv 0 \pmod 2,\quad k_2\equiv 1 \pmod 2,\quad k_1+k_2 = 2n+3\,.$$

Then we have two cases:

(5.1.2) when $k_1<4n_2+4$, we have $n_1=n_2$ and

$$P(G/K_1;\, t) = (1+t^{k_2-1})(1+t^4+\cdots+t^{4n_1})\,,$$
$$P(G/K_2;\, t) = (1+t^{k_1-1})(1+t^4+\cdots+t^{4n_2});$$

(5.1.3) when $k_1=4n_2+4$, we have $k_2=2(n_1-n_2)+1$ and

$$P(G/K_1;\,t) = 1+t^4+\cdots+t^{4n_1}+t^{k_2-1}(1+t^4+\cdots+t^{4n_2})\,,$$
$$P(G/K_2;\,t) = (1+t^{2n+1})(1+t^4+\cdots+t^{4n_2})\,.$$

We shall show

Theorem 5.1.4. *Under the above assumption, such a* $(G,\,M)$ *is essentially isomorphic to one of the following:*

$$(U(n+1),\,P_n(\boldsymbol{H})), \qquad n\geqq 1\,,$$
$$(SU(n+1),\,P_n(\boldsymbol{H})), \qquad n\geqq 2\,,$$
$$(SU(3),\,\boldsymbol{G}_2/SO(4)), \qquad n=2\,,$$

where $U(n+1)$ *(resp.* $SU(n+1)$*) acts naturally via the natural inclusion* $U(n+1)\subset Sp(n+1)$ *(resp.* $SU(n+1)\subset Sp(n+1)$*) on* $P_n(\boldsymbol{H})=Sp(n+1)/Sp(n)\times Sp(1)$ *and* $SU(3)$ *acts on* $\boldsymbol{G}_2/SO(4)$ *naturally via the natural inclusion* $SU(3)\subset\boldsymbol{G}_2$.

As in the previous section, we suppose that G acts on M almost effectively and $G=G_1\times T^h$, where G_1 is a compact simply connected Lie group and T^h is an h-dimensional toral group.

5.2. First, consider the case $n=1$. Then $k_1=2$, $k_2=3$ and $G/K_1=S^2$, $G/K_2=S^1$. Hence we can write

$$G = T\times Sp(1)\times G'\,,\quad K_1 = T\times U(1)\times G'\,,\quad K_2^0 = 1\times Sp(1)\times G'\,,$$

where G' is a compact connected Lie group and $T=U(1)$. Since $K^0\subset K_1\cap K_2^0 =1\times U(1)\times G'$ and $\dim(G/K_1\cap K_2)=\dim G/K=3$, we have $K^0=K_1\cap K_2^0= 1\times U(1)\times G'$. Then we have $G'=\{1\}$ by (1.4.6). Hence we can write

$$G = T\times Sp(1)\,,\quad K_1 = T\times U(1)\,,\quad K_2 = F\times Sp(1)\,,$$

where F is a finite subgroup of T. Then we have $K=F\times U(1)$ from $K_2=K_2^0K$. Here $F\times 1$ is a closed normal subgroup of G and acts trivially on M. Therefore we can consider the induced action of $G/F\times 1$ on M. Then we have

$$G = T\times Sp(1)\,,\quad K_1 = T\times U(1)\,,\quad K_2 = 1\times Sp(1)\,,\quad K = 1\times U(1)\,.$$

It follows that

$$\begin{aligned} N(K;\,G) &= N(1\times U(1);\,T\times Sp(1)) \\ &= T\times N(U(1);\,Sp(1)) \end{aligned}$$

and

$$\frac{N(K;\,G)}{N(K;\,G)^0}\cong\frac{N(K;\,K_2)}{N(K;\,K_2)^0}\cong \boldsymbol{Z}_2\,,$$

which is generated by the class of the antipodal involution of $K_2/K=S^2$. The slice representations

$$\sigma_1\colon K_1 = T\times U(1) \to O(2)\,,$$
$$\sigma_2\colon K_2 = 1\times Sp(1) \to O(3)$$

are uniquely determined up to conjugation. Therefore, (G, M) is uniquely determined up to essential isomorphism. On one hand, as is seen in 3.2, the pair $(U(2), P_1(\boldsymbol{H}))$, where $U(2)$ acts on $P_1(\boldsymbol{H})=Sp(2)/Sp(1)\times Sp(1)$ naturally, is an example of this type. Therefore, Theorem 5.1.4 is proved in case $n=1$.

5.3. Next, we consider the case $n\geqq 2$. Since $k_2\geqq 3$, G/K_1 is simply connected by (2.1.3). Note that $\operatorname{rank} K_1=\operatorname{rank} G$ by our assumption (5.1.2) or (5.1.3). Decompose

$$G = G'\times G''\,,$$

where G' is a compact simply connected semi-simple Lie group which acts on G/K_1 almost effectively and G'' is a compact connected Lie group which acts on G/K_1 trivially. Let

$$p\colon G = G'\times G'' \to G'$$

be a natural projection, and let

$$K'_s = p(K_s)\,, \qquad s = 1, 2\,.$$

Then

$$K_1 = K'_1\times G''\,, \qquad \operatorname{rank} K'_1 = \operatorname{rank} G'\,.$$

By the same way as in [7, Lemma 9.2.2], we can see that

(5.3.1) K'_1 acts on K_1/K transitively

and hence

(5.3.2) G' acts on G/K transitively.

From an observation on the structure of the cohomology ring of G/K_1 (cf. (5.1.2), (5.1.3) and (2.1.1)), it follows that either

(I) G' is simple

or

(II) G' is a product of two simple groups.

