Carroll, R.
Osaka J. Math.
14 (1977), 609–618

SOME REMARKS ON DEGENERATE CAUCHY PROBLEMS IN GENERAL SPACES

ROBERT CARROLL

(Received November 15, 1976)

1. Introduction. We will consider problems of the form

$$u''+s(t)u'+Ar(t)u-A^2a(t)u+b(t)u=f \tag{1.1}$$

$$u(0)=u'(0)=0 \tag{1.2}$$

where A is the generator of a locally equicontinuous group $T(t)$ in a complete separated locally convex space E (cf. [8; 14]), $u\in C^2(E), f\in C^0(E)$, s, r, a, and b are continuous real valued functions, while $a(t)>0$ for $t>0$ with $a(0)=0$. This is an extension of the Cauchy problem for Tricomi equations and various general versions of (1.1)–(1.2) have been considered for example in [1; 2; 7; 8; 10; 15; 16; 18; 22; 23; 24]; for an extensive bibliography see [8]. We will adapt a method of Hersh [13] as extended by the author in [4, 5; 6; 8], to solve (1.1)–(1.2) and prove some uniqueness theorems. The behavior of $\int_\tau^T (r^2/a)(\xi)d\xi$ as $\tau\to 0$ again turns out to play a critical role in uniqueness (as in [7; 8; 23; 24]) and is related to conditions of Krasnov [15] and Protter [18] in their specific contexts. Let us note that a typical case involves $A^2=\Delta$ in a suitable space E (cf. [8]).

2. Following [4; 5; 6; 8; 13] we replace A by $-d/dx$ in (1.1) and consider

$$w''+s(t)w'-r(t)w_x-a(t)w_{xx}+b(t)w=0 \tag{2.1}$$

where $w(t)\in\mathcal{S}_x'$ (detailed properties are indicated below). Let us Fourier transform (2.1) in the x variable, writing formally $\hat{w}(t)=\mathcal{F}w(t)=\int_{-\infty}^{\infty} w(t)\exp ixy\, dx$, to obtain

$$\hat{w}''+s(t)\hat{w}'+iyr(t)\hat{w}+a(t)y^2\hat{w}+b(t)\hat{w}=0 \tag{2.2}$$

It will be convenient to elminate the $b(t)$ term as follows. Let $\hat{w}(t)=\hat{v}(t)\exp \int_0^t\gamma(\xi)d\xi$ where $\gamma(t)$ satisfies the Riccati equation

$$\gamma' + s\gamma + \gamma^2 + b = 0; \quad \gamma(0) = 0 \tag{2.3}$$

(see below for details). Then $\hat{v}$ satisfies

$$\hat{v}'' + (2\gamma(t) + s(t))\hat{v}' + (a(t)y^2 + \mathrm{i}y\mathrm{r}(t))\hat{v} = 0 \tag{2.4}$$

and it will be easier to deal with (2.4). In order to produce a suitable function $\gamma(t)$ we note that if one sets $\gamma = \alpha'/\alpha$ then α satisfies

$$\alpha'' + s(t)\alpha' + b(t)\alpha = 0 \tag{2.5}$$

(cf. [12]) and we choose α to be the unique solution of (2.5) satisfying $\alpha(0)=1$ with $\alpha'(0)=0$. Then $\gamma(0)=0$ and the continuous function γ will remain finite on some interval $0 \leq t \leq T < t_0 < \infty$ where t_0 is the first zero of $\alpha(t)$. It is sufficient for us to solve (1.1) on such an interval since for $t \geq T$ the equation (1.1) is not degenerate and can be handled by standard techniques (cf [3; 17]). Now following [11] we write (2.4) as a system

$$\vec{v}'(t) = P(y, t)\vec{v}(t); \quad \vec{v}(t) = \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \end{bmatrix}; \tag{2.6}$$

$$P(y, t) = \begin{bmatrix} 0 & y \\ -ir - ay & -s - 2\gamma \end{bmatrix}$$

where $\hat{v}_1 = y\hat{v}$ and $\hat{v}_2 = \hat{v}'$. We look for solutions $\vec{Y}$ and $\vec{Z}$ of (2.6) satisfying

$$\vec{Y}(\tau) = \begin{bmatrix} y\hat{Y} \\ \hat{Y}_t \end{bmatrix}(\tau) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \tag{2.7}$$

$$\vec{Z}(\tau) = \begin{bmatrix} y\hat{Z} \\ \hat{Z}_t \end{bmatrix}(\tau) = \begin{bmatrix} y \\ 0 \end{bmatrix}$$