5.4. Here we shall show

(5.4.1) The case (II) cannot occur.

To prove this, it suffices to consider the case $G''=\{1\}$. Hence, we suppose that

$$G = H_1 \times H_2 ,$$
$$K_1 = H_{(1)} \times H_{(2)} ,$$

where H_s is a compact simply connected simple Lie group and $H_{(s)}$ is its closed connected subgroup for $s=1, 2$, and that $H_1/H_{(1)}$ is a rational cohomology (k_2-1)-sphere and $H/H_{(2)}$ is a rational cohomology quaternion projective m-space, where $m=n_1$, in the case (5.1.2) or $m=n_1-1$, $k_2=5$, in the case (5.1.3). Since $K_1=H_{(1)}\times H_{(2)}$ acts transitively on a sphere K_1/K, either $H_{(1)}$ or $H_{(2)}$ acts transitively on K_1/K.

(i) Suppose first that $H_{(1)}$ acts transitively on K_1/K. Let

$$p_s\colon G = H_1 \times H_2 \to H_s , \qquad s = 1, 2 ,$$

be the natural projection, and let

$$N = (\ker p_2 \,|\, K_2)^0 .$$

Then there exists a connected closed normal subgroup L of K_2 such that

$$K_2^0 = N \circ L .$$

Note that p_2 maps L isomorphically onto $p_2(K_2^0)$. Since $K_2^0/K^0=S^{k_2-1}$ is an even-dimensional sphere, rank $K^0=$rank K_2^0. Therefore, if we denote $K^0=N'\circ L'$, where N', L' are connected closed subgroup of N, L, respectively, we have $K_2^0/K^0=N/N'\times L/L'$ and hence $N=N'$ or $L=L'$. If $K^0=N'\circ L$, then $H_{(2)}=p_2(K^0)=p_2(L)=p_2(K_2^0)$ and hence $p_2(K_2)=p_2(K_2^0)p_2(K)=p_2(K^0)p_2(K)=p_2(K)$ from $K_2=K_2^0K$. Then the projection $H_1\backslash G/K \to H_1\backslash G/K_2$ is a homeomorphism and hence $H_1\backslash M$ is naturally homeomorphic to a mapping cylinder of the projection $H_1\backslash G/K \to H_1\backslash G/K_1$. Therefore, $H_1\backslash G/K_1 = H_2/H_{(2)}$ is a deformation retract of $H_1\backslash M$. Consider the commutative diagram

$$\begin{array}{ccc} G/K_1 & \xrightarrow{i_1} & M \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q} \\ H_2/H_{(2)} = H_1\backslash G/K_1 & \xrightarrow[j_1]{} & H_1\backslash M , \end{array}$$

where i_1, j_1 are the natural inclusions and q, q_1 are the natural projections. Since j_1 is a homotopy equivalence, q induces an isomorphism of rational cohomology rings. But, this is impossible by our assumption. Therefore, $K^0=N\circ L'$, and $H_{(2)}=p_2(K^0)=p_2(L')\subset p_2(L)=p_2(K_2^0)$. Since p_2 gives a local isomorphism $(L, L')\to(p_2(L), H_{(2)})$, we have $p_2(K_2^0)/H_{(2)}=L/L'=K_2/K=S^{k_2-1}$. Consider the fibration

$$p_2(K_2^0)/H_{(2)} \to H_2/H_{(2)} \to H_2/p_2(K_2^0) .$$

Since $\chi(H_2/H_{(2)})\neq 0$, we have $\chi(H_2/p_2(K_2^0))\neq 0$ and hence rank $H_2=$rank $p_2(K_2^0)$. It follows that $H^{\text{odd}}(H_2/p_2(K_2^0))=0$ and the homomorphism

$$H^*(H_2/H_{(2)}) \to H^*(p_2(K_2^0)/H_{(2)})$$

is surjective. Therefore, $k_2=5$, that is, $H_1/H_{(1)}$ is a rational cohomology 4-sphere and by (1.2.1) $(H_1, H_{(1)})$ is pairwise locally isomorphic to $(Sp(2), Sp(1)\times Sp(1))$. Since $H_{(1)}$ acts transitively on $K_1/K=S^{k_1-1}$, we have $k_1=4$. By (2.1.3), K_1 and K_2 are connected. Hence, $n=3$ and $m=1$ in both cases (5.1.2) and (5.1.3). Therefore, $(H_2, H_{(2)})$ is pairwise locally isomorphic to $(Sp(2), Sp(1)\times Sp(1))$ and $p_2(K_2)=H_2$. We can write $K_2=A\circ B$, where A, B are closed connected normal subgroup of K_2 such that $A\subset H_1=Sp(2)$, $\dim A=3$ and $p_2(B)=H_2$. Then, considering the centralizer of A in $G=H_1\times H_2$, we can see $B=H_2$. Since K is connected and rank $K=$rank K_2 by $K_2/K=S^4$, we may wirte $K=A\circ B'$, where B' is a connected closed subgroup of B with codimension 4. It is easy to see that $B'=H_{(2)}$. Thus we have

$$\begin{aligned} &K_2 = A\times Sp(2)\,, \\ &K = A\times H_{(2)}\,, \qquad A\subset H_{(1)}\,. \end{aligned}$$

By taking a conjugation in K_1 if necessary, we may assume that A $(\subset Sp(2))$ has the form

$$Sp(1)\times 1\,, \quad 1\times Sp(1)\,, \quad \text{or} \quad \Delta Sp(1)\,,$$

where

$$Sp(1)\times 1 = \left\{\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}\right\}, \quad 1\times Sp(1) = \left\{\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}\right\},$$

$$\Delta Sp(1) = \left\{\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}\right\}.$$

We can see that $Sp(2)/A$ in 2-connected and

$$\begin{aligned} &\pi_3(Sp(2)/Sp(1)\times 1) = \pi_3(Sp(2)/1\times Sp(1)) = 0\,, \\ &\pi_3(Sp(2)/\Delta Sp(1)) = \boldsymbol{Z}_2\,. \end{aligned}$$

Therefore, $Sp(2)/A$ is a rational cohomology 7-sphere. It follows that only in the case (5.1.3), $H_{(1)}$ may act transitively on K_1/K. This implies $n_1=2$. But we can see that it is impossible, by an observation on the Mayer-Vietoris cohomology sequence of $(X_1\cup X_2, X_1, X_2)$, with $X_s\approx G/K_s$, $X_1\cup X_2=M$, $X_1\cap X_2=G/K$ (see, 1.4). Thus we see that $H_{(1)}$ cannot act transitively on K_1/K.

(ii) We assume that $H_{(2)}$ acts transitively on K_1/K. Then, $p_1(K_2)$ is equal to either H_1 or $H_{(1)}$ by the same argument as the case (i). First, suppose that $p_1(K_2)=H_{(1)}$. Then, $H_2\backslash G/K_1=H_1/H_{(1)}$ is a deformation retract of $H_2\backslash M$.

Consider the commutative diagram

$$\begin{array}{ccccc} & & G/K_1 & \xrightarrow{\;i_1\;} & M \\ & \overset{i}{\nearrow} & \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q} \\ S^{k_2-1}=H_1/H_{(1)} & = & H_2\backslash G/K_1 & \xrightarrow[j]{} & H_2\backslash M\,, \end{array}$$

where, i, i_1, j are the natural inclusions and q, q_1 are the natural projections. Since j is a homotopy equivalence, $q\circ i_1\circ i$ induces a cohomology isomorphism $(q\circ i_1\circ i)^*=(i_1\circ i)^*\circ q^*$. This implies that q^* is injective and $(i_1\circ i)^*$ is surjective. Since M is a rational cohomology quaternion projective n-space, we have $k_2=5$ and $n=1$. This contradicts our assumption $n=2$. Thus we may assume that $p_1(K_2)=H_1$. We shall show

(5.4.2) $H_{(1)}\not\subset K$.

Suppose that $H_{(1)}\subset K$. Then,

$$H_{(1)}=K\cap H_1\subset K_2\cap H_1\subset p_1(K_2)=H_1\,.$$

Since H_1 is simple and $K_2\cap H_1$ is a normal subgroup of $H_1=p_1(K_2)$, we have

$$K_2=H_1\times N\,,$$

where N is a closed subgroup of H_2, and

$$K=H_{(1)}\times N\,.$$

Therefore, $p_2(K)=p_2(K_2)$. It follows that $H_2/H_{(2)}=H_1\backslash G/K_1$ is a deformation retract of $H_1\backslash M$. In the commutative diagram

$$\begin{array}{ccccc} & & G/K_1 & \xrightarrow{\;i_1\;} & M \\ & \nearrow & \downarrow & & \downarrow{\scriptstyle q} \\ H_2/H_{(2)} & = & H_1\backslash G/K_1 & \xrightarrow[j_1]{} & H_1\backslash M\,, \end{array}$$

j_1 is a homotopy equivalence and $H_2/H_{(2)}$ (resp. M) is a rational cohomology quaternion projective m- (resp. n-) space. Hence it follows that $m=n$, which is a contradiction. Thus we obtain (5.4.2).

Since $H_{(1)}/K\cap H_1=p_1(K)/K\cap H_1\cong K/(K\cap H_1)\times(K\cap H_2)=p_2(K)/K\cap H_2$, and $p_2(K)/K\cap H_2$ acts freely from the right on the sphere $H_{(2)}/K\cap H_2=K_1/K$, it follows that

(5.4.3) $K\cap H_1$ is a normal subgroup of $H_{(1)}=p_1(K)$, with codimension $\leqq 3$.

Now, since k_2 is odd, $(H_1, H_{(1)})$ is pairwise locally isomorphic to $(SO(2a+1), SO(2a))$, $a=(k_2-1)/2$, or to $(G_2, SU(3))$ if $k_2=7$. Hence, by (5.4.2) and (5.4.3), we see that $(H_1, H_{(1)})$ is pairwise locally isomorphic to $(SO(2a+1), SO(2a))$, $a \leqq 2$. Let $a=2$. Then $\dim(K \cap H_1) \geqq 3$. Since $K \cap H_1 \subset K_2 \cap H_1$ and $K_2 \cap H_1$ is a closed normal subgroup of $H_1 = p_1(K_2)$, H_1 is simple and $K \cap H_1$ is not finite, we have

$$K_2 = H_1 \times N, \qquad N \subset H_2,$$

and hence

$$K = H_{(1)} \times N.$$

But this contradicts (5.4.2). Therefore, $a=1$. Thus only the case (5.1.2) is possible. Then, since $m=n_1=n_2$ and since $k_1+k_2=2n+3$, $k_2=3$, we have that $k_1=4m+2$. By (1.2.1), $(H_2, H_{(2)})$ is pairwise locally isomorphic to $(Sp(m+1), Sp(m) \times Sp(1))$ or to $(G_2, SO(4))$ when $m=2$. But in every case, $H_{(2)}$ cannot act transitively on $K_1/K = S^{4m+1}$ by (1.3.3). This contradicts our assumption. Thus the proof of (5.4.1) is completed.