where $0 \leq \tau \leq t \leq T$. The functions $\hat{Z}(t, \tau, y)$ and $\hat{Y}(t, \tau, y)$, together with their inverse Fourier transforms, will be called resolvants. It is easily shown following [7; 8; 19] that

$$\hat{Z}_\tau = (ay^2 + iyr)(\tau)\hat{Y} \tag{2.8}$$

$$\hat{Y}_\tau = -\hat{Z} + (s + 2\gamma)(\tau)\hat{Y}. \tag{2.9}$$

Now by well known theorems (cf [3; 9; 12] there exist solutions $\hat{Y}(t, \tau, y)$ and $\hat{Z}(t, \tau, y)$ of (2.4) (i.e. (2.6)), satisfying the prescribed initial conditions, which are continuous in (t, τ, y) and analytic in y for $0 \leq \tau \leq t \leq T < \infty$ and $y \in \boldsymbol{C}$. Moreover by a clever argument in [11] if one writes the solution of (2.6) in the form

$$\vec{v}(t, \tau, y) = Q(t, \tau, y)\vec{v}(\tau, \tau, y) \tag{2.10}$$

where $Q(\tau, \tau, y) = I$ then $\|Q(t, \tau, y)\| \leq |c \exp \hat{c}|y|(t-\tau)$ where $\| \ \|$ denotes the matrix operator norm (so $|q_{ij}| \leq \|Q\|$ in particular when $Q = (q_{ij})$). Thus the entries in Q are entire analytic functions of y of exponential type $\leq \hat{c}(t-\tau) \leq \hat{c}T$. This proves

Lemma 2.1. *The functions $\hat{Y}(t, \tau, y)$ and $\hat{Z}(t, \tau, y)$ are continuous in (t, τ, y) for $0 \leq \tau \leq t \leq T$ and $y \in \boldsymbol{C}$ while, for (t, τ) fixed, $y\hat{Y}$, $y\hat{Z}$, $\hat{Y}_t$, and $\hat{Z}_t$ are entire analytic functions of exponential type $\leq \hat{c}T$.*

In order to invoke the Paley-Wiener-Schwartz theorem later (cf. [8; 11; 20]) we examine the growth of $\hat{Y}$, $\hat{Z}$, etc. for real y. Thus writing first $\hat{Y} = \varphi + i\psi$ we obtain from (2.4)

$$\varphi'' + (2\gamma + s)\varphi' + ay^2\varphi - yr\psi = 0; \qquad (2.11)$$
$$\psi'' + (2\gamma + s)\psi' + ay^2\psi + yr\varphi = 0 .$$

Multiply the first equation in (2.11) by φ' and the second by ψ' and add, observing that $\hat{Y}\overline{\hat{Y}'} = \varphi\varphi' + \psi\psi' + i(\psi\varphi' - \varphi\psi')$ for example so that in particular $d/dt|\hat{Y}|^2 = 2 \text{ Re } \hat{Y}\overline{\hat{Y}'} = 2(\varphi\varphi' + \psi\psi')$ while $|yr(\psi\varphi' - \varphi\psi')| = |yr \text{ Im } \hat{Y}\overline{Y}|$ $\leq {}^{1/2}(y^2r^2|\hat{Y}|^2 + |\hat{Y}'|^2)$. This yields then

$$\frac{d}{dt}|\hat{Y}'|^2 + 2(2\gamma + s)|\hat{Y}'|^2 + ay^2\frac{d}{dt}|\hat{Y}|^2 \leq (y^2r^2|\hat{Y}|^2 + |\hat{Y}'|^2) \qquad (2.12)$$

Integrating (2.12) now under the assumption that $a \in C^1$ we obtain for $0 < \tau \leq t \leq T$

$$|\hat{Y}'|^2 + 2\int_\tau^t (2\gamma + s)|\hat{Y}'|^2 d\xi + a(t)y^2|\hat{Y}|^2 \leq 1 + \int_\tau^t [(a'y^2 + y^2r^2)|\hat{Y}|^2 + |\hat{Y}'|^2] d\xi \qquad (2.13)$$

where $\hat{Y} = \hat{Y}(\xi, \tau, y)$ etc. in the integrations. This type of inequality can be treated by use of Gronwall type lemmas as in [7; 8; 23]. Thus set $P = a'y^2 + y^2r^2$ and $\tilde{Q} = 1 - 2(2\gamma + s)$ so that $|\tilde{Q}| \leq \tilde{c}$ on $[0, T]$ by the continuity of γ and s. Then add $\tilde{c}\int_\tau^t a^2y^2|\hat{Y}|^2 d\xi$ to the right side of (2.13), without changing the inequality, and setting $\Xi = |\hat{Y}'|^2 + ay^2|\hat{Y}|^2$ we have