5.5. Now we consider the case (I) in 5.3. Let

$$G = H \times G'', \quad K_1 = H_1 \times G'',$$

where H is a compact connected simple Lie group and H_1 is its closed connected subgroup. We recall (5.1.2) and (5.1.3). That is,

$$\begin{aligned} P(H/H_1; t) &= P(G/K_1; t) \\ &= (1+t^{k_2-1})(1+t^4+\cdots+t^{4n_1}), && n_1 = n_2, \\ \text{or} \quad &= 1+t^4+\cdots+t^{4n_1}+t^{k_2-1}(1+t^4+\cdots+t^{4n_2}), && k_2 = 2(n_1-n_2)+1. \end{aligned}$$

Consider the case $k_2 \neq 3$. By making use of the table of maximal subgroup in [2, p. 219], we can see that there is no homogeneous space with a Poincaré polynomial

$$1+t^4+\cdots+t^{4n_1}+t^{k_2-1}(1+t^4+\cdots+t^{4n_2}), \qquad k_2 = 2(n_1-n_2)+1.$$

Hence, by (1.2.3), (H, H_1) is pairwise locally isomorphic to one of the following:

$(SO(k_2+2), SO(3) \times SO(k_2-1))$, when $k_2 = n \geqq 5$,
$(Sp(3), Sp(1) \times Sp(1) \times Sp(1))$, when $k_2 = 5$, $n = 5$,
$(Sp(4), Sp(2) \times Sp(2))$, when $k_2 = 9$, $n = 5$,
$(Sp(5), Sp(2) \times Sp(3))$, when $k_2 = 9$, $n = 9$,
$(\boldsymbol{F}_4, H_1)$, where H_1 is locally isomorphic to $SU(2) \times Sp(3)$,
when $k_2 = 9$, $n=11$.

Suppose that (H, H_1) is pairwise locally isomorphic to $(Sp(5), Sp(2) \times Sp(3))$. Then we can write

$$G = Sp(5)\times G'' ,$$
$$K_1 = Sp(2)\times Sp(3)\times G'' .$$

From the transitivity of $Sp(3)$ ($\subset K_1$)-action on $K_1/K=S^{11}$, it follows that K is locally isomorphic to $Sp(2)\times Sp(2)\times G''$. Therefore, we see that $G=Sp(5)$ and $K=Sp(2)\times Sp(2)$. But, on one hand, K must contain $SO(8)$ as a normal subgroup, since $K_2/K=S^8$. This is a contradiction. Thus (H, H_1) cannot be pairwise locally isomorphic to $(Sp(5), Sp(2)\times Sp(3))$. The other cases are all impossible, since H_1 acts non-transitively on $K_1/K=S^{k_1-1}$. Therefore, we suppose $k_2=3$. By (1.2.2), (H, H_1) is pairwise locally isomorphic to one of the following:

$(SU(n+1), S(U(n)\times U(1)))$,
$(SO(n+2), SO(n)\times SO(2))$,
$(Sp((n+1)/2), Sp((n-1)/2)\times U(1))$,
(G_2, U), where U is locally isomorphic to $U(2)$, when $n=5$.

Except the first case, these cases are impossible, since by (1.3.3) H_1 acts on the $(2n-1)$-sphere K_1/K non-transitively. Hence, it suffices to observe the case that (H, H_1) is pairwise locally isomorphic to $(SU(n+1), S(U(n)\times U(1)))$.

5.6. Suppose that

$$G = SU(n+1)\times G'' ,$$
$$K_1 = S(U(n)\times U(1))\times G'' ,$$

where $n\geqq 2$ and G'' is a connected closed normal subgroup of G, and that G acts on M almost effectively. In this case, $k_1=2n\geqq 4$, and hence K_2 is connected by (2.1.3). Let

$$\sigma_1\colon K_1 \to O(2n)$$

be the slice representation. Then there exists a representation

$$\tau\colon K_1 \to U(n) ,$$

such that the diagram

$$\begin{array}{ccc} K_1 & \xrightarrow{\sigma_1} & O(2n) \\ & {\scriptstyle\tau}\searrow \quad \nearrow & \\ & U(n) & \end{array}$$

is commutative up to conjugation. This is a consequence from the fact that σ_1 is non-trivial on the center of $S(U(n)\times U(1))$ by (1.3.2) and (1.3.3). Moreover, we have $G''=\{1\}$ or $G''=T$ by (1.4.6). First, let us consider

(i) the case $G''=T$.
The representation

$$\tau\colon K_1=S(U(n)\times U(1))\times T\to U(n)$$

in (5.6.1) is given by

$$\tau\left(\left(\begin{array}{c|c} X & \\ \hline & z\end{array}\right)\times w\right)=z^a w^b X, \tag{5.6.2}$$

for some integers a and b, where $X\in U(n)$, $z\in U(1)$, $w\in U(1)=T$ and $(\det X)z=1$. Since we can assume that $(I_{n+1}\times T)\cap K=I_{n+1}\times\{1\}$ in G, we have $b=\pm 1$. By changing the orientation of T if necessary, we can assume that $b=1$. Note that since $k_2=3$, we can write

$$\begin{aligned} K_2&=A\circ N\,,\\ K&=A'\circ N\,,\end{aligned} \tag{5.6.3}$$

where A, N are closed connected normal subgroups of K_2, A is locally isomorphic to $SO(3)$ and A' is a closed connected subgroup of A. Note that

$$K=\tau^{-1}(U(n-1))=\left\{\left(\begin{array}{c|cc} X' & & \\ \hline & z^a\overline{w} & \\ & & z\end{array}\right)\times w\right\}, \tag{5.6.4}$$

where $X'\in U(n-1)$.

Now assume $n\geqq 3$. Then the semi-simple part of K is $SU(n-1)$, which has codimension 2 in K and is contained in N. Hence, $SU(n-1)$ is a closed normal subgroup of K_2 and

$$K_2\subset S(U(n-1)\times U(2))\times T\,.$$

Since $SU(n-1)$ is a normal subgroup of K_2 with codimension 4, K_2 has the form

$$K_2=(SU(n-1)\times SU(2))\circ T'\,,$$

where

$$T'=\left\{\left(\begin{array}{c|c} u^pI_{n-1} & 0\\ \hline 0 & u^qI_2\end{array}\right)\times u^r,\quad u\in U(1),\ (n-1)p+2q=0\right\}.$$

Hence

$$A=SU(2)\times\{1\}$$

and

$$K=(SU(n-1)\times S(U(1)\times U(1)))\circ T'\,.$$

By comparison with (5.6.4), we can see that $a=1$ in (5.6.2). Thus the slice representations

$$\sigma_1\colon K_1 \to O(2n)\,,$$
$$\sigma_2\colon K_2 \to O(3)$$

are uniquely determined up to conjugation. (Note that σ_2 is tivial on $SU(n-1)$). Moreover,

$$\frac{N(K;\,G)}{N(K;\,G)^0} \cong \frac{N(K;\,K_2)}{N(K;\,K_2)^0} = \frac{N(K;\,K_2)}{K} \cong \mathbf{Z}_2\,,$$

whose generator is the class of the antipodal involution of $K_2/K = S^2$. Therefore, when $n \geqq 3$, (G, M) is uniquely determined up to essential isomorphism.

Next, let $n=2$. As above, we can assume $b=1$ in (5.6.2). Hence

$$(5.6.4)' \qquad K = \left\{ \begin{pmatrix} wz^{a-1} & & \\ & z^a\bar{w} & \\ & & z \end{pmatrix} \times w \right\}.$$

Note that since $\dim K=2$ we have that $\dim N=1$. Therefore, A' in (5.6.3) is of the form

$$A' = \left\{ \begin{pmatrix} z^{a-1} & & \\ & z^a & \\ & & z \end{pmatrix} \times 1 \right\}.$$

On one hand, since A' is a maximal torus of A in (5.6.3) (which is conjugate to $SO(3)$ or $SU(2)$ in $SU(3)$), A' is conjugate to the group

$$\left\{ \begin{pmatrix} u & & \\ & \bar{u} & \\ & & 1 \end{pmatrix} \right\} \times \{1\}$$

in $SU(3) \times T$. Hence, in (5.6.4)′, $a=0$ or $a=1$. Denote by τ_0 resp. τ_1 the representation τ in (5.6.2) for $a=0$, $b=1$, resp. $a=1$, $b=1$. Define an isomorphism

$$\phi\colon SU(3) \to SU(3)$$

by

$$\phi(Y) = P\bar{Y}P^{-1}\,,$$

where

$$P = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the diagram

$$\begin{array}{ccccc} SU(3)\times T & \supset & S(U(2)\times U(1))\times T & \searrow^{\tau_1} & \\ \phi\times 1\downarrow & & \phi\times 1\downarrow & & U(2) \\ SU(3)\times T & \supset & S(U(2)\times U(1))\times T & \nearrow_{\tau_0} & \end{array}$$

is commutative. Therefore, we shall discuss only the case $a=1$, and assume that

$$K=\left\{\begin{pmatrix} w & & \\ & z\bar{w} & \\ & & z \end{pmatrix}\times w\right\} \quad \text{and} \quad A'=\left\{\begin{pmatrix} 1 & & \\ & u & \\ & & \bar{u} \end{pmatrix}\times 1\right\}.$$

Now let $Z(A)$ be the centralizer of A in G. Note that $N\subset Z(A)^0$ since $K_2=A\circ N$. If $A=SO(3)$ up to conjugation in $SU(3)$, then $Z(A)^0=1\times T$. From $\dim N=1$, it follows that $N=1\times T\subset K$ which contradicts the almost effectivity of G-action on M. Hence, we assume that $A=SU(2)$ up to conjugation in $SU(3)$. Then

$$Z(A)^0=\left\{\begin{pmatrix} \bar{u}^2 & & \\ & u & \\ & & u \end{pmatrix}\right\}\times T$$

up to conjugation in $SU(3)$. Therefore,

$$N=\left\{\begin{pmatrix} \bar{u}^2 & & \\ & u & \\ & & u \end{pmatrix}\times \bar{u}^2\right\} \quad \text{and} \quad A=\left\{\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & X \end{array}\right)\times 1,\ \ X\in SU(2)\right\}.$$

Thus the slice representation

$$\sigma_2\colon K_2\to O(3)$$

can be determined uniquely up to conjugation. The other slice representation $\sigma_1\colon K_1\to O(4)$ has already determined and

$$\frac{N(K;G)}{N(K;G)^0}\cong \boldsymbol{Z}_2,$$

whose generator is the class of the antipodal involution of $S^2=K_2/K$. Therefore, in the present case, (G, M) is determined uniquely up to essential isomorphism.

(ii) The case $G''=\{1\}$.

When $n\geqq 3$, by the similar argument as in the case (i), we can see that (G, M) is uniquely determined up to essential isomorphism.