$$\Xi \leq 1 + \int_\tau^t P|\hat{Y}|^2 d\xi + \tilde{c}\int_\tau^t \Xi d\xi$$

A straightforward application of the Gronwall lemma (cf. [3]) yields

$$\Xi \leq E(t, \tau) + \int_\tau^t P|\hat{Y}|^2 E(t, \xi) d\xi \qquad (2.15)$$

where $E(t, \xi)=\exp\tilde{c}\,(t-\xi)$. Now forget the $|\hat{Y}'|^2$ term in Ξ and following a Gronwall type procedure written out in [8] we get immediately from (2.15) for $P\geq 0$

$$ay^2|\hat{Y}|^2\leq E(t, \tau)\exp\int_\tau^t \hat{P}d\xi \tag{2.16}$$

where $\tilde{P}=a'/a+r^2/a$. Integrating the a'/a term and rearranging these results

Lemma 2.2. *Given* $a\in C^1$, b, r, $s\in C^0$, $\hat{P}\geq 0$, *and* $\hat{Y}$ *the solution of* (2.4) *satisfying* $\hat{Y}(\tau, \tau, y)=0$ *with* $\hat{Y}_t(\tau, \tau, y)=1$ *it follows that*

$$a(\tau)y^2|\hat{Y}(t, \tau, y)|^2\leq E(t, \tau)\exp\int_\tau^t (r^2/a)d\xi \tag{2.17}$$

for y real and $0<\tau\leq t\leq T$.

Let now $F(t, \tau)=\exp(-\int_\tau^t (r^2/a)\,d\xi)$ and $F(\tau)=F(T, \tau)$ so $F(\tau)\leq F(t, \tau)$. Then since $E(t, \tau)\leq\exp\tilde{c}T=k$ we have from (2.17) the inequality

$$a(\tau)F(\tau)y^2|\hat{Y}(t, \tau, y)|^2\leq k\,. \tag{2.18}$$

Note that $F(\tau)$ may tend to zero as $\tau\to 0$ while $a(\tau)\to 0$ by assumption, but for $\tau>0$ both $F(\tau)$ and $a(\tau)$ are positive. Similarly, as in [2], we obtain from (2.14)–(2.16)

$$|\hat{Y}_t(t, \tau, y)|^2 a(\tau)F(\tau)\leq\tilde{k} \tag{2.19}$$

where $\tilde{k}=k\max a(t)$ on $[0, T]$, and going back to (2.4) we have for $Q(\tau)=(a(\tau)F(\tau))^{1/2}$

$$\begin{aligned} Q(\tau)|\hat{Y}_{tt}(t, \tau, y)| &\leq |2\gamma(t)+s(t)|Q(\tau)|\hat{Y}_t| \\ &\quad +(|yr(t)|+a(t)y^2)Q(\tau)|\hat{Y}|+k_1+k_2|y| \end{aligned} \tag{2.20}$$

(upon using (2.18)–(2.19) and the continuity of a, r, s, and γ). Next, setting $\hat{W}(t, \tau, y)=Q(\tau)\hat{Y}(t, \tau, y)$, from Lemma 2.1 and the estimate (2.18) arising from Lemma 2.2 we know that the functions $y\to y\hat{W}(t, \tau, y)$ are entire of exponential type $\leq\hat{c}T$ and are bounded uniformly by a constant for y real and $0\leq\tau\leq t\leq T$. Further we know that the $\hat{W}(t, \tau, \cdot)$ are analytic in the same region (note that the $Q(\tau)$ factor arising from (2.18) is only needed to produce a uniform bound for y real as $\tau\to 0$—the function $\hat{Y}(t,\tau,y)$ is continuous in (t,τ,y) for $0\leq\tau\leq t\leq T$ and $y\in\boldsymbol{C}$). Writing $\hat{Y}(t, \tau, y)=\sum_0^\infty a_n(t, \tau)y^n$ we have $y\hat{Y}(t, \tau, y)=\sum_0^\infty a_n(t, \tau)y^{n+1}=\sum_1^\infty a_{k-1}y^k$ and by definition one has then $1=\limsup k\log k/-\log|a_{k-1}|$ as $k\to\infty$ (cf. [8; 20]). Consequently we can write $\limsup(n+1)\log(n+1)/-\log|a_n|=1$ which implies $\limsup n\log n/-\log|a_n|=1$ so $\hat{Y}(t, \tau, \cdot)$ is of exponential type along with $y\hat{Y}(t, \tau, \cdot)$. Further, since the type of such a function $g(y)$ is defined by $\limsup \log|g(y)|/|y|$ as $|y|\to\infty$, we see from $\limsup \log|yg(y)|/|y| = \limsup(\log|y|+\log|g(y)|)/|y| = \limsup \log|g(y)|/|y|$ that the functions $\hat{Y}(t, \tau, \cdot)$ are also of exponential type $\leq \hat{c}T$ for $0\leq\tau\leq t\leq T$. Now for y real with $|y|\leq R_0$ say $|\hat{W}(t, \tau, y)|$ is bounded by continuity in (t, τ, y) and by (2.18) $|\hat{W}(t,\tau,y)|\leq k^{1/2}/|y|$ is bounded for $|y|>R_0$. From the Paley-Wiener-Schwartz theorem it then follows that $W(t, \tau, \cdot)=\mathcal{F}^{-1}\hat{W}(t, \tau, y)\in\mathcal{E}_x'$ with supp W contained in a fixed compact set for $0\leq\tau\leq t\leq T$. Similar conclusions apply to W_t and W_{tt} from Lemma 2.1, (2.4), and the estimates (2.19)–(2.20). Reasoning as in [8] one can verify that W_t and W_{tt} indeed represent the derivatives of W in $\mathcal{E}_x'$ and we can state