Now assume that $n=2$. Then

$$G = SU(3)\,,$$
$$K_1 = S(U(2)\times U(1))\,,$$

and for the slice representation

$$\sigma_1 \colon K_1 \to O(4)\,,$$

there exists a representation

$$\tau_a \colon K_1 \to U(2)\,,$$

so that the diagram

$$\begin{array}{ccc} K_1 & \xrightarrow{\sigma_1} & O(4) \\ & {\scriptstyle\tau_a}\searrow & \cup \\ & & U(2) \end{array}$$

is commutative. Since τ_a is given by

$$\tau_a\begin{pmatrix} X & 0 \\ 0 & z \end{pmatrix} = z^a X\,,$$

where $X \in U(2)$, $z \in U(1)$, $(\det X)z=1$ and a is an integer, we have that

$$\text{(5.6.4)}'' \qquad K = \left\{ \begin{pmatrix} z^{a-1} & & \\ & z^{-a} & \\ & & z \end{pmatrix},\ z \in U(1) \right\}.$$

Since $\dim K_2=3$, K_2 is isomorphic to $SO(3)$ or $SU(2)$ up to conjugation in $G=SU(3)$, and

$$K = SO(2)\,, \qquad \text{if } K_2 = SO(3)\,,$$
$$K = S(U(1)\times U(1))\,, \qquad \text{if } K_2 = SU(2)\,.$$

Hence, $a=0$ or $a=1$ in (5.6.4)$''$. As in the case (i), it suffices to discuss only the case $a=1$, i.e., we can assume that

$$K = \left\{ \begin{pmatrix} 1 & & \\ & z & \\ & & z \end{pmatrix},\ z \in U(1) \right\}.$$

Define

$$H = \begin{cases} \left\{ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & X \end{array} \right) \in SU(3) \right\}, & \text{if } K_2 = SU(2)\,, \\ BSO(3)B^{-1}\,, & \text{if } K_2 = SO(3)\,; \end{cases}$$

where

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & i/\sqrt{2} \\ 0 & i/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

Then $(G, K_2, K)=(G, H, T)$ up to conjugation, where T is a maximal torus of H. Thus the slice representations

$$\sigma_1: K_1 \to O(4)$$
$$\sigma_2: K_2 \to O(3)$$

are uniquely determined up to conjugation and

$$\frac{N(T; SU(3))}{N(T; SU(3))^0} \cong \frac{N(T; SU(2))}{N(T; SU(2))^0} \cong \boldsymbol{Z}_2 ,$$
$$\frac{N(SO(2); SU(3))}{N(SO(2); SU(3))^0} \cong \frac{N(SO(2); SO(3))}{N(SO(2); SO(3))^0} \cong \boldsymbol{Z}_2 ,$$

which are generated by the classes of the antipodal involutions of 2-dimensional spheres $SU(2)/T$ and $SO(3)/SO(2)$ respectively. Therefore, in each of the case $K_2=SU(2)$, or $SO(3)$, (G, M) is uniquely determined up to essential isomorphism. On one hand, we have seen in 3.3, that $(U(n+1), P_n(\boldsymbol{H}))$ is an example of (G, M) of the case (i), $(SU(n+1), P_n(\boldsymbol{H}))$ is an example of (G, M) of the case (ii) and $(SU(3), \boldsymbol{G}_2/SO(4))$ is an example of (G, M) of the case (ii), $n=2$. Thus the proof of Theorem 5.1.4 is completed.

6. Pairs (G, M) with non-orientable singular orbits

6.1. The purpose of this section is to classify (G, M) up to essential isomorphism, when both singular orbits G/K_1 and G/K_2 are non-orientable. We shall prove

Theorem 6.1.1. *Such a (G, M) is essentially isomorphic to $(SO(3), S^4)$. Here, S^4 is considered as the unit sphere of the 5-dimensional irreducible real representation space of $SO(3)$ given in 3.4.*

As in the previous sections, we suppose that G acts on M almost effectively and $G=G_1\times T^h$, where G_1 is a compact simply connected Lie group and T^h is an h-dimensional toral group.

6.2. Consider the pair (G, M) with non-orientable singular orbits G/K_1, G/K_2 and a principal orbit G/K. By Theorem 2.1.4, M is 4-dimensional and

$$P(G/K_s; t) = 1 , \quad P(G/K_s^0; t) = 1+t^2 ,$$

for $s=1, 2$. First, we shall show

(6.2.1) $G=Sp(1)$, and K is a finite subgroup of G.

We can assume that

$$G = Sp(1)\times G'\times T^h ,$$
$$K = T\times G'\times T^h ,$$

where G' is semi-simple, T^h is an h-dimensional toral subgroup and T is a maximal torus of $Sp(1)$. Since K^0 is a closed connected subgroup of a compact connected Lie group K_1^0 with $\dim K_1^0/K^0=1$, we can see that K^0 is a normal subgroup of K_1^0. Therefore, by (1.4.6), $G'=\{1\}$ and $h\leqq 1$. Now we see that

$$G = Sp(1)\times T^h , \qquad h\leqq 1$$

and that K_s^0 is a maximal torus of G. Since G/K_s is non-orientable and $N(K_s^0; G)/K_s^0\cong \boldsymbol{Z}_2$, we have

$$K_s = N(K_s^0; G) , \qquad s = 1, 2 .$$

Now we suppose that $h=1$. Then, since G acts almost effectively on G/K by our assumption, $1\times T^1$ is not contained in K and is mapped onto $SO(2)$ by the slice representation

$$\sigma_s\colon K_s = N(K_s^0; G) \to O(2) .$$

Since the centralizer of $SO(2)$ in $O(2)$ is $SO(2)$ and since $1\times T^1$ is a central subgroup of K_s, $\sigma_s(K_s)=SO(2)$. This contradicts the non-orientability of G/K_s. Hence h must be 0, that is, $G=Sp(1)$. Since $\dim G/K=3$, K is a finite subgroup of G.

Note that

(6.2.2) $\sigma_s\colon K_s \to O(2)$ is surjective.

For, since $\ker \sigma_s\subset K$ and K is finite, we have $\sigma_s(K_s^0)=SO(2)$. Therefore, $\sigma_s(K_s)=O(2)$ follows from the non-orientability of G/K_s.

6.3. We shall observe the normalizer of a maximal torus of $Sp(1)$. Let $q=a+bi+cj+dk$ (a, b, c and d are real numbers) be a quaternion number. It can be written in the form

$$q = \alpha+\beta j ,$$

where $\alpha=\alpha(q)$ and $\beta=\beta(q)$ are complex numbers. We assume that $q\in Sp(1)$, i.e., the norm of q, $|q|=\sqrt{|\alpha^2|+|\beta|^2}$, is equal to 1, throughout this section. Define

$$T_q = \{qe^{i\theta}q^{-1} | \theta \in \boldsymbol{R}\} .$$

This is a maximal torus of $Sp(1)$. It is clear that $T_q = qT_1q^{-1}$. Let NT_q be the normalizer of T_q in $Sp(1)$. Note that

$$NT_1 = T_1 \cup jT_1$$

and

$$q = \alpha + \beta j \in NT_1 , \qquad \text{if and only if } \alpha\beta = 0 .$$

The following propositions are easily verified:
For $q = \alpha + \beta j$,

(6.3.1) if $\alpha\beta \neq 0$, then $T_1 \cap T_q = \{\pm 1\}$,
if $\alpha\beta = 0$, then $T_1 = T_q$;

(6.3.2) if $|\alpha| = |\beta|$, then
$(NT_1 - T_1) \cap T_q = \{\pm 2\alpha\beta k\}$,
$(NT_q - T_q) \cap T_1 = \{\pm i\}$;

(6.3.3) $|\alpha| = |\beta| \Leftrightarrow (NT_1 - T_1) \cap T_q \neq \phi$
$\Leftrightarrow (NT_q - T_q) \cap T_1 \neq \phi$;

(6.3.4) if $\alpha\beta \neq 0$, then
$(NT_1 - T_1) \cap (NT_q - T_q) = \{\pm \alpha\beta j / |\alpha\beta|\}$.

From these propositions, it follows

(6.3.5) for $q = \alpha + \beta j$, $\alpha\beta \neq 0$,

$$NT_1 \cap NT_q \cong \begin{cases} D_8^* = \{\pm 1, \pm i, \pm j, \pm k\} , & \text{if } |\alpha| = |\beta| , \\ \boldsymbol{Z}_4 , & \text{if } |\alpha| \neq |\beta| . \end{cases}$$

Let $N = N(D_8^*; Sp(1))$ be the normalizer of D_8^* in $Sp(1)$. A quaternion $q \in Sp(1)$ is in N if and only if both qiq^{-1} and qjq^{-1} are in D_8^*. We can see that N consists of 48 elements and is isomorphic to the binary octahedral group

$$O^* = \{a, b \,|\, a^2 = (ab)^3 = b^4, a^4 = 1\} ,$$

under the correspondence

$$a \leftrightarrow (i+k)/\sqrt{2} ,$$
$$b \leftrightarrow (1-k)/\sqrt{2} .$$

Moreover, we can see

$$N(D_8^*; NT_1) = D_8^* \cup \{(\pm 1 \pm i)/\sqrt{2}, (\pm j \pm k)/\sqrt{2}\} .$$

6.4. For a surjective representation

$$NT_1 \to O(2)\,,$$

we can find an equivalent representation σ, satisfying

$$\sigma(e^{i\theta}) = \begin{pmatrix} \cos 2t\theta & -\sin 2t\theta \\ \sin 2t\theta & \cos 2t\theta \end{pmatrix},$$
$$\sigma(j) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

where t is a positive integer. Let the inclusion $O(1)\subset O(2)$ be given by

$$\pm 1 \mapsto \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then there is an isomorphism

(6.4.1) $$\sigma^{-1}(O(1)) = D^*_{4t} = \{x, y \mid x^t = y^2 = (xy)^2,\, y^4 = 1\}$$

defined by

$$\exp(i\pi)/t \leftrightarrow x\,.$$
$$j \leftrightarrow y\,.$$

Next, consider the homomorphism

(6.4.2) $$\pi_1(Sp(1)/D^*_{4t}) \to \pi_1(Sp(1)/NT_1)$$

induced by the natural projection. Note that

$$\pi_1(Sp(1)/D^*_{4t}) \cong D^*_{4t}\,,$$
$$\pi_1(Sp(1)/NT_1) \cong \boldsymbol{Z}_2\,.$$

Since the diagram, which consists of the natural projections,

$$\begin{array}{ccc} Sp(1) & \longrightarrow & Sp(1)/T_1 \\ \downarrow & & \downarrow \\ Sp(1)/D^*_{4t} & \underset{p}{\longrightarrow} & Sp(1)/NT_1 \end{array}$$

is commutative and the right vertical map is a double covering, we can see

(6.4.3) $$p_*(x) = 1\,,\quad p_*(y) \neq 1\,,$$

where 1 means the unit element of $\pi_1(Sp(1)/NT_1)\cong \boldsymbol{Z}_2$.

6.5. Now we go back to the consideration on (G, M) and claim

(6.5.1) (G, K_1, K_2) is uniquely determined up to conjugation by elements of G, and $K=K_1\cap K_2$.