Theorem 2.3. *Let the hypotheses of Lemma 2.2 hold with $Q(\tau)=(a(\tau)F(\tau))^{1/2}$ where $F(\tau)=\exp(-\int_\tau^T(r^2/a)d\xi)$ and set $\hat{W}(t, \tau, y)=Q(\tau)\hat{Y}(t, \tau, y)$ where $\hat{Y}$ is the unique solution of (2.4) satisfying $\hat{Y}(\tau, \tau, y)=0$ and $\hat{Y}_t(\tau, \tau, y)=1$. Then $W=\mathcal{F}^{-1}\hat{W}$, W_t, and W_{tt} belong to $\mathcal{E}_x'$ and have supports contained in a fixed compact set for $0\leq\tau\leq t\leq T$. Moreover $(t, \tau)\to W$, W_t, and W_{tt} are continuous with values in $\mathcal{E}_x'$ for $0\leq\tau\leq t\leq T$ with $t\to W(t, \tau)\in C^2(\mathcal{E}_x')$.*

3. Going back to (1.1) and (2.1) we omit the $b(t)$ term in view of (2.3) and replace $s(t)$ by $s(t)+2\gamma(t)=\tilde{s}(t)$. Let us write $h(t)=f(t)/Q(t)$ and assume $h(\cdot)\in C^0(E)$ with $f(t)\in D(A^2)$ for fixed t, while $Ah(\cdot)$ and $A^2h(\cdot)\in C^0(E)$ on $[0, T]$. We define a bracket $\langle W(t, \xi, \cdot), T(\cdot)h(\xi)\rangle$ as in [4; 5; 6; 8] for fixed (t, ξ) and observe that $(\xi, x)\to T(x)h(\xi)\in C^0(E)$ since $x\to T(x)\in C^0(L_s(E))$ and, for any continuous seminorm p on E, there is a continuous seminorm q such that $p(T(x)e)\leq q(e)$ for $|x|\leq x_1$ suitably large and $e\in E$ (cf. [14]). The operation $\langle\ ,\ \rangle$ indicates a pairing between distributions $S\in\mathcal{E}_x'$ of order ≤ 2 with supp $S\subset K$ compact and functions $g\in C_x^2(E)$ on $\boldsymbol{R}$ (recall here that $T(x)$ is a group). Given this situation we can think of $K\subset\hat{K}=\{x; |x|\leq x_0\}$ and represent $C^2(E)$ on $\hat{K}$ as $C^2\otimes_\varepsilon E$ (cf. [4; 5; 21]) for details in the present discussion). Then $S\in C^2(\hat{K})^2$ and the pairing $\langle S, g\rangle$ is well defined with $S\to\langle S, g\rangle$ continuous $C^2(\hat{K})'\to E$. The map $\Delta=\Delta\otimes 1=d^2/dx^2\otimes 1$: $C^2(E)\to C^0(E)$ is defined by extension from $C^2\hat{\otimes}E\to C^0\otimes E$ and is continuous; it can be transported around under $\langle\ ,\ \rangle$ in a distribution sense for suitable S and g as above (i.e. $\langle\Delta S, g\rangle=\langle S, \Delta g\rangle$ for S of order zero, the bracket for $\langle S, \Delta g\rangle$ being defined in the same way). We remark that in fact $(S,g)\to\langle S,g\rangle$: $\mathcal{E}'\times C^2(E)\to E$ is easily seen to be separately continuous for S restricted as indicated and since $\mathcal{E}'$ is barreled $(S, g)\to\langle S, g\rangle$ will be hypocontinuous on bounded sets in $C^2(E)$ (cf. [21]). Consider then for $\tau>0$

$$u(t) = \int_\tau^t\langle W(t, \xi, \cdot), T(\cdot)h(\xi)\rangle d\xi \tag{3.1}$$

We calculate formally in remarking that all the operations are legitimate. First

$$u'(t) = \int_{\tau}^{t} \langle W_t(t, \xi, \cdot), T(\cdot)h(\xi)\rangle d\xi \tag{3.2}$$

since $W(t, t, \cdot)=0$ and since $W_t(t, t, \cdot)=\underset{\sim}{Q}(t)\delta$ there results

$$u''(t) = f(t)+\int_{\tau}^{t} \langle W_{tt}(t, \xi, \cdot), T(\cdot)h(\xi)\rangle d\xi \tag{3.3}$$

Now look at our new version of (1.1) and observe that for example

$$\begin{aligned} Au(t) &= \int_{\tau}^{t} \langle W(t, \xi, \cdot), AT(\cdot)h(\xi)\rangle d\xi \\ &= \int_{\tau}^{t} \langle W(t, \xi, \cdot), \frac{d}{dx}T(\cdot)h(\xi)\rangle d\xi \\ &= -\int_{\tau}^{t} \langle \frac{d}{dx}W(t, \xi, \cdot), T(\cdot)h(\xi)\rangle d\xi \end{aligned} \tag{3.4}$$

Similarly $A^2u(t)=\int_{\tau}^{t}\langle \Delta W(t, \xi, \cdot), T(\cdot)h(\xi)\rangle d\xi$ where $\Delta=d^2/dx^2$. Putting $u(t)$, defined by (3.1), in the modified equation (1.1) we obtain

$$\begin{aligned} u''+\tilde{s}(t)u'+Ar(t)u-A^2a(t)u &= f(t) \\ &+\int_{\tau}^{t} \langle W_{tt}+\tilde{s}(t)W_t-r(t)\frac{d}{dx}W-a(t)\Delta W, Th\rangle \, d\xi \end{aligned} \tag{3.5}$$

and the integral term vanishes because W, along with Y, satisfies the correspondingly modified equation (2.1). There is no trouble now in passing to the limit $\tau=0$ under our hypotheses and, using γ to transform back to the original equation (1.1), we have proved

Theorem 3.1. *Let $a(t)>0$ for $t>0$ with $a(0)=0$ and $a\in C^1$; let b, r, and s belong to C^0, $\hat{P}\geq 0$ and choose T as in (2.3)–(2.4); let $\underset{\sim}{Q}$ and F be defined as in Theorem 2.3 and assume $h(\cdot)=f(\cdot)/\underset{\sim}{Q}(\cdot)\in C^0(E)$ on $[0, T]$ with $Ah(\cdot)$ and $A^2h(\cdot)$ $\in C^0(E)$ on $[0, T]$, where A generates a locally equicontinuous group $T(x)$ in E. Then, after modification by a factor $\exp\int_0^t \gamma(\xi)d\xi$, $u(t)$ given by (3.1) with $\tau=0$ is a solution of (1.1)–(1.2) on $[0, T]$.*

4. We go now to questions of uniqueness and will have to determine some properties of the other resolvant $\hat{Z}(t, \tau, y)$. First we duplicate our procedure (2.11)–(2.12) in order to estimate $|\hat{Z}|$ and $|\hat{Z}_t|$ for y real. This yields

$$\begin{aligned} \frac{d}{dt}|\hat{Z}'|^2+2\tilde{s}(t)|\hat{Z}'|^2+a(t)y^2\frac{d}{dt}|\hat{Z}|^2 \\ \leq y^2r^2(t)|\hat{Z}|^2+|\hat{Z}'|^2 \end{aligned} \tag{4.1}$$

$$|\hat{Z}'|^2+2\int_{-}^{t}\tilde{s}(\xi)|\hat{Z}'|^2d\xi+a(t)y^2|\hat{Z}|^2 \tag{4.2}$$