Recall that $G=Sp(1)$, K_s^0 is a maximal torus of G and $K_s=N(K_s^0;G)$, $s=1, 2$. First, we assume $K_1=K_2$. Then, since we can put $K_s^0=T_1$ and $K_s=NT_1$ (see, 6.3.),

$$p_{s*}\colon \pi_1(G/K)\to\pi_1(G/K_s)\,,\qquad s=1,2\,,$$

is identified with the homomorphism (6.4.2). Hence $\ker p_{1*}=\ker p_{2*}$ is a proper normal subgroup of K. On the other hand, by [7, (2.4.2)], $\pi_1(G/K)=(\ker p_{1*})\times(\ker p_{2*})$, which shows that our assumption fails. Therefore, $K_1\neq K_2$. Since $K_s=N(K_s^0;G)$, $K_1^0\neq K_2^0$. Thus we can suppose

$$(G, K_1, K_2)=(Sp(1), NT_1, NT_q)\,,$$

for some $q\in Sp(1)$. Since $T_1\neq T_q$, we have $\alpha(q)\beta(q)\neq 0$ by (6.3.1).

Suppose $|\alpha|\neq|\beta|$. Then by (6.3.5) we have

$$K_1\cap K_2\cong \boldsymbol{Z}_4\,.$$

In the commutative diagram

$$\begin{array}{ccc} \pi_1(G/K) & \longrightarrow & \pi_1(G/K_1\cap K_2)\cong K_1\cap K_2 \\ & {}_{p_{s*}}\searrow & \downarrow \\ & & \pi_1(G/K_s) \end{array}\quad,$$

p_{s*} is surjective by (6.4.3). Hence, the generator of $K_1\cap K_2$ goes into the non-trivial element of $\pi_1(G/K_s)\cong\boldsymbol{Z}_2$. It follows that $K=K_1\cap K_2$ and $\ker p_{1*}=\ker p_{2*}$ is a proper normal subgroup of K. This contradicts [7, (2.4.2)]. Thus we have $|\alpha|=|\beta|$. Then $2\alpha\beta i=u^{-2}$ for some $u\in U(1)$, and hence

$$(Sp(1), uNT_1u^{-1}, uNT_qu^{-1})=(Sp(1), NT_1, NT_r)\,,$$

where $r=(1+k)/\sqrt{2}$. Therefore, we can assume $q=(1+k)/\sqrt{2}$. Then we have

$$K_1\cap K_2\cong D_8^*=\{\pm 1, \pm i, \pm j, \pm k\}\,.$$

Consider the commutative diagram

$$\begin{array}{ccccc} & & \pi_1(G/K) & & \\ & {}^{p_{1*}}\swarrow & & \searrow^{p_{2*}} & \\ \pi_1(G/K_1) & & \downarrow & & \pi_1(G/K_2)\,. \\ & {}_{p'_{1*}}\nwarrow & & \nearrow_{p'_{2*}} & \\ & & \pi_1(G/K_1\cap K_2) & & \end{array}$$

Here, each homomorphism is induced by the corresponding natural projection. By (6.4.1), $D_{4t}^*\cong K\subset K_1\cap K_2\cong D_8^*$, and hence $t\leqq 2$. Suppose $t=1$. Then $K=\boldsymbol{Z}_4$ is generated by y, and $p_{1*}(y)\neq 1$, $p_{2*}(y)\neq 1$ by (6.4.3). It follows that

$\ker p_{1*} = \ker p_{2*} \cong \boldsymbol{Z}_2$ is a proper normal subgroup of $K = \boldsymbol{Z}_4$. This contradicts [7, (2.4.2)]. Therefore, $t=2$ and $K = K_1 \cap K_2$. Hence the slice representations of K_1, K_2 are uniquely determined up to conjugation by (6.4.1).

Now, let us define

$$X = Sp(1) \underset{NT_1 \circlearrowright \sigma}{\times} D^2 ,$$

where

$$\sigma \colon NT_1 \to O(2)$$

is given by

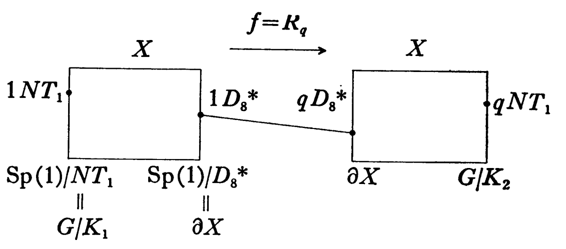
$$\sigma(e^{i\theta}) = \begin{pmatrix} \cos 4\theta & -\sin 4\theta \\ \sin 4\theta & \cos 4\theta \end{pmatrix},$$

$$\sigma(j) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and NT_1 acts on D^2 via σ. By the above consideration, we can assume

$$M(f) = X \underset{f}{\cup} X$$

as $Sp(1)$-manifold, where f is an $Sp(1)$-diffeomorphism on $\partial X = Sp(1)/D_8^*$. There exists $q = \alpha + \beta j \in N(D_8^*; Sp(1))$ such that $f = R_q$ (right translation by q). (See, (1.4.5).) Since the isotropy group at qNT_1 is $qNT_1 q^{-1} = NT_q$, we have $|\alpha| = |\beta|$.



Then there exist u, $v \in N(D_8^*; NT_1)$ such that $q = u \dfrac{1+k}{\sqrt{2}} v$. Therefore

$$M(R_q) = M(R_{(1+k)/\sqrt{2}})$$

as $Sp(1)$-manifold, because they are identified by $Sp(1)$-diffeomorphism

$$(\text{extension of } R_{u^{-1}}) \cup (\text{extension of } R_v) .$$

Thus the pair (G, M) of the type (C) of Theorem 2.1.4 is unique up to essential isomorphism. On one hand, there is an example of this type as is seen in 3.4. Therefore, the proof of Theorem 6.1.1 is completed.

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