We will develop now a uniqueness procedure based on [6; 8] which uses the following formal calculations, valid for $\tau>0$. Define first

$$(4.9)\qquad \begin{aligned} R(t,\xi) &= \langle Z(t,\xi,\cdot),\ T(\cdot)u(\xi)\rangle\ ;\\ S(t,\xi) &= \langle Y(t,\xi,\cdot),\ T(\cdot)u'(\xi)\rangle \end{aligned}$$

where u is *any* solution of our modified equation (1.1) (i.e. $s(t)$ is replaced by $\tilde{s}(t)=s(t)+2\gamma(t)$ and $b(t)=0$) with $f=0$. For $\tau>0$, Y, Z, Y_τ and Z_τ belong to $\mathcal{E}_x'$ with supports contained in a fixed compact set so (4.9) makes sense, as do the following computations (cf. (2.8)–(2.9)), but we will mercifully omit detailed examination of each step. Thus

$$(4.10)\qquad \begin{aligned} R_\xi &= \langle Z_\xi,\ Tu\rangle+\langle Z,\ Tu'\rangle = \langle Z,\ Tu'\rangle\\ &-\langle a(\xi)\Delta Y,\ Tu\rangle-\langle r(\xi)\frac{d}{dx}Y,\ Tu\rangle = \langle Z,\ Tu'\rangle\\ &+\langle Y,\ r(\xi)ATu\rangle-\langle Y,\ a(\xi)A^2Tu\rangle \end{aligned}$$

$$(4.11)\qquad \begin{aligned} S_\xi &= \langle Y_\xi,\ Tu'\rangle+\langle Y,\ Tu''\rangle = \langle Y,\ Tu''\rangle\\ &-\langle Z,\ Tu'\rangle+\langle \tilde{s}(\xi)Y,\ Tu'\rangle = \langle Y,\ Tu''\rangle\\ &+\langle Y,\ \tilde{s}(\xi)Tu'\rangle-\langle Z,\ Tu'\rangle\,. \end{aligned}$$

Letting $\varphi(t,\xi)=R(t,\xi)+S(t,\xi)$ we have from (4.10)–(4.11)

$$(4.12)\qquad \varphi_\xi = \langle Y,\ T(u''+\tilde{s}u'+rAu-aA^2u)\rangle = 0\,.$$

Consequently $\varphi(t,t)=\varphi(t,\tau)$ which implies that

$$(4.13)\qquad \begin{aligned} u(t) &= \langle Z(t,\tau,\cdot),\ T(\cdot)u(\tau)\rangle\\ &+\langle Y(t,\tau,\cdot),\ T(\cdot)u'(\tau)\rangle = \langle F^{1/2}(\tau)Z(t,\tau,\cdot),\ T(\cdot)F^{-1/2}(\tau)u(\tau)\rangle\\ &+\langle \underset{\sim}{Q}(\tau)Y(t,\tau,\cdot),\ T(\cdot)\underset{\sim}{Q}^{-1}(\tau)u'(\tau)\rangle \end{aligned}$$

Now let $\tau\to 0$ and if $F^{-1/2}(\tau)u(\tau)$ and $\underset{\sim}{Q}^{-1}(\tau)u'(\tau)\to 0$ we have $u(t)\equiv 0$. Hence, referring back to the original equation (1.1) via γ as before we have proved

Theorem 4.3. *Let u satisfy* (1.1) (*modified*) *under the stipulations that $F^{-1/2}(\tau)u(\tau)\to 0$ and $\underset{\sim}{Q}^{-1}(\tau)u'(\tau)\to 0$ as $\tau\to 0$. Assume the hypotheses of Lemma* 2.2. *Then u is unique.*

REMARK 4.4. The condition $\hat{P}\geq 0$ has been discussed in [7; 8; 23; 24].

In general the requirements of Theorem 4.3 regarding the growth of $u(\tau)$ and $u'(\tau)$ as $\tau\to 0$ are too strong (cf. [7]) although the solution u of (1.1) given by (3.1) could be made to satisfy them by imposing further hypotheses on f. It is therefore of some interest to consider the case when $F(\tau)\not\to 0$ as $\tau\to 0$ and the relation of this to certain conditions of Krasnov [15] and Protter [18] has been

$$\leq a(\tau)y^2+\int_{-}^{t}[(a'y^2+y^2r^2)|\hat{Z}|^2+|\hat{Z}'|^2]d\xi .$$

Setting $P=a'y^2+y^2r^2$ as before and $\tilde{Q}=1-2\tilde{s}$ with $|\hat{Q}|\leq\tilde{c}$ on $[0, T]$, we write $\Xi=|\hat{Z}'|^2+ay^2|\hat{Z}|^2$ and add. $\tilde{c}\int_{\tau}^{t}ay^2|\hat{Z}|^2d\xi$ to the right side of (4.2) to obtain

$$\tilde{\Xi}\leq a(\tau)y^2+\int_{\tau}^{t}P|\hat{Z}|^2d\xi+\tilde{c}\int_{\tau}^{t}\tilde{\Xi}d\xi \tag{4.3}$$

Consequently as in (2.15) there results

$$\tilde{\Xi}\leq a(\tau)y^2E(t, \tau)+\int_{\tau}^{t}P|\hat{Z}|^2E(t, \xi)d\xi \tag{4.4}$$

and as in (2.16) we obtain

$$a(t)y^2|\hat{Z}|^2\leq a(\tau)y^2E(t, \tau) \exp\int_{\tau}^{t}\hat{P}d\xi \tag{4.5}$$

which yields

Lemma 4.1. *Given the hypothesis of Lemma 2.2 on $a, b, r, s, \hat{P}$, with $\hat{Z}(t, \tau, y)$ the unique solution of (2.4) satisfying $\hat{Z}(\tau, \tau, y)=1$ and $\hat{Z}_t(\tau, \tau, y)=0$ it follows that for y real and $0\leq\tau\leq t\leq T$*

$$|\hat{Z}(t, \tau, y)|^2\leq E(t, \tau) \exp \int_{\tau}^{t}(r^2/a)d\xi \tag{4.6}$$

which can be written as $F(\tau)|\hat{Z}(t, \tau, y)|^2\leq E(t, \tau)$.

Similarly, as in (2.19)–(2.20), we could estimate $|\hat{Z}_t|$ and $|\hat{Z}_{tt}|$ but this will not be needed here. Instead we want estimates on $\hat{Y}_t$ and $\hat{Z}_\tau$ which will follow from (2.8)–(2.9). Thus, from (2.8) one obtains, using (2.18),

$$|Q(\tau)\hat{Z}_\tau|\leq\hat{k}+\hat{k}_1|y| \tag{4.7}$$

while, using (2.18) and (4.6), we get from (2.9)

$$|yQ(\tau)\hat{Y}_\tau|\leqslant\hat{k}_2+\hat{k}_3|y| . \tag{4.8}$$

From their expressions (2.8)–(2.9) (and reasoning about $\hat{Z}$ from Lemma 2.1 as was done for $\hat{Y}$ before Theorem 2.3) we know that $\hat{Y}_\tau$ and $\hat{Z}_\tau$ are entire functions in y of exponential type $\leq\hat{c}T$. The estimates (4.7)–(4.8) and an argument as in Theorem 2.3 then proves (cf. Lemma 4.1)

Theorem 4.2. *Under the hypothesis of Theorem 2.3, $F^{1/2}(\tau)Z=F^{1/2}(\tau)\mathcal{F}^{-1}\hat{Z}$, $Q(\tau)Z_\tau$ (and $Q(\tau)Z$), and $Q(\tau)Y_\tau$ belong to $\mathcal{E}_x'$ with supports contained in a fixed compact set for $0\leq\tau\leq t\leq T$. The derivatives in τ can be taken in $\mathcal{E}_x'$ for $\tau>0$ and $(t, \tau)\rightarrow F^{1/2}Z$ or QZ, QZ_τ, and $Q(\tau)Y_\tau$ are continuous with values in $\mathcal{E}_x'$.*

discussed in [7; 8]. In this event the requirements of Theorem 4.3 on u are only that $u(0)=0$ and $a^{-1/2}(\tau)u'(\tau)\to 0$ as $\tau\to 0$. To examine the feasibility of this let u satisfy the modified equation (1.1) with $f=0$, $u(0)=0$, and $u'(0)=0$. Multiply this equation by $\exp\int_0^t \tilde{s}(\xi)d\xi$ and integrate to obtain (cf. [7; 8])

$$u'(t) = -\int_0^t [Ar(\xi)u - A^2a(\xi)u]e^{-\int_\xi^t \tilde{s}(\eta)d\eta}d\xi\,. \tag{4.14}$$

Let p be any continuous seminorn in E so that, since $\exp(-\int_\xi^t \tilde{s}(\eta)d\eta)\le M$ on $[0, T]$,

$$p(u'(t))\le \int_0^t [r(\xi)p(Au)+a(\xi)p(A^2u)]M\,d\xi \tag{4.15}$$

Now $\int_0^t r(\xi)d\xi=\int_0^t a^{1/2}(r/a^{1/2})d\xi\le(\int_0^t a(\xi)d\xi)^{1/2}(\int_0^t (r^2/a)d\xi)^{1/2}$ whereas $\int_0^t a(\xi)d\xi=((\int_0^t a(\xi)d\xi)^{1/2})^2$. Since $p(Au)$ and $p(A^2u)$ will be bounded for a solution $u\in C^2(E)$ on $[0, T]$ we have for $\int_0^t (r^2/a)d\xi$ bounded

$$\begin{aligned} p(a^{-1/2}(t)u'(t)) &\le a^{-1/2}(t)p(u'(t)) \\ &\le M_1a^{-1/2}(t)\,(\int_0^t ad\xi)^{1/2}+M_2a^{-1/2}(t)\int_0^t ad\xi \\ &\le M_3a^{-1/2}(t)\,(\int_0^t ad\xi)^{1/2} \end{aligned} \tag{4.16}$$

Hence $a^{-1/2}(t)u'(t)\to 0$ if $a^{-1/2}(t)\,(\int_0^t ad\xi)^{1/2}\to 0$. This condition is examined in [7; 8; 23; 24] and since oscillations in $a(t)$ are permitted by the stipulation $\hat{P}\ge 0$ (or $a'\ge -r^2$) it is not automatically satisfied. However if a is monotone increasing near $t=0$ it is obviously valid since then $(\int_0^t ad\xi)^{1/2}\le a(t)^{1/2}t^{1/2}$. Thus it makes sense to state the result (after modification) as

Theorem 4.5. *Assume the hypothesis of Lemma* 2.2 *and suppose* $F(\tau)>0$ *on* $[0, T]$ *with* $a^{-1/2}(t)\,(\int_0^t a(\xi)d\xi)^{1/2}\to 0$ *as* $t\to 0$. *Then* $a^{-1/2}(\tau)u'(\tau)\to 0$ *as* $\tau\to 0$ *and if* u *satisfies* (1.1)–(1.2) *with* $f=0$ *it follows that* $u(t)\equiv 0$ *on* $[0, T]$.

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

References

[1] L. Bers: Mathematical aspects of subsonic and transonic gas dynamics, Wiley, N.Y., 1958.

[2] R. Carroll: *Some degenerate Cauchy problems with operator coefficients*, Pacific J. Math. **13** (1963), 471–485.

[3] R. Carroll: Abstract methods in partial differential equations, Harper-Row, N.Y., 1969.

[4] R. Carroll: *On some hyperbolic equations with operator coefficients*, Proc. Japan Acad. **49** (1973), 233–238.

[5] R. Carroll: *On a class of canonical singular Cauchy problems*, Anal. fonct. appl., Act. Sci. Ind. 1367, Hermann, Paris, 1975, pp. 71–90.

[6] R. Carroll: *A uniqueness theorem for EPD type equations in general space*, Applicable Anal. to appear.

[7] R. Carroll and C. Wang: *On the degenerate Cauchy problem*, Canad. J. Math. **17** (1965), 245–256.

[8] R. Carroll and R. Showalter: Singular and degenerate Cauchy problems, Academic Press, N.Y., 1976.

[9] E. Coddington and N. Levinson: Theory of ordinary differential equations, McGraw-Hill, N.Y., 1955.

[10] G. Gangeux: These 3me cycle, Paris, to appear.

[11] I. Gelfand and G. Šilov: Some questions of the theory of differential equations, Generalized functions, Vol. 3, Moscow, 1958.

[12] P. Hartman: Ordinary differential equations, Wiley, N.Y., 1964.

[13] R. Hersh: *Explicit solution of a class of higher order abstract Cauchy problems*, J. Differential Equations **8** (1970), 570–579.

[14] T. Komura: *Semigroups of operators in locally convex spaces*, J. Functional Analysis **2** (1968), 258–296.

[15] M. Krasnov: *Mixed boundary value problems for degenerate linear hyperbolic differential equations of second order*, Mat. Sbornik **91** (1959), 29–84.

[16] C. Lacomblez: *Une équation d'évolution du second order en t à coefficients dégénérés ou singuliers*, Pub. Math. Univ. Bordeux **4** (1974), 33–64.

[17] J. Lions: Equations différentielles-opérationnelles, Springer, Berlin, 1961.

[18] M. Protter: *The Cauchy problem for a hyperbolic second order equation with data on the parabolic line*, Canad. J. Math. **6** (1954), 542–553.

[19] L. Schwartz: *Les équations d'évolution liées au produit de composition*, Ann. Inst. Fourier **2** (1950), 19–49.

[20] L. Schwartz: Théorie des distributions, Edition "Papillon", Hermann, Paris, 1966.

[21] F. Treves: Topological vector spaces, distributions, and kernels, Academic Press, N.Y., 1967.

[22] W. Walker: A nonsymmetric singular Cauchy problem, Report 99, Math. Dept. Univ. Auckland, 1976.

[23] C. Wang: *On the degenerate Cauchy problem for linear hyperbolic equations of the second order*, Thesis, Rutgers Univ., 1964.

[24] C. Wang: *A uniqueness theorem on the degenerate Cauchy problem*, Canad. Math. Bull. **18** (1975), 417–421